EQUIVARIANT COHOMOLOGY AND THE SUPER RECIPROCAL PLANE OF A HYPERPLANE ARRANGEMENT

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ABSTRACT. In this paper, we investigate certain graded-commutative rings which are related to the reciprocal plane compactification of the coordinate ring of a complement of a hyperplane arrangement. We give a presentation of these rings by generators and defining relations. This presentation was used by Holler and I. Kriz [8] to calculate the $\mathbb{Z}$-graded coefficients of localizations of ordinary $RO((\mathbb{Z}/p)^n)$-graded equivariant cohomology at a given set of representation spheres, and also more recently by the author [10] in a generalization to the case of an arbitrary finite group. We also give an interpretation of these rings in terms of superschemes, which can be used to further illuminate their structure.

1. INTRODUCTION

G-equivariant generalized homology and cohomology theory for a compact lie group G is best behaved when the (co)-homology groups are graded by elements of the real representation ring $RO(G)$. In this case (see Lewis, May, Steinberger [15] for background), the theory enjoys many of the properties of non-equivariant (co)-homology, for example, Spanier-Whitehead duality. Explicit calculations of equivariant cohomology groups, however, are much harder than in the non-equivariant case. A telling example is the case of "ordinary" G-equivariant cohomology theories, defined by Lewis, May and McClure [14]. These theories satisfy a "dimension axiom" in the sense that the $\mathbb{Z}$-graded part of their coefficients (i.e. (co)-homology of a point) are zero except in dimension 0 for all (closed) subgroups of G.

However, calculation of the $RO(G)$-graded coefficients of these "ordinary" G-equivariant cohomology theories has been an open problem since the 1980s, and these groups carry some deep information. For example, for the "constant" $\underline{\mathbb{Z}}$ Mackey functor coefficients, (which means that restrictions to subgroups are identities), a partial calculation of the $RO(G)$-graded coefficients for $G = \mathbb{Z}/8$ was a key ingredient in the solution by Hill, Hopkins and Ravenel [7] of the Kervaire invariant 1 problem.

The algebraic calculations made in the present paper are relevant to the ordinary $RO(G)$-graded (co)homology theory with constant $\underline{\mathbb{Z}/p}$ coefficients for $G=(\mathbb{Z}/p)^n$. We denote this theory by $H\underline{\mathbb{Z}/p}_{(\mathbb{Z}/p)^n}$. In the paper [9], Holler and I. Kriz calculated the "positive" part of these coefficients, meaning the groups

$$H\underline{\mathbb{Z}/p}^V_{(\mathbb{Z}/p)^n}(*) \tag{1}$$

with V an actual (not virtual) representation for $p=2$. A key ingredient in this calculation was the geometric fixed point ring

$$(\Phi^{(\mathbb{Z}/p)^n}H\underline{\mathbb{Z}/p})_*, \tag{2}$$

which is the localization of the full $RO((\mathbb{Z}/p)^n)$-graded coefficient ring by inverting the inclusions $S^0\rightarrow S^\alpha$ for all non-trivial irreducible representations α (see Tom Dieck [22] and [15], chapter 11, Def. 9.7).

Holler and I. Kriz [9] calculated the ring (2) for $p=2$ by hand using a spectral sequence, and commented that the rings seemed to have an unusual algebraic structure, and asked about its geometric significance. They also did not know how to complete the same computation for $p>2$, where the structure seemed much more complicated.

Answering the second question is the main purpose of the present paper. Using our main theorem (Theorem 2 below), Holler and I. Kriz [8] then generalized their calculations of the geometric fixed point coefficient ring (2) to $p>2$, and also answered the following more general question:

What is the structure of the $\mathbb{Z}$-graded coefficient ring R_S of the $(\mathbb{Z}/p)^n$-fixed point spectrum given by localizing $H\underline{\mathbb{Z}/p}_{(\mathbb{Z}/p)^n}$ by inverting the maps $S^0\rightarrow S^\alpha$ for a given set S of irreducible $(\mathbb{Z}/p)^n$-representations?

Symbolically, we may write

$$R_S=((\bigwedge_{i=1}^{m}S^{\infty\alpha_i})\wedge H\underline{\mathbb{Z}/p})^{(\mathbb{Z}/p)^n}_* \tag{3}$$

where $S=\{\alpha_1,\dots,\alpha_m\}$.

Then, in particular, the geometric fixed point coefficient ring (2) is equal to R_S where

$$S=\{\alpha_1,\dots,\alpha_{p^n-1}\}$$

consists of all non-trivial irreducible representations of $(\mathbb{Z}/p)^n$.

In [9] Theorem 2, Holler and I. Kriz proved that

$$\begin{gathered}(\Phi^{(\mathbb{Z}/2)^n} H\underline{\mathbb{Z}/2})_* = \\ \mathbb{Z}/2[t_\alpha | \alpha \in (\mathbb{Z}/2)^n \setminus \{0\}]/(t_\alpha t_\beta + t_\alpha t_\gamma + t_\beta t_\gamma | \alpha + \beta + \gamma = 0),\end{gathered} \tag{4}$$

where t_α are in degree 1. They proved this by counting the dimension of the submodule of homogeneous elements of a given degree and matching it with a spectral sequence. But what do these relations mean?

Consider the affine space

$$\mathbb{A}^n_{\mathbb{F}_2} = \mathrm{Spec}(\mathbb{F}_2[x_1, \dots, x_n]).$$

Then consider a set of elements z_α which are non-zero linear combinations of the coordinates $x_1, \dots, x_n$ with coefficients in $\mathbb{F}_2$. Such linear combinations can, in turn, be identified with equations of hyperplanes through the origin in $\mathbb{A}^n_{\mathbb{F}_2}$. (In the case of (4), all possible rational hyperplanes, as it turns out.) If we remove these hyperplanes from $\mathbb{A}^n_{\mathbb{F}_2}$, we obtain an affine variety with coordinate ring

$$\Big(\prod_{\alpha \in (\mathbb{Z}/2)^n \setminus \{0\}} z_\alpha^{-1}\Big)\mathbb{F}_2[x_1, \dots, x_n]. \tag{5}$$

The ring (4) is isomorphic to the subring of the ring (5) generated by the elements $t_\alpha = z_\alpha^{-1}$. This result turned out to be known (for example, [18], Theorem 4). In fact, the affine variety with coordinate ring (4) is known as the *reciprocal plane* of the hyperplane arrangement $\{z_\alpha\}$ (see [3]).

The main contribution of the present paper is finding an analog of this story for $p > 2$. From the point of view of algebraic geometry, there is no difference: As we already mentioned, the reciprocal plane construction is independent of characteristic.

In algebraic topology, however, when we are dealing with characteristic $p \neq 2$, coefficient rings become *graded-commutative*, i.e.

$$xy = (-1)^{|x||y|} yx$$

where $|x|$ denotes the degree of x. So to solve the structure of the rings (2), (3) for $p > 2$, it was necessary to discover the *appropriate graded-commutative analogue* of the reciprocal plane, and to prove structure results analogous to [18]. This is the main result of the present paper.

Very briefly, we consider the ring

$$\mathbb{F}_p[x_1, \dots, x_n] \otimes \Lambda_{\mathbb{F}_p}[dx_1, \dots, dx_n]$$

where Λ denotes the exterior algebra. In this ring, invert a set of linear combinations z_α of the elements x_i. The right ring turns out to be the subring generated by $t_\alpha = z_\alpha^{-1}$ and $u_\alpha = z_\alpha^{-1}dz_\alpha$. Topologically, the element t_α has degree 2 and the element u_α has degree 1, corresponding to the fact that we are dealing with complex, not real, representations for $p > 2$.

In this paper, I determine the structure of these subrings in a way analogous to (but more complicated than) the commutative case. Holler and I. Kriz [8] then used my structure theorems to prove that these rings are isomorphic to the rings (3) for $p > 2$. In a recent follow-up paper [10], I also used these results to obtain a generalization to all finite groups. These are the main topological applications of the results of the present paper.

On the geometric side, the *Spec* of a graded-commutative ring is a *superscheme* (for a survey, see [23]). In Section 4, I develop the superscheme analog of some of the known geometric structures associated with the reciprocal plane, which correspond to my algebraic generalization to graded-commutative rings. Again, the algebraic geometry side of the story is independent of characteristic.

