# ON THE FROBENIUS TYPE OF SEMISIMPLE PRE-TANNAKIAN CATEGORIES IN CHARACTERISTIC $p > 0$

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## 1. Introduction

Let $k$ be a field of characteristic $p > 2$. In this paper, we consider the *Verlinde category* $Ver_p$ associated to $k$, used by V. Ostrik [7] for the purpose of generalizing the result of P. Deligne, [2], Theorem 0.6, which states that, in characteristic 0, a pre-Tannakian category has a symmetric tensor functor to the category $sVec$ of super vector spaces if and only if it is of sub-exponential growth. Ostrik [7] proposed the conjecture that, in characteristic $p > 0$, a semisimple pre-Tannakian category has a symmetric tensor functor to the Verlinde category if and only if it is of sub-exponential growth, and proved this in the case when $\mathcal{C}$ has finitely many simple objects. (For recent developments, see also [5].) For this purpose, he introduced a "Frobenius functor"

$$Fr_0 : \mathcal{C} \to \mathcal{C} \boxtimes Ver_p \tag{1}$$

where $\boxtimes$ is Deligne's external tensor product (it turns out that it is actually helpful to compose this functor with a "Frobenius twist," which, however, is not important for our purposes). Ostrik observed that a consequence of his conjecture would be that all semisimple pre-Tannakian categories of subexponential growth are of Frobenius type $Vec$ or $Ver_p^+$, which is equivalent to saying that the image of (1) lands in $\mathcal{C} \boxtimes Ver_p^+$ where $Ver_p^+$ is a certain symmetric tensor subcategory of $Ver_p$, which we recall below. In this note, we prove that this conclusion actually holds for all semisimple pre-Tannakian categories.

To give a rigorous discussion of $Ver_p$, we will need some broader category theory concepts. In general, the semisimplification ([3], Exercise 8.18.9) of a symmetric tensor category is defined by taking the same objects as the original category, but quotienting out the negligible morphisms (a morphism is called negligible if every composition with it with the same source and target has trace 0). It is a general result that the simple objects of the semisimplification are the images of the quotient functor of the indecomposables of the original category (which are not sent 0).

Now let us apply these general concepts to the context we are interested in for this paper. The Verlinde category $Ver_p$ is defined as the semisimplification of the category of finite dimensional $\mathbb{Z}/p$-representations over $k$. Now the indecomposable representations are vector spaces of dimensions $1, \dots, p$ where the generator acts by the Jordan blocks with eigenvalue 1. It can be seen that all of these representations are non-negligible, except in dimension $p$. Thus, the simple objects of $Ver_p$ are the images of the $\mathbb{Z}/p$-representations corresponding to the Jordan blocks with eigenvalue 1 of dimensions $1, \dots, p-1$. We call these objects $L_1, \dots, L_{p-1}$, respectively. (Note that this category can also be considered a categorification in characteristic $p > 0$ of the $s\ell_2$-Verlinde algebra of level $p$.)

We also consider the symmetric tensor subcategory $Ver_p^+ \subseteq Ver_p$ generated by $L_i$ for $i$ odd. Using this subcategory, we can also describe the Verlinde category as the Deligne tensor product

$$Ver_p = Ver_p^+ \boxtimes sVec.$$

For a more detailed discussion, see for example [8].

Our main result is

**Theorem 1.** *The Frobenious type of a semi-simple pre-Tannakian category $\mathcal{C}$ in characteristic $p$ is contained in $Ver_p^+$.*

This answers a question of Coulembier, Etingof, and Ostrik ([1], Question 7.3).

## 2. A Key Result

To prove Theorem 1, we will use the following key result of Etingof and Ostrik [4]:

**Theorem 2.** [4] *The simple objects $D_{(p-1,1)}$, $S_{(p-1,1)}$ generate $\overline{Rep(\Sigma_p)}$ with respect to $\oplus$, $\otimes$. More specifically, the simple objects of $\overline{Rep(\Sigma_p)}$ are $\Lambda^i(D_{(p-1,1)}) \otimes \Lambda^i(S_{(p-1,1)})$ for $0 \leq i, j \leq p-2$. The fusion rules are determined by*

$$\Lambda^j(S_{(p-1,1)}) \otimes S_{(p-1,1)} = \Lambda^{j+1}(S_{(p-1,1)}) \in Obj(\overline{Rep(\Sigma_p)}) \tag{2}$$

$$\Lambda^i(D_{(p-1,1)}) \otimes D_{(p-1,1)} = \Lambda^{i+1}(D_{(p-1,1)}) \oplus \Lambda^{i-1}(D_{(p-1,1)}) \in Obj(\overline{Rep(\Sigma_p)}) \tag{3}$$

$$\Lambda^{p-2}(D_{(p-1,1)}) = \Lambda^{p-1}(S_{(p-1,1)}) = \sigma \in Obj(\overline{Rep(\Sigma_p)}) \tag{4}$$

We include a proof here to make this paper self-contained. This will also allow us to establish notation needed in the proof of Theorem 1.

Let $\widetilde{V}_p$ be the kernel of the augmentation map of $V_p = k\Sigma_p/\Sigma_{p-1}$. Then, in particular, we have

$$\widetilde{V}_p = S_{(p-1,1)}.$$

**Lemma 3.** *Consider the $\Sigma_p$-representation $Ind_{\Sigma_p}^{\mathbb{Z}/p}(1)$ given by taking the induction of the trivial 1-dimenstional $\mathbb{Z}/p$-representation to $\Sigma_p$. This $\Sigma_p$-representation can be expressed as a direct sum of indecomposable $\Sigma_p$-representations of the form*

$$S_{(p-2i,1^{2i})} \text{ and } S_{(p-2i,1^{2i})} \otimes \sigma \tag{5}$$

*for $0 \leq i < \frac{p-1}{2}$ (where $\sigma$ denotes the sign repesentation) and representations that are negligible in $Rep(\Sigma_p)$.*

*Proof of Lemma 3.* We have, be definition

$$Ind_{\Sigma_p}^{\mathbb{Z}/p}(1) = k(\Sigma_p/\mathbb{Z}/p)$$

(as a $\Sigma_p$-representation). Thus,

$$Res_{\mathbb{Z}/p}^{\Sigma_p}(Ind_{\Sigma_p}^{\mathbb{Z}/p}(1)) = Res_{\mathbb{Z}/p}^{\Sigma_p}(k\Sigma_p/\mathbb{Z}/p)$$

(where $Res$ denotes restriction) is a permutation representation and therefore only has orbits of cardinalities 1 and $p$. An orbit of cardinality $p$ is negligible in $Rep(\mathbb{Z}/p)$. On the other hand, the orbits of cardinality 1 correspond (by definition) exactly to the elements of the Weyl group of $\mathbb{Z}/p$ in $\Sigma_p$, which is $\mathbb{Z}/(p-1)$ with its elements given by permutations sending $i \mapsto ki$ where we consdier $i \in \mathbb{Z}/p$ for a fixed $k \in (\mathbb{Z}/p)^\times = \mathbb{Z}/(p-1)$.

Thus, $Res_{\mathbb{Z}/p}^{\Sigma_p}(k\Sigma_p/\mathbb{Z}/p)$ has $p-1$ direct summands of dimension 1 (i.e. non-negligible $\mathbb{Z}/p$-fixed points). Hence, $k\Sigma_p/\mathbb{Z}/p = Ind_{\Sigma_p}^{\mathbb{Z}/p}(1)$ (as $\Sigma_p$-representation) has at most $p-1$ non-negligible direct summands.