The present paper is organized as follows: In the next section, I give precise statements of the algebraic results of this paper. In Section 3, I give a proof of the main theorem and also prove that the relation ideal K is also generated by the relation polynomials $P_{L,S}$ where the L's are restricted to "minimal" relations. In Section 4, I discuss the geometric interpretation, including the construction of the superscheme corresponding to the graded-commutative case (Theorem 3).

2. Statement of the results

Following Terao [21], consider an n-dimensional affine space

$$\mathbb{A}^n_F = Spec(F[x_1, \dots, x_n])$$

over a field F. Let $z_1, \dots, z_m$ be non-zero linear combinations of the coordinates $x_1, \dots, x_n$ with coefficients in F. We can think of the z_i's as equations of hyperplanes in $\mathbb{A}^n_F$. Then the coordinates $t_i = z_i^{-1}$ define a morphism of affine varieties

$$\pi : \mathbb{A}^n_F \setminus Z(z_1 \dots z_m) \to \mathbb{A}^m_F$$

where $ZI = Z(I)$ is the set of zeros of an ideal I. The morphism π is an embedding if the z_j's linearly span the x_i's. Consider the Zariski closure of $\mathrm{Im}(\pi)$. As we shall see, this variety is a cone, so we can speak of the corresponding projective variety. This construction, called the reciprocal plane, has been studied extensively (see [18, 16, 11, 17, 20, 19, 12, 13]). For a survey, see [3].

To understand this construction better, we must describe it algebraically, which will also bring us closer to the motivation of the present paper. Algebraically, let

$$R = z_1^{-1} \dots z_m^{-1} F[x_1, \dots, x_n] = F[x_1, \dots, x_n][z_1^{-1}, \dots, z_m^{-1}].$$

Then we have a homomorphism of rings

$$h : F[t_1, \dots, t_m] \to R$$

with $h(t_i) = z_i^{-1}$ (which is, of course, not onto). Consider the ideal $I = \mathrm{Ker}(h)$. Denote $\mathcal{A} = \{z_1, \dots, z_m\}$, and put

$$R_{\mathcal{A}, \mathbb{A}_F^n} = F[t_1, \dots, t_m]/I.$$

Then $\mathrm{Spec}(R_{\mathcal{A}, \mathbb{A}_F^n})$ is, by definition, the Zariski closure of $\mathrm{Im}(\pi)$. Also by the homomorphism theorem, $R_{\mathcal{A}, \mathbb{A}_F^n}$ is a subring of R. Observe that I is a prime ideal (therefore a radical) since R is an integral domain, and hence so are its subrings. Further, if the z_i's generate the x_j's, then

$$R = (t_1 \cdot \dots \cdot t_m)^{-1} R_{\mathcal{A}, \mathbb{A}_F^n}.$$

Thus, in particular, in this case π is an open embedding of the hyperplane arrangement complement into the Zariski closure of its image.

The ideal I is non-zero when there are linear dependencies among the hyperplane equations z_i. Suppose, then,

$$L = a_1 z_{i_1} + \dots + a_k z_{i_k} = 0 \in F[x_1, \dots, x_n] \tag{6}$$

where $a_1, \dots, a_k \in F$ are not 0, and

$$1 \le i_1 < \dots < i_k \le m.$$

So, in R, we have $\frac{a_1}{t_{i_1}} + \dots + \frac{a_k}{t_{i_k}} = 0$ where $k > 1$ (where, in the rest of this paper, we indentify $t_j = z_j^{-1}$). Thus,

$$\frac{a_1 t_{i_2} \dots t_{i_k} + \dots + a_j t_{i_1} \dots \widehat{t_{i_j}} \dots t_{i_k} + \dots + a_k t_{i_1} \dots t_{i_{k-1}}}{t_{i_1} \dots t_{i_k}} = 0 \in R, \tag{7}$$

where the hat means an omitted term.

Hence, the numerator P_L of the left hand side of (7) is in I.

Theorem 1. *(*[18], [3], *(5.3)) Let $\mathcal{Z}$ be the set of all nonzero linear relations L among the hyperplane equations z_i. Then*

$$I = (P_L(t_1, \dots, t_m) | L \in \mathcal{Z}), \tag{8}$$

or in other words,

$$R_{\mathcal{A},\mathbb{A}^n_F} = F[t_1,\dots,t_m]/(P_L(t_1,\dots,t_m)|L\in\mathcal{Z}).$$

Corollary 2. *([8, 9]) For $p=2$, the $\mathbb{Z}$-graded coefficient ring (6) of the constant $\underline{\mathbb{Z}/2}$-Mackey functor ordinary $(\mathbb{Z}/2)^n$-equivariant cohomology spectrum with the inclusion $S^0\to S^{\alpha_i}$ inverted where α_i are real irreducible representations corresponding to the hyperplanes z_i is*

$$R_S = R_{\mathcal{A},\mathbb{A}^n_{\mathbb{F}_2}}.$$

Example: Formula (4) is a special case of Corollary 2 when $\mathcal{A}$ contains all the non-zero linear combinations of the variables x_i (corresponding to all non-zero irreducible real representations of $(\mathbb{Z}/2)^n$).

To give a simple example of the generalization, consider $n=4$ and the hyperplanes

$$z_1 = x_1+x_2,\ \ z_2 = x_2+x_3,\ \ z_3 = x_3+x_4,\ \ z_4 = x_1+x_4.$$

Then the only relation among them is

$$L = z_1+z_2+z_3+z_4,$$

giving rise to

$$P_L = t_2t_3t_4+t_1t_3t_4+t_1t_2t_4+t_1t_2t_3,$$

so we have

$$R_S = R_{\mathcal{A},\mathbb{A}^n_{\mathbb{F}_2}} = \mathbb{F}_2[t_1,t_2,t_3,t_4]/(t_2t_3t_4+t_1t_3t_4+t_1t_2t_4+t_1t_2t_3).$$

For the graded-commutative case, consider

$$\Omega = F[x_1,\dots,x_n]\otimes\Lambda[dx_1,\dots,dx_n]$$

where Λ denotes the exterior algebra over the field F. Then the non-zero F-linear combinations z_i of the x_i's are in the center of Ω. Now consider

$$T = z_1^{-1}\dots z_m^{-1}\Omega\supset\Omega.$$

This is the graded-commutative analog of the ring R. We are interested in the subring $T_{\mathcal{A},\mathbb{A}^n_F}$ of T generated by $z_1^{-1},\dots,z_m^{-1},z_1^{-1}dz_1,\dots,z_m^{-1}dz_m$. Put $t_i=z_i^{-1}$and $u_i=z_i^{-1}dz_i$. Then we have a canonical homomorphism of rings

$$\psi:\Xi = F[t_1,\dots,t_m]\otimes\Lambda[u_1,\dots,u_m]\to T. \tag{9}$$

Let $K=\mathrm{Ker}(\psi)$. By the Homomorphism Theorem, we have

$$T_{\mathcal{A},\mathbb{A}^n_F} = \Xi/K.$$

We want to find the generators of the ideal K. Recalling (8), note that $I \subseteq K$, but in general, equality does not arise, so we need to look for additional relations. If L is again the left hand side of (6), then

$$dL = a_{i_1} dz_{i_1} + \cdots + a_{i_k} dz_{i_k} = 0 \in T.$$

If we multiply

$$P_L = a_{i_1} t_{i_2} \dots t_{i_k} + a_{i_2} t_{i_1} \widehat{t_{i_2}} \dots t_{i_k} + \\ \cdots + a_{i_j} t_{i_1} \dots \widehat{t_{i_j}} \dots t_{i_k} + \cdots + a_{i_k} t_{i_1} \dots t_{i_{k-1}}$$

by $dz_{j_1} \dots dz_{j_l}$ where

$$S = \{j_1 < \cdots < j_l\} \subseteq \{i_1, \dots, i_k\}, \tag{10}$$

then some monomial summands can be expressed in terms of the u_j's. If a monomial summand does not contain t_{j_s} but does contain dz_{j_s}, then use $dL = a_{i_1} dz_{i_1} + \cdots + a_{i_k} dz_{i_k}$ to eliminate dz_{j_s}. Explicitly, let

$$\begin{aligned} P_{L,S} &= P_L dz_{j_1} \dots dz_{j_l} \\ &- \textstyle\sum_{s=1}^{l} t_{i_1} \dots \widehat{t_{j_s}} \dots t_{i_k} dz_{j_1} \dots \widehat{dz_{j_s}} dL \dots dz_{j_l}. \end{aligned} \tag{11}$$

We have $P_{L,S} \in \Xi$. Note that, by definition, $P_{L,\emptyset} = P_L$. Our main result is

Theorem 3. *Let $\mathcal{Y}$ be the set of all pairs (L, S) where L is a linear relation among hyperplanes equations as in (6), and S is a subset of the index set as in (10). Then*

$$K = (P_{L,S} \mid (L, S) \in \mathcal{Y}).$$

In other words,

$$T_{\mathcal{A}, \mathbb{A}^n_F} = \Xi / (P_{L,S} \mid (L, S) \in \mathcal{Y}).$$

This algebraic Theorem, along with Theorem 7 below, was used in [8] to prove the following result:

Corollary 4. *([8]) For $p > 2$, the $\mathbb{Z}$-graded coefficient ring (6) of the constant $\underline{\mathbb{Z}/p}$-Mackey functor ordinary $(\mathbb{Z}/p)^n$-equivariant cohomology spectrum with inclusions $S^0 \to S^{\alpha_i}$ inverted where α_i are complex irreducible representations corresponding to the hyperplanes z_i is*

$$R_S = T_{\mathcal{A}, \mathbb{A}^n_F}.$$

Example: Let $L = z_1 + z_2 + z_3 = 0 \in \Omega$. Then we have

$$P_L = P_{L,\emptyset} = \frac{z_1 + z_2 + z_3}{z_1 z_2 z_3} = t_2 t_3 + t_1 t_3 + t_1 t_2.$$

Now to compute $P_{L,\{2\}}$, write

$$P_L dz_2 = t_2 t_3 dz_2 + t_1 t_2 dz_2 + t_1 t_3 dz_2 = u_2 t_3 + t_1 u_2 + t_1 t_3 dz_2. \tag{12}$$

Now use

$$dL = dz_1 + dz_2 + dz_3 = 0 \tag{13}$$

to express $dz_2 = -dz_1 - dz_3$, which we use to conclude

$$t_2t_3dz_2 = -t_1t_3(dz_1 + dz_3) = u_1t_3 + u_3t_1.$$

Substituting this into (12) gives the relation

$$P_{L,\{2\}} = u_2(t_1 + t_3) - u_1t_3 - u_3t_1.$$

To calculate $P_{L,\{1,2\}}$, we start with the expression

$$\begin{aligned} p_L dz_1dz_2 &= t_2t_3dz_1dz_2 + t_1t_2dz_1dz_2 + t_1t_3dz_1dz_2 = \\ &= t_2t_3dz_1dz_2 + u_1u_2 + t_1t_3dz_1dz_3. \end{aligned}$$

Using (13) again, we get

$$t_2t_3dz_1dz_2 = t_2t_3(-dz_2 - dz_3)dz_2 = t_2t_3dz_2dz_3 = u_2u_3$$

and

$$t_1t_3dz_1dz_2 = t_1t_3dz_1(-dz_1 - dz_3) = u_3u_1.$$

Thus, we obtain the relation

$$P_{L,\{1,2\}} = u_1u_2 + u_2u_3 + u_3u_1.$$

The reader should keep in mind that the above derivation of examples of the relations $P_{L,S}$ is used simply to explain our definition of these relations. Nevertheless, they illustrate the fact that $P_{L,S}$ is a relation in $t_1^{-1} \dots t_m^{-1}\Xi$ which is contained in Ξ, and thus is valid in Ξ.

3. The proof of the main result

In this section, we will prove Theorem 3. In this proof, we will use the notion of a Gröbner basis of a module. Let F be a field and let $R = F[x_1, \dots, x_n]$. Consider a free module

$$R\{e_1, \dots, e_k\}. \tag{14}$$

By a *monomial* we mean a product of some powers of the x_i's with one e_j and possibly a non-zero coefficient from F. On monomials (ignoring the coefficients) we have the TOP (term over position) lexicographic order given by

$$x_n > \dots > x_1 > e_k > \dots > e_1.$$

A nonzero element p of (14) can be expressed as a sum of monomials which are not F-multiples of each other, the greatest of which is called the *leading term* $LT(p)$. Let M be a submodule of (14). Note that by the Hilbert Basis Theorem, M must be finitely generated. A set of

R-module generators g_i of M is called a *Gröbner basis* if their leading terms $LT(g_i)$ generate the submodule of (14) generated by the leading terms of all elements of M. (We allow the set to include 0, which does not affect whether it is a Gröbner basis or not).

Then we have the following fact known as the *Buchberger criterion*:

Theorem 5. *(Theorem 15.8,* [4]*) Let $g_1, \dots, g_n$ be nonzero elements of M. Let $f_{i,j}$, $f_{j,i}$ be the monomials of minimal possible degree such that the leading terms of $f_{j,i} \cdot g_i$ and $f_{i,j} \cdot g_j$ are equal for every $i \neq j \in \{1, \dots, n\}$ for which the leading terms of g_i and g_j involve the same basis element e_ℓ of M. Then the set $\{g_1, \dots g_n\}$ is a Gröbner basis for M if and only if there exist, for all applicable $i, j \in \{1, \dots, n\}$, polynomials $h_{i,j,s}$ such that $f_{j,i} \cdot g_i - f_{i,j} \cdot g_j = 0$ or*

$$f_{j,i} \cdot g_i - f_{i,j} \cdot g_j = \sum_{s=1}^{n} h_{i,j,s} g_s$$

where the summands on the right hand side have leading terms less than or equal to the leading term of $f_{j,i} \cdot g_i - f_{i,j} \cdot g_j$.

□

To apply this to our situation, first define

$$\Psi_{\mathcal{A},\mathbb{A}^n_F} = R_{\mathcal{A},\mathbb{A}^n_F} \otimes \Lambda_F[dz_1, \dots, dz_m]/(dL \mid L \in \mathcal{Z}).$$

Note that this is a graded $R_{\mathcal{A},\mathbb{A}^n_F}$-module by Grassmannian degree. In other words, it can be expressed as the direct sum of the free $R_{\mathcal{A},\mathbb{A}^n_F}$-modules

(15)
$$M_r = \Lambda^r_{R_{\mathcal{A},\mathbb{A}^n_F}}[dz_1, \dots, dz_m]/(\Lambda^{r-1}_{R_{\mathcal{A},\mathbb{A}^n_F}}[dz_1, \dots, dz_m] \wedge F\{dL \mid L \in \mathcal{Z}\}).$$

Now, without loss of generality, $z_1, \dots, z_{m_0}$ is a basis of the F-vector space generated by $z_1, \dots, z_m$, for some $1 \leq m_0 \leq m$. For any

$$I = \{i_1 < \cdots < i_r\} \subseteq \{1, \dots, m\},$$

we can write

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_r} = a_1 dz_{I'_1} + \cdots + a_k dz_{I'_k}$$

for some $I'_1, \dots, I'_k \subseteq \{1, \dots, m_0\}$ of cardinality r, which are unique if we insist the sets I'_j be different and the coefficients be non-zero. In other words, the $\binom{m_0}{r}$ elements $dz_{I'}$ for subsets $I' \subseteq \{1, \dots, m_0\}$ of cardinality r form a basis of the free $R_{\mathcal{A},\mathbb{A}^n_F}$-module M_r. We can write

$$M_r = \Lambda^r_{R_{\mathcal{A},\mathbb{A}^n_F}}[dz_1, \dots, dz_{m_0}].$$

(We write $=$ instead of $\cong$ to indicate that the isomorphism is canonical.) We shall also write

$$e_I = dz_I$$

for $I \subseteq \{1, \dots, m\}$.