Now suppose $V$ is any indecomposable non-negligible $\Sigma_p$-representation whose restriction to $\mathbb{Z}/p$ has a non-negligible fixed point, i.e. there exist maps

$$k \xrightarrow{\alpha} V \xrightarrow{\phi} k$$

such that $\phi \circ \alpha = a \neq 0 \in \mathbb{Z}/p$. Then we can form the diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] & k \arrow[r, "\alpha"] & V \\ & k\Sigma_p/\mathbb{Z}/p. \arrow[u] & \end{tikzcd}$$

This then lifts to maps

$$V \to k\Sigma_p/\mathbb{Z}/p \to V \tag{6}$$

such that the diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[dr, dotted] & k \arrow[r, "\alpha"] & V \\ & k\Sigma_p/\mathbb{Z}/p \arrow[u] \arrow[ur, dotted] & \end{tikzcd}$$

commutes. Then the composition of (6) is multiplication by $a(p-1)!$ which is not $0 \in \mathbb{Z}/p$. Thus, any such $V$ is a direct summand of $k\Sigma_p/\mathbb{Z}/p$.

We claim

$$S_{(p-2i,1^{2i})} = \Lambda^{2i}(\widetilde{V}_p)$$
and
$$S_{(p-2i,1^{2i})} \otimes \sigma = \Lambda^{2i}(\widetilde{V}_p) \otimes \sigma$$

satisfy this for $0 \leq i < \frac{p-1}{2}$.

First note that, for $i < p-1$, the short exact sequence

$$0 \to K \to \Lambda^i \widetilde{V}_p \otimes \widetilde{V}_p \to \Lambda^{i+1}\widetilde{V}_p \to 0 \tag{7}$$

(where here $K$ simply denotes the kernel of the quotient map

$$\Lambda^i \widetilde{V}_p \otimes \widetilde{V}_p \to \Lambda^{i+1}\widetilde{V}_p)$$

splits since we can take the map

$$\Lambda^{i+1}\widetilde{V}_p \to \Lambda^i \widetilde{V}_p \otimes \widetilde{V}_p$$

given by averaging.

We also have the isomorphism

$$\widetilde{V}_p \cong_{\mathbb{Z}/p} L_{p-1}$$

where both sides are consdiered as $\mathbb{Z}/p$-representations. Then in the semisimplification $Ver_p = \overline{Rep(\mathbb{Z}/p)}$, we have $\widetilde{V}_p = L_{p-1}$, so

$$\widetilde{V}_p \otimes \widetilde{V}_p = L_{p-1} \otimes L_{p-1} = 1.$$

Thus, in $Rep(\mathbb{Z}/p)$, $\widetilde{V}_p \otimes \widetilde{V}_p$ has exactly 1 non-negligible summand, which is a fixed point. Therefore, $\Lambda^2(\widetilde{V}_p)$, which is a direct summand of $\widetilde{V}_p \otimes \widetilde{V}_p$ by (7) and has dimension

$$dim(\Lambda^2(\widetilde{V}_p)) = \binom{p-1}{2} = \frac{(p-1)(p-2)}{2} = \frac{(-1)(-2)}{2} = 1,$$

contains the non-negligible fixed point as well (in $Rep(\mathbb{Z}/p)$). Thus, $\Lambda^2(\widetilde{V}_p)$ satisfies the condition.

Thus, for $0 \leq i < \frac{p-1}{2}$, $(\Lambda^2(\widetilde{V}_p))^{\otimes i}$ also contains a non-negligible $\mathbb{Z}/p$-summand which is a fixed point. For each $0 \leq i < \frac{p-1}{2}$, this has $\Lambda^{2i}(\widetilde{V}_p)$ as a non-negligible summand, and therefore each $\Lambda^{2i}(\widetilde{V}_p)$ has the non-negligible $\mathbb{Z}/p$-fixed point, too.

We can also tensor with the sign representation $\sigma$ to obtain that each $\Lambda^{2i}(\widetilde{V}_p) \otimes \sigma$ has a non-negligible direct summand that is a fixed point.