Now $T_{\mathcal{A},\mathbb{A}^n_F}$ is the graded $R_{\mathcal{A},\mathbb{A}^n_F}$-submodule of $\Psi_{\mathcal{A},\mathbb{A}^n_F}$ whose degree r submodule is generated by the elements

$$u_{i_1} \wedge \cdots \wedge u_{i_r} = t_{i_1} \cdot \cdots \cdot t_{i_r} dz_{i_1} \wedge \cdots \wedge dz_{i_r}$$

with $I = \{i_1 < \cdots < i_r\} \subseteq \{1, \dots, m\}$. Denote

$$t_I = t_{i_1} \cdot \cdots \cdot t_{i_r}.$$

Now consider the $F[t_1, \dots, t_m]$-module

$$\Phi_r = F[t_1, \dots, t_m] \otimes \Lambda^r_F[dz_1, \dots, dz_{m_0}].$$

Then consider the submodules of Φ_r defined by

$$P_r = \langle t_I \cdot dz_I \mid |I| = r,\ I \subseteq \{1, \dots, m\}\rangle$$

$$N_r = \langle P_L \cdot dz_{I'} \mid |I'| = r,\ I' \subseteq \{1, \dots, m_0\}, L \in \mathcal{Z}\rangle.$$

We have (by the Homomorphism Theorem)

$$T_{\mathcal{A},\mathbb{A}^n_F} = \bigoplus_{r=1}^{m_0} (P_r / N_r \cap P_r).$$

Hence, it suffices to calculate the submodule of relations $N_r \cap P_r$ for a given r and verify that it is generated by the $\bigwedge_F[u_1, \dots, u_m]$-multiples of the $P_{L,S}$'s contained in M_r.

We will use the Buchberger criterion (Theorem 5) above to calculate $N_r \cap P_r$. In general, for two submodules $\langle f_1, \dots, f_n\rangle$, $\langle g_1, \dots, g_m\rangle$ of a free module over a polynomial ring, their intersection can be calculated by introducing another polynomial variable s greater than the other variables in the lexicographic order; it is then generated by all elements of a Gröbner basis of

$$\langle f_1 \cdot s, \dots, f_n \cdot s, g_1 \cdot (1 - s), \dots, g_m \cdot (1 - s)\rangle$$

that do not contain s.

Letting

$$L = a_1 z_{i_1} + \cdots + a_k z_{i_k}$$

where $1 \leq i_1 < \cdots < i_k \leq m$, $a_i \neq 0 \in F$, put

$$|L| := \{i_1, \dots, i_k\}. \tag{16}$$

Then Theorem 3 follows from the following

Lemma 6. *The set* $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$*, where*

$$\mathcal{S}_1 = \{s \cdot t_{I_1\cup\cdots\cup I_k} \cdot (a_1 e_{I_1} + \cdots + a_k e_{I_k}) \mid \\ a_i \in F,\ I_1, \dots I_k \subseteq \{1, \dots m\}, \forall i\ |I_i| = r\},$$

$$\mathcal{S}_2 = \{(1-s) \cdot P_L \cdot e_{I'} \mid L \in \mathcal{Z},\ I' \subseteq \{1, \dots, m_0\},\ |I'| = r\}, \tag{17}$$

$$\mathcal{S}_3 = \{P_L \cdot t_{(I_1\cup\cdots\cup I_k)\smallsetminus|L|} \cdot (a_1 e_{I_1} + \cdots + a_k e_{I_k}) \mid \\ a_i \in F,\ L \in \mathcal{Z},\ I_1, \dots I_k \subseteq \{1, \dots m\}, \forall i\ |I_i| = r\}.$$

forms a Gröbner basis of the $F[s, t_1, \dots, t_m]$*-submodule of* $\Phi_r \otimes_F F[s]$ *generated by*

$$\begin{gathered}\{s \cdot t_I \cdot e_I \mid I \subseteq \{1, \dots, m\},\ |I| = r, \\ (1-s) \cdot P_L \cdot e_{I'} \mid I' \subseteq \{1, \dots, m_0\},\ |I'| = r,\ L \in \mathcal{Z}\}\end{gathered} \tag{18}$$

with respect to the TOP lexicographic order specified above.

Remark: Note that in the ring $\Psi_{\mathcal{A},\mathbb{A}^n_F}$, the elements of $\mathcal{S}_3$ are in the ideal $(P_{L,S})$. Concretely,

$$P_L t_{I\smallsetminus|L|} e_I = t_{I\smallsetminus|L|} P_{L,I} \in \Psi_{\mathcal{A},\mathbb{A}^n_F}$$

(see (11)).

Proof of Lemma 6. First observe that all elements of (17) are generated by (18). Note that this is only nontrivial for $\mathcal{S}_3$, in which case it was already observed in Section 2 (where we introduced $P_{L,S}$). We will prove Lemma 6 by verifying the assumptions of the Buchberger criterion for any applicable pair of elements from (17). This gives six cases:

Case 1: $\mathcal{S}_1$ vs. $\mathcal{S}_1$.

Suppose we have two elements of $\mathcal{S}_1$, i.e. two nonzero elements

$$s \cdot t_{I_1\cup\cdots\cup I_k} \cdot (a_1 e_{I_1} + \dots a_k e_{I_k})$$

$$s \cdot t_{J_1\cup\cdots\cup J_\ell} \cdot (b_1 e_{J_1} + \dots b_\ell e_{J_\ell})$$

whose leading terms involve the same basis element $e_{I'}$, $I' \subseteq \{1, \dots, m_0\}$. Without loss of generality, $a_i, b_j \in F^\times$ and e_{I_1}, e_{J_1} involve the basis element of the highest degree. So, we must multiply the two elements by $b_1 t_{J_1\cup\cdots\cup J_\ell\smallsetminus I_1\cup\cdots\cup I_k}$ and $a_1 t_{I_1\cup\cdots\cup I_k\smallsetminus J_1\cup\cdots\cup J_\ell}$ to match their leading terms. Then the difference is

$$s \cdot t_{I_1\cup\cdots\cup I_k\cup J_1\cup\cdots\cup J_\ell} \cdot (b_1 \cdot (a_1 e_{I_1} + a_2 e_{I_2} + \cdots + a_k e_{I_k}) - \\ a_1 \cdot (b_1 e_{J_1} + b_2 e_{J_2} + \cdots + b_\ell e_{J_\ell})),$$

which is still an element of $\mathcal{S}_1$.

Case 2: $\mathcal{S}_2$ vs. $\mathcal{S}_2$.

Suppose we have two elements of $\mathcal{S}_2$, say

$$(1-s)\cdot P_L \cdot e_{I'} \tag{19}$$

$$(1-s)\cdot P_M \cdot e_{J'} \tag{20}$$

for $I', J' \subseteq \{1,\dots,m_0\}$, $|I'| = |J'| = r$. The Buchberger criterion gives a condition only when $e_{I'} = e_{J'}$, i.e. $I' = J'$. To match the leading terms of these two elements, we therefore multiply (19), (20) by monomials $a \cdot t_K$, $b \cdot t_{K'}$. Then $LT(a\cdot t_K \cdot P_L) = LT(b \cdot t_{K'} \cdot P_M)$. However, by the results of Proudfoot and Speyer (Theorem 2 of [18]), the P_L's form a universal Gröbner basis. Thus,

$$a \cdot t_K \cdot P_L - b \cdot t_{K'} \cdot P_M = \sum_N t_{I_N} \cdot P_{L_N},$$

for some t_{I_N}, P_{L_N} where the summands have lesser or equal leading terms than the left hand side, and hence,

$$a\cdot t_K\cdot(1-s)\cdot P_L\cdot e_{I'} - b\cdot t_{K'}\cdot(1-s)\cdot P_M\cdot e_{J'} = (1-s)\cdot e_{I'}\cdot(\sum_N P_{L_N} t_{I_N}),$$

which is a linear combination of elements of $\mathcal{S}_2$.

Case 3: $\mathcal{S}_3$ vs. $\mathcal{S}_3$.

Suppose we have two different nonzero elements of $\mathcal{S}_3$, say

$$\begin{gathered} P_L \cdot t_{(I_1\cup\dots\cup I_k)\smallsetminus|L|}\cdot(a_1\cdot e_{I_1}+\dots+a_k\cdot e_{I_k}) \\ P_{L'} \cdot t_{(J_1\cup\dots\cup J_\ell)\smallsetminus|L'|}\cdot(b_1\cdot e_{J_1}+\dots+b_\ell\cdot e_{J_\ell}). \end{gathered} \tag{21}$$

Again, the condition of the Buchberger criterion only applies when the leading terms of

$$\begin{gathered} a_1\cdot e_{I_1}+\dots+a_k\cdot e_{I_k} \\ b_1\cdot e_{J_1}+\dots+b_\ell\cdot e_{J_\ell} \end{gathered} \tag{22}$$

are equal up to non-zero scalar multiple, which, without loss of generality, we can assume to be equal to 1.

First of all, note that without loss of generality, we have

$$\begin{gathered} min(|L|) \notin I_1\cup\dots\cup I_k \\ min(|L'|) \notin J_1\cup\dots\cup J_\ell, \end{gathered} \tag{23}$$

since otherwise the elements (21) are $t_{min(|L|)}$- resp. $t_{min(|L'|)}$-monomial multiples of other elements of $\mathcal{S}_3$ which satisfy (23) by applying the relations dL, respectively dL', to (22), using (15). (Note that this is where our definition of $P_{L,S}$ in Section 2 comes from.)

To simplify notation, from now on, we shall write

$$I = I_1\cup\dots\cup I_k$$

$$J = J_1 \cup \cdots \cup J_\ell$$

and abbreviate the elements (22) as $e_{(I)}$, $e_{(J)}$. By assumption, we have

$$LT(e_{(I)}) = LT(e_{(J)}). \tag{24}$$

Our strategy will be to verify the condition of the Buchberger criterion separately on the pairs of elements

$$P_{L'} \cdot t_{I \smallsetminus |L'|} \cdot e_{(I)}, P_L \cdot t_{I \smallsetminus |L|} \cdot e_{(I)} \tag{25}$$

and

$$P_{L'} \cdot t_{I \smallsetminus |L'|} \cdot e_{(I)}, P_{L'} \cdot t_{J \smallsetminus |L'|} \cdot e_{(J)}. \tag{26}$$

First let us verify that this is valid. The elements (21) can be brought to a minimal common leading term by multiplying by t_i-monomials. Denote those elements with a common leading term by

$$A = \overline{P_L \cdot t_{I \smallsetminus |L|}} \cdot e_{(I)}$$
$$C = \overline{P_{L'} \cdot t_{J \smallsetminus |L'|}} \cdot e_{(J)}.$$

First we need to show that

$$LT(P_{L'} \cdot t_{I \smallsetminus |L'|} \cdot e_{(I)}) \mid LT(A). \tag{27}$$

To this end, note that the t_i-monomial factor of $LT(A) = LT(C)$ is of the form t_H with

$$H = ((|L| \smallsetminus \{min(|L|)\}) \cup (I \smallsetminus |L|)) \cup$$
$$((|L'| \smallsetminus \{min(|L'|)\}) \cup (J \smallsetminus |L'|))$$

while $P_{L'} \cdot t_{I \smallsetminus |L'|}$ has leading term (up to scalar multiple) t_G with

$$G = (|L'| \smallsetminus \{min(|L'|)\}) \cup (I \smallsetminus |L'|).$$

Thus, we need to show $G \subseteq H$.

To this end, note that H is $I \cup J \cup |L| \cup |L'|$ with possible exclusion of the elements $min(|L|)$, $min(|L'|)$, while $G = (|L'| \cup I) \smallsetminus \{min(|L'|)\}$. Thus, the only possible element of $G \smallsetminus H$ could be $min(|L|)$. But by (23), $min(|L|) \notin I$, so in that case

$$min(|L|) \in |L'| \smallsetminus \{min(|L'|)\} \subseteq H.$$

Contradiction.

Thus, (27) is proved. However, this also proves

$$LT(P_{L'} \cdot t_{I \smallsetminus |L'|}) \mid LT(\overline{P_{L'} \cdot t_{J \smallsetminus |L'|}})$$

and hence

$$P_{L'} \cdot t_{I \smallsetminus |L'|} \mid \overline{P_{L'} \cdot t_{J \smallsetminus |L'|}},$$

since both sides are equal up to a t_i-monomial multiple. Put

$$B = \overline{P_{L'} \cdot t_{J \smallsetminus |L'|}} \cdot e_{(I)}.$$

Thus,

$$LT(A) = LT(B) = LT(C).$$

To verify that the pairs (25), (26) can be considered separately, by Theorem 5, it therefore suffices to show that

$$LT(A - B),\ LT(B - C) \leq LT(A - C). \tag{28}$$

Clearly, it now suffices to assume $e_{(I)} \neq e_{(J)}$ (since otherwise $B = C$). Similarly, we can assume $A \neq B$. Then we have

$$LT(A - B) = LT(\overline{P_L \cdot t_{I \smallsetminus |L|}} - \overline{P_{L'} \cdot t_{J \smallsetminus |L'|}}) \cdot LT(e_{(I)})$$

and

$$LT(A - C) = LT(\overline{P_L \cdot t_{I \smallsetminus |L|}}) \cdot LT(e_{(I)} - e_{(J)})$$

by the TOP ordering. Now we have

$$LT(\overline{P_L \cdot t_{I \smallsetminus |L|}} - \overline{P_{L'} \cdot t_{J \smallsetminus |L'|}}) < LT(\overline{P_{L'} \cdot t_{J \smallsetminus |L'|}}) = LT(\overline{P_L \cdot t_{I \smallsetminus |L|}}),$$

and thus again, since we are using the TOP ordering,

$$LT(A - B) = LT(\overline{P_L \cdot t_{I \smallsetminus |L|}} - \overline{P'_L \cdot t_{J \smallsetminus |L'|}}) \cdot LT(e_{(I)}) <$$
$$LT(\overline{P_L \cdot t_{I \smallsetminus |L|}}) \cdot LT(e_{(I)} - e_{(J)}) = LT(A - C).$$

Also,

$$LT(B - C) = LT(\overline{P_{L'} \cdot t_{J \smallsetminus |L'|}}) \cdot LT(e_{(I)} - e_{(J)}),$$

which equals $LT(A - C)$. Thus, (28) is proved.

Subcase 3a: The pair (25).

We must verify the condition of the Buchberger criterion for $P_L \cdot t_{I \smallsetminus |L|} \cdot e_{(I)}$ and $P_{L'} \cdot t_{I \smallsetminus |L'|} \cdot e_{(I)}$. Multiplying by t_i-monomials to get equal leading terms and applying the Buchberger criterion to P_L, $P_{L'}$, since the P_M's form a universal Gröbner basis by [18], the difference is, again, a linear combination

$$e_{(I)} \cdot \sum t_{Q_{i,j}} \cdot P_{M_i} \tag{29}$$

for different relations M_i (not scalar multiples of one another) where the summands have lower leading terms. Note that several different monomials $t_{Q_{i,j}}$ can be multiplied by the same element P_{M_i}, which is why the second index is needed. Additionally, we have

$$I \smallsetminus (|L| \cup |L'|) \subseteq Q_{i,j}, \tag{30}$$

since all the polynomials we started with were multiples of $t_{I \smallsetminus (|L| \cup |L'|)}$.

Now each M_i will be some linear combination of L and L' and thus can be written as

$$M_i = a_i \cdot L + b_i \cdot L'. \tag{31}$$

To see this, take a set of variables z_s modulo the linear relations L and L'. By the results of Proudfoot and Speyer [18], applying the Buchberger criterion to P_L and $P_{L'}$ (thought of as elements of the polynomial ring on the t_s) then gives a linear combination of some of the polynomials P_{M_i}. Those are necessarily of the form (31) since those are the only relations present. However, the resulting equality (of the form "a monomial multiple of P_L plus a monomial multiple of $P_{L'}$ equals a linear combination of P_{M_i}") will remain valid when other linear relations among the z_s are added (since this will only enlarge the ideal in the polynomial ring on the generators t_s) and hence, after being multiplied by an appropriate monomial, can be used for (30). This proves (31).

We need to show

$$|(|L| \cup |L'|) \smallsetminus (Q_{i,j} \cup M_i)| \leq 1. \tag{32}$$

The reason why (32) suffices is that we have linear relations between the basis elements e_J. Recall that those elements are exterior multiples of the elements e_j. Now if, say, $j \in |L|$, then the relation dL expresses e_j as a linear combination of other elements e'_j by (15). Under the assumption (32), such elements are accompanied by a $t_{j'}$ factor, and thus are linear combinations of the basis elements $\mathcal{S}_3$. The argument of L' is analogous.

To prove (32), consider again the relation (31). Let us study the possible sets of indices s such that the coefficient at z_s of a given non-zero linear combination of the relations L, L' is 0. Those are sets of such indices s on which the ratio of the coefficients of L and L' at z_s is constant. There are only finitely many such ratios for which this set of indices is non-empty. I denote these ratios by q_k ($k \in K$ where K is some finite set). Let us denote the set of indicies where the ratio is q_k by S_{q_k}.

More precisely, let $L = \sum \alpha_s \cdot z_s$, $L' = \sum \beta_s \cdot z_s$, $(\alpha_s, \beta_s) \neq (0,0)$ for $s \in |L| \cup |L'|$, and the disjoint sets

$$S_{q_k} = \{s \in |L| \cup |L'| \mid [\alpha_s : \beta_s] = q_k\}$$

for different ratios q_k. The possible $|M_i|$'s are of the form $|L| \cup |L'| \smallsetminus S_{q_k}$ (which includes $|L|$, $|L'|$, and $|L| \cup |L'|$). Each $S_{q_k} \neq \emptyset$ is associated with at most one M_i in (31) such that $|M_i| = |L| \cup |L'| \smallsetminus S_{q_k}$ (since we already know the ratio $[a_i : b_i]$ and we chose the M_i not to be multiples of each other). Thus, we have proven (32) if we can rule out

$$|S_{q_k} \smallsetminus Q_{i,j}| \geq 2. \tag{33}$$

However, (33) is impossible since if it occurred, then the monomial terms of $t_{Q_{i,j}} \cdot P_{M_i}$ could not be multiples of any of the monomial terms of $t_{Q_{i',j'}} \cdot P_{M_{i'}}$ for $i' \neq i$, $t_{I\smallsetminus |L|} \cdot P_L$, or $t_{I\smallsetminus |L'|} \cdot P_{L'}$ each of which miss at most one variable t_ℓ for $\ell \in Q_{i,j}$. Selecting a minimal $Q_{i,j}$ for a given i (with respect to inclusion), the monomial terms of $t_{Q_{i,j}} \cdot P_{M_i}$ also cannot be multiples of any monomial terms of $t_{Q_{i,j'}} \cdot P_{M_i}$ for $j' \neq j$. This contradicts the assumption that (29) was obtained by applying the Buchberger criterion to the elements indicated. This concludes the case of (25).

Subcase 3b: The pair (26).

We must verify the condition of the Buchberger criterion for

$$P_{L'} \cdot t_{I_1 \cup \dots \cup I_k \smallsetminus |L'|} \cdot (a_1 e_{I_1} + \dots a_k e_{I_k}),$$

$$P_{L'} \cdot t_{J_1 \cup \dots \cup J_\ell \smallsetminus |L'|} \cdot (b_1 e_{J_1} + \dots b_\ell e_{J_\ell}).$$

The Buchberger algorithm step gives

$$P_{L'} \cdot t_{I_1 \cup \dots \cup I_k \cup J_1 \cup \dots \cup J_\ell \smallsetminus |L'|} \cdot (a_1 e_{I_1} + \dots a_k e_{I_k} - b_1 e_{J_1} - \dots - b_\ell e_{J_\ell}),$$

which is an element of $\mathcal{S}_3$.

Case 4: $\mathcal{S}_1$ vs. $\mathcal{S}_2$.

Now we will show that the Buchberger criterion's assumptions hold for an element of $\mathcal{S}_1$ and an element of $\mathcal{S}_2$. Suppose we have two nonzero elements of the form

$$s \cdot t_{I_1 \cup \dots \cup I_k} \cdot (a_1 e_{I_1} + \dots + a_k e_{I_k}),\ I_i \subseteq \{1, \dots, m\},\ |I_i| = r \tag{34}$$

$$(1-s) \cdot P_L \cdot e_{I'},\ I' \subseteq \{1, \dots, m_0\},\ |I'| = r.$$

Then there exist some unique nonzero $b_1, \dots, b_\ell$'s and distinct $I'_1, \dots, I'_\ell \subseteq \{1, \dots, m_0\}$ such that

$$a_1 \cdot e_{I_1} + \dots + a_k \cdot e_{I_k} = b_1 \cdot e_{I'_1} + \dots + b_\ell \cdot e_{I'_\ell}.$$

Without loss of generality, they are ordered $e_{I'_1} > \dots > e_{I'_\ell}$. First, this implies, by definition,

$$P_L \cdot t_{(I_1 \cup \dots \cup I_k) \smallsetminus |L|} \cdot (b_1 e_{I'_1} + \dots + b_\ell e_{I'_\ell}) \in \mathcal{S}_3.$$

Take the minimal degree t_i-monomials f, g, such that the leading term of

$$f \cdot (1-s) \cdot P_L \cdot e_{I'} \tag{35}$$

equals the leading term of

$$g \cdot s \cdot t_{I_1 \cup \dots \cup I_k} \cdot (b_1 e_{I'_1} + \dots + b_\ell e_{I'_\ell}).$$

Then we must have $I' = I'_1$. In addition, $t_{(I_1\cup\cdots\cup I_k)\smallsetminus|L|}$ must also divide the monomial f. Thus, for some t_i-monomial h, we have

$$LT(f\cdot(1-s)\cdot P_L\cdot e_{I'}) =$$

$$LT(h\cdot(1-s)\cdot t_{(I_1\cup\cdots\cup I_k)\smallsetminus|L|}\cdot P_L\cdot(b_1e_{I'_1}+\cdots+b_\ell e_{I'_\ell})),$$

and the terms including each $b_ie_{I'_i}$ are in $\mathcal{S}_2$. Also, we have

$$\begin{aligned}LT(h\cdot(1-s)\cdot t_{(I_1\cup\cdots\cup I_k)\smallsetminus|L|}\cdot P_L\cdot(b_2e_{I'_2}+\cdots+b_\ell e_{I'_\ell}))\\ = LT(g\cdot s\cdot t_{I_1\cup\cdots\cup I_k}\cdot(b_2e_{I'_2}+\cdots+b_\ell e_{I'_\ell})).\end{aligned}$$

So we can replace $(1-s)\cdot P_L\cdot e_{I'}$ by

$$(1-s)\cdot P_L\cdot t_{I_1\cup\cdots\cup I_k\smallsetminus|L|}(b_1e_{I'_1}+\cdots+b_\ell e_{I'_\ell}) \tag{36}$$

in (34).

Now in verifying the Buchberger criterion, if we expand the P_L into a sum of t_i-monomials in the expression

$$-s\cdot t_{(I_1\cup\cdots\cup I_k)\smallsetminus|L|}\cdot P_L\cdot(b_1e_{I'_1}+\cdots+b_\ell e_{I'_\ell}), \tag{37}$$

the term of the leading t_i-monomial times g equals the leading term of (35), and the terms of (37) of the other monomials containing $t_{|L|\smallsetminus\{i\}}$, $i\in L$ are in $\mathcal{S}_1$ by eliminating i from $I_1\cup\cdots\cup I_k$ (if $i\in I_1\cup\cdots\cup I_k$) using the relation L, as above in proving the sufficiency of the assumption (23). Therefore, the difference of (34) and (36) is a sum of multiples of elements of $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3$ of lower or equal leading terms as required.

Case 5: $\mathcal{S}_1$ vs. $\mathcal{S}_3$.

Suppose we have a nonzero element of $\mathcal{S}_1$ and a nonzero element of $\mathcal{S}_3$ of the forms

$$s\cdot t_{I_1\cup\cdots\cup I_k}\cdot(a_1e_{I_1}+\ldots a_ke_{I_k})$$
$$P_L\cdot t_{(J_1\cup\cdots\cup J_\ell)\smallsetminus|L|}\cdot(b_1e_{J_1}+\ldots b_\ell e_{J_\ell}).$$

Take the minimal degree t_i-monomials f, g, such that the leading terms of

$$f\cdot s\cdot t_{I_1\cup\cdots\cup I_k}\cdot(a_1e_{I_1}+\ldots a_ke_{I_k})$$
$$g\cdot P_L\cdot t_{(J_1\cup\cdots\cup J_\ell)\smallsetminus|L|}\cdot(b_1e_{J_1}+\ldots b_\ell e_{J_\ell})$$

are equal. Then s must divide g. Then, again, after expanding P_L into a sum of t_i-monomials, in

$$g\cdot P_L\cdot t_{(J_1\cup\cdots\cup J_k)\smallsetminus|L|}\cdot(b_1e_{J_1}+\ldots b_\ell e_{J_\ell})-f\cdot s\cdot t_{I_1\cup\cdots\cup I_k}\cdot(a_1e_{I_1}+\ldots a_ke_{I_k})$$

the term corresponding to the greatest monomial will cancel and the terms corresponding to all other monomials can be expressed as multiples of elements of $\mathcal{S}_1$ by again using the relation L, as above.

Case 6: $\mathcal{S}_2$ vs. $\mathcal{S}_3$.

Finally, suppose we have an element of $\mathcal{S}_2$ and a nonzero element of $\mathcal{S}_3$ of the forms

$$(1-s)\cdot P_{L'}\cdot e_{I'}$$

$$P_L\cdot t_{(I_1\cup\cdots\cup I_k)\smallsetminus|L|}\cdot(a_1e_{I_1}+\cdots+a_ke_{I_k})$$

Take the minimal degree t_i-monomials f, g, such that the leading terms of

$$f\cdot(1-s)\cdot P_{L'}\cdot e_{I'} \tag{38}$$

and

$$g\cdot P_L\cdot t_{(I_1\cup\cdots\cup I_k)\smallsetminus|L|}\cdot(a_1e_{I_1}+\dots a_ke_{I_k}) \tag{39}$$

are equal. Then $g=-s\cdot h$ for some t_i-monomial h. If the difference (38) and (39) does not include s, then

$$f\cdot P_{L'}\cdot e_{I'}=h\cdot P_L\cdot t_{(I_1\cup\cdots\cup I_k)\smallsetminus|L|}\cdot(a_1e_{I_1}+\dots a_ke_{I_k})$$

which is a t_i-monomial multiple of an element of $\mathcal{S}_3$. Otherwise, we can replace (39) by

$$(1-s)\cdot h\cdot P_L\cdot t_{(I_1\cup\cdots\cup I_k)\smallsetminus|L|}\cdot(a_1e_{I_1}+\dots a_ke_{I_k}) \tag{40}$$

because the additional term is a t_i-monomial multiple of an element of $\mathcal{S}_3$ (which in particular does not involve s, thus having a lower leading term than the difference of (38) and (39)). Now the difference of (38) and (40) is a sum of t_i-monomial multiples of elements of $\mathcal{S}_2$. This concludes the proof of Lemma 6.

□

Call L *minimal* if there do not exist relations L_1, L_2 such that

$$L_1+L_2=L$$

$$|L_1|,|L_2|\subsetneq|L|.$$

Define shuffle permutations as follows: for sets of natural numbers $S_1=\{i_1<\cdots<i_k\}$, $S_2=\{j_1<\cdots<j_l\}$, $S_1\cap S_2=\emptyset$, denote by σ_{S_1,S_2} the permutation which puts the sequence $(i_1,\dots,i_k,j_1,\dots,j_l)$ in increasing order. Also define for $S=\{i_1<\cdots<i_k\}$:

$$t_S:=t_{i_1}\dots t_{i_k}$$

$$u_S:=u_{i_1}\dots u_{i_k}$$

$$dz_S:=dz_{i_1}\dots dz_{i_k}$$

Theorem 7. *The ideals I and K are generated as follows:*

$$I = (P_L | L \text{ is a minimal relation})$$

$$K = (P_{L,S} | L \text{ is a minimal relation and } S \subseteq |L|)$$

(For the case of I, see [18], *Theorem 4.)*

Let L be as in (6). Recalling the notation of (16), let $S \subseteq |L|$. Put

$$Q_{L,S} := t_{|L|} dL dz_S.$$

So obviously, $Q_{L,S} \in K$.

Lemma 8. $Q_{L,S} \in (P_{L,T} | T \subseteq |L|)$

Proof. If $S \neq \emptyset$, let $i \in S$. Then $i \in S \subseteq |L|$, and hence, $t_i | t_{|L|}$ and $dz_i | dz_S$. Thus, $u_i = t_i dz_i$ must divide $t_{|L|} dL dz_S = Q_{L,S}$. Then $Q_{L,S} = u_i P_{L,S}$. On the other hand, by applying (11), we have, by definition,

$$u_{i_1} P_L - t_{i_1} P_{L,\{i_1\}} = u_{i_1} P_L - t_i P_L dz_i + t_{i_1} \dots t_{i_\ell} dL = t_{|L|} dL = Q_{L,\emptyset}.$$

In the first summand the surviving term is the term of $P_{L,\emptyset}$ which omits t_{i_1}. In the second summand the surviving terms are the "error terms" of the summand of P_L which omits t_{i_1}. All remaining terms cancel. □

Proof of Theorem 7: Even case : Suppose we know

$$L_1 + L_2 = L$$

$$|L_1|, |L_2| \subsetneq |L|.$$

Then

$$P_L = t_{L \setminus L_1} P_{L_1} + t_{L \setminus L_2} P_{L_2}.$$

Odd case : If L is not minimal we know $L_1 + L_2 = L$ and $|L_1|, |L_2| \subsetneq |L|$. Based on the even case, the first guess for $P_{L,S}$ could be

$$\begin{aligned} P'_{L,S} &:= P_{L_1,S_1} u_{S \setminus S_1} t_{|L| \setminus (|L_1| \cup S)} sign(\sigma_{S, S \setminus S_1}) \\ &+ P_{L_2,S_2} u_{S \setminus S_2} t_{|L| \setminus (|L_2| \cup S)} sign(\sigma_{S_2, S \setminus S_2}), \end{aligned}$$

for

$$S_1 = S \cap |L_1|, \; S_2 = S \cap |L_2|.$$

The terms that match are those when we omit t_i from t_L with $i \in |L| \setminus S$ or $i \in |L_1| \cap |L_2| \cap S$. The terms which do not match are for $i \in (|L_1| \cap S) \setminus |L_2|$ or $(|L_2| \cap S) \setminus |L_1|$. For $i \in (|L_1| \cap S) \setminus |L_2|$, the term missing in our first guess is

$$q_i := u_{S \setminus S_2 \setminus \{i\}} Q_{L_2,S_2} t_{|L| \setminus (|L_2| \cup S)} sign(\sigma_{S \setminus S_2 \setminus \{i\}, \{i\}}) sign(\sigma_{S \setminus S_2, S_2}).$$

Symmetrically, denote the missing term by r_j for $j \in |L_2| \cap S \setminus |L_1|$. Thus, we have

$$P_{L,S} = P'_{L,S} + \sum_{i \in S \smallsetminus S_2} q_i + \sum_{j \in S \smallsetminus S_1} r_j.$$

Use Lemma 8. $\square$

4. The geometric interpretation

Since the well known paper by W. Fulton and R. MacPherson [5], compactifications of configuration spaces, and complements of hyperplane arrangements [2], became an important topic of algebraic geometry. For a good survey, see [3]. Our geometric interpretation is related to a compactification known as the *reciprocal plane* [3], Section 5.1, and its super analog.

Let us assume the z_j's linearly span the vector space $\mathbb{A}^n_F$ (otherwise, we can replace $x_1, \dots, x_n$ by a basis of the span of $z_1, \dots, z_m$). Denote

$$\mathcal{A} = \{z_1, \dots, z_m\}, \mathcal{A}_S = \{z_i | i \in S\}.$$

Let $R_{\mathcal{A},\mathbb{A}^n_F} = F[t_1, \dots, t_m]/I$ (see Theorem 1). We can then similarly write $R_{\mathcal{A},W}$ where $\mathcal{A}$ is a set of vectors spanning the dual of an F-vector space W. A stratification of $\mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F})$ can be described as follows. Recall that we have a canonical embedding

$$\mathbb{A}^n_F \setminus Z(z_1 \dots z_m) \subseteq \mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F}). \tag{41}$$

Call a vector subspace $V \subseteq \mathbb{A}^n_F$ *special* if $V = Z(\mathcal{A}_S)$ for some $S \subseteq \{1, \dots, m\}$. (Note: S can be empty.) Put also

$$S_V = \{i \in \{1, \dots, m\} | V \subseteq Z(z_i)\}.$$

(Note [3] that the sets of i's for which the z_i's are linearly independent are the independent sets of a matroid. Then the sets S_V are precisely what is called the *flats* of this matroid.) For a scheme X, denote by $|X|$ the underlying topological space.

Theorem 9. *(*[18]*, Remark 6) For $V \subseteq \mathbb{A}^n_F$ special, there is a canonical embedding*

$$\mathrm{Spec}(R_{\mathcal{A}_{S_V},\mathbb{A}^n_F/V}) \to \mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F}). \tag{42}$$

Composing (42) with

$$\mathbb{A}^n_F/V \setminus \bigcup_{i \in S} Z(z_i) \subseteq \mathrm{Spec}(R_{\mathcal{A}_{S_V},\mathbb{A}^n_F/V}),$$

(see (41)), induces a decomposition of sets (not topological spaces),

$$|\mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F})| = \coprod_{V\subseteq \mathbb{A}^n_F \ special} |(\mathbb{A}^n_F/V) \setminus \bigcup_{i\in S_V} Z(z_i)|. \tag{43}$$

Proof. We have

$$R_{\mathcal{A},\mathbb{A}^n_F}/(t_i | i \notin S_V) = R_{\mathcal{A}_S,\mathbb{A}^n_F/V},$$

which gives the maps (42). (The point is that there is no linear relation between the z_i's in which all but one term would have $i \in S_V$. Thus, all the relations P_L where L contains a term not in S_V are in $(t_i | i \notin S_V)$.)

To prove (43), first note that the images of the inclusions of the components of the right hand side of (43) are clearly disjoint since they correspond to imposing relations t_i with $i \notin S_V$ for some special vector subspace V, and inverting all other t_i's. Thus, our task is to show that the canonical map from the right hand side to the left hand side of (43) is onto. To this end, let $Q \in \mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F})$ and let

$$S = \{j \in \{1, \dots, m\} | Q \in (t_j)\}.$$

Let

$$V = \bigcap_{j\in S} Z(z_j).$$

We want to prove that $S = S_V$. The fact that $S \subseteq S_V$ is automatic. Suppose $j \in S_V \setminus S$. Then $z_j = a_1 z_{j_1} + \dots a_k z_{j_k}$ with $j_1 < \dots < j_k \in S$, $a_1, \dots, a_k \neq 0 \in F$. Let

$$L = z_j - a_1 z_{j_1} - \dots - a_k z_{j_k}.$$

By assumption, $Q \in (t_j)$. But in $R_{\mathcal{A},\mathbb{A}^n_F}/(t_j)$, P_L is a non-zero multiple of

$$t_{j_1} \cdot \dots \cdot t_{j_k}.$$

This implies $Q \in (t_{j_i})$ for some $i = 1, \dots, k$. Contradiction. $\square$

Theorem 9 suggests that $\mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F})$ should have a compactification where on the right hand side of (43) we replace each

$$(\mathbb{A}^n_F/V) \setminus \bigcup_{i\in S_V} Z(z_i)$$

with the corresponding affine space $(\mathbb{A}^n_F/V)$. In fact, there is such a compactification $X_{\mathbb{A}^n_F,\mathcal{A}}$ and it can be described as the Zariski closure of the image of the embedding

$$\mathbb{A}^n_F \setminus Z(z_1 \dots z_m) \xrightarrow{(z_1,\dots,z_m)} \prod_{i=1}^m \mathbb{P}^1_F. \tag{44}$$

In the terminology of [3], this is an example of what is called a *toric compactification*. It was also studied, from a different point of view, in [1]. Note that while (44) resembles superficially the formula for the De Concini-Procesi wonderful compactification [2], (44) is in fact quite different. While the wonderful compactification uses projections to (typically) higher-dimensional projective spaces, (44) uses inclusions of the affine coordinates z_i into $\mathbb{P}^1_F$.

The projective variety $X_{\mathbb{A}^n_F,\mathcal{A}}$ is covered by a system of affine open sets, closed under intersection,

$$U_{V,T} = \operatorname{Spec} \prod_{j \in T} z_j^{-1} F[t_i, z_j | i \notin S_V, j \in S_V]/(\frac{P_L}{t_{S_V \cap |L|}})$$

where V runs through special subspaces of $\mathbb{A}^n_F$, L runs through all linear relations among the z_i's, and T is any subset of S_V. The following fact follows from the definitions:

Lemma 10. *We have*

$$U_{V,T} \bigcap U_{V',T'} = U_{W, T \cup T' \cup (S_V - S_{V'}) \cup (S_{V'} - S_V)}$$

where

$$V + V' \subseteq W = \bigcap_{i \in S_V \cap S_{V'}} Z(z_i)$$

so

$$S_V \bigcap S_{V'} = S_W.$$

□

It follows from Theorem 9 that $|U_{V,T}|$ are open subsets covering $X_{\mathbb{A}^n_F,\mathcal{A}}$. To show the affine schemes $U_{V,T}$ are reduced (their coordinate rings have no nilpotent elements), we have the following generalization of Theorem 1:

Theorem 11. *Let V be a special subspace of $\mathbb{A}^n_F$. The kernel of the homomorphism of rings*

$$F[t_i, z_j | i \notin S_V, j \in S_V] \to \prod_{i \notin S_V} z_i^{-1} F[z_1, \dots, z_m]/(\mathcal{Z}_V)$$

given by $t_i \mapsto z_i^{-1}$, where $\mathcal{Z}_V$ is the set of all linear relations among the z_i's, $i \in S_V$, is

$$(\frac{P_L}{t_{S_V \cap |L|}}).$$

Proof. Note that by the proof of Theorem 9, any linear relation among the z_i's which involves a z_i for $i \notin S_V$ involves at least two of them. Therefore, we can repeat the induction in Section 3 with $\{1, \dots, m\}$ replaced by $\{1, \dots, m\} \setminus S_V$. $\square$

We also have a similar analog of Theorem 3:

Theorem 12. *Let V be a special subspace of $\mathbb{A}^n_F$. The kernel of the homomorphism of rings*

$$\begin{array}{c} F[t_i, z_j | i \notin S_V, j \in S_V] \otimes \Lambda[u_i, dz_j | i \notin S_V, j \in S_V] \\ \Big\downarrow \\ \prod_{i \notin S_V} z_i^{-1} F[z_1, \dots, z_m] \otimes \Lambda[dz_i, \dots, dz_m]/(\mathcal{Y}_V) \end{array}$$

given by $t_i \mapsto z_i^{-1}$, $u_i \mapsto z_i^{-1} dz_i$, where $\mathcal{Y}_V = \mathcal{Z}_V \cup \{dL | L \in \mathcal{Z}_V\}$, is

$$\left(\frac{P_{L,S}}{t_{S_V \cap |L|}}\right)$$

where L runs through the linear relations among the z_i's and $S \subseteq |L|$.

Accordingly, we have a superscheme analog $\widetilde{X}_{\mathbb{A}^n_F, \mathcal{A}}$ of $X_{\mathbb{A}^n_F, \mathcal{A}}$. Here by a superscheme, we mean a locally ringed space by $\mathbb{Z}/2$-graded commutative rings which is locally isomorphic to Spec of a $\mathbb{Z}/2$-graded commutative ring (see e.g. [23]). $\widetilde{X}_{\mathbb{A}^n_F, \mathcal{A}}$ is covered by super-affine open subsets

$$\begin{array}{c} \widetilde{U}_{V,T} = \operatorname{Spec} \prod_{j \in T} z_j^{-1} F[t_i, z_j | i \notin S_V, j \in S_V] \\ \otimes \Lambda[u_i, dz_j || i \notin S_V, j \in S_V]/(\frac{P_{L,S}}{t_{T \cap |L|}}). \end{array}$$

We clearly have

$$|\widetilde{U}_{V,T}| = |U_{V,T}|$$

and for $|U_{V',T'}| \subseteq |U_{V,T}|$, $\widetilde{U}_{V',T'}$ is a complement of the zero set of an (even) principal ideal in $\widetilde{U}_{V,T}$. Therefore, $\widetilde{X}_{\mathbb{A}^n_F, \mathcal{A}}$ can be defined as the colimit of the $\widetilde{U}_{V,T}$'s in the category of superschemes.

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