Using out condition, this gives us $p-1$ spaces ($\Lambda^{2i}(\widetilde{V}_p)$, $\Lambda^{2i}(\widetilde{V}_p) \otimes \sigma$ for $0 \leq i < \frac{p-1}{2}$) that are non-isomorphic, indecomposable, and are direct summands of $k\Sigma_p/\mathbb{Z}/p$. By the discussion at the beginning of the proof, these are all of its summands. So

$$\begin{aligned} k\Sigma_p/\mathbb{Z}/p &= \bigoplus_{0\leq i<\frac{p-1}{2}} \Lambda^{2i}(\widetilde{V}_p) \oplus (\Lambda^{2i}(\widetilde{V}_p) \otimes \sigma) = \\ &= \bigoplus_{0\leq i<\frac{p-1}{2}} S_{(p-2i,1^{2i})} \oplus (S_{(p-2i,1^{2i})} \otimes \sigma) \end{aligned} \tag{8}$$

up to direct sum with $\Sigma_p$-representations that are negligible. $\square$

This Lemma directly implies the part of the Proposition stating that $\Lambda^i(D_{(p-1,1)}) \otimes \Lambda^j(S_{(p-1,1)})$ generate $\overline{Rep(\Sigma_p)}$, since each $L_i$ is a direct summand of a tensor product of copies of $L_{p-1}$ and copies of $L_{p-2}$, and because $L_{p-1}$ and $L_{p-2}$ are the restrictions to $\mathbb{Z}/p$ of $\Sigma_p$-representations $S_{(p-1,1)}$ and $D_{(p-1,1)}$, respectively, $Ind_{\Sigma_p}^{\mathbb{Z}/p}(L_i)$ is a direct summand of

$$Ind_{\Sigma_p}^{\mathbb{Z}/p}(L_{p-1}^{\otimes j} \otimes L_{p-2}^{\otimes \ell}) = S_{(p-1,1)}^{\otimes j} \otimes D_{(p-1,1)}^{\otimes \ell} \otimes k\Sigma_p/\mathbb{Z}/p.$$

Thus, by Lemma 3 and its proof, $Ind_{\Sigma_p}^{\mathbb{Z}/p}(L_i)$ is a direct summand of a tensor product of copies of $S_{(p-1,1)}$, $D_{(p-1,1)}$. This proves the first claim of the Theorem.

To prove the fusion rules, we also claim the following

**Lemma 4.** *In $Ver_p^+$, we have*

$$\begin{aligned}\Lambda^i(L_{p-2}) &= L_{p-i-1} \quad \text{if } i \text{ odd}\\ \Lambda^i(L_{p-2}) &= L_{i+1} \quad \text{if } i \text{ even}\end{aligned} \tag{9}$$

*and*

$$\Lambda^i(L_{p-2}) \otimes L_{p-2} = \Lambda^{i+1}(L_{p-2}) \oplus \Lambda^{i-1}(L_{p-2}), \tag{10}$$

*for $i < p-1$.*

*Proof.* Induction on $i$. We have

$$L_i \otimes L_2 = L_{i+1} \oplus L_{i-1} \tag{11}$$

Also $L_{p-2} = L_{p-1} \otimes L_2$. In particular, we have

$$L_{p-2} \otimes L_{p-2} = L_1 \oplus L_3$$

which implies (9), (10) for $i = 0, 1, 2$. Next, note that since $L_{p-2}$ is self-dual, the left hand side of (10) has the two summands on the right hand side for $i < p-1$ (the first one given by multiplication, the second one by differentiation).

By the induction hypothesis, (9) is true for all the objects in (10) except $\Lambda^{i+1}(L_{p-2})$. Thus, it follows from (11) for $\Lambda^{i+1}(L_{p-2})$ also (modulo negligibles). This completes the induction step.

$\square$

Using this we can complete the proof of Theorem 2 as follows:

*Proof of Theorem 2.* Now $D_{(p-1,1)}$ is also self-dual and thus the left hand side of (3) has the two summands on the right hand side. This implies (3) modulo negligibles by Lemma 4.

To compute (4), let $v_1, \ldots, v_p$ be the standard basis of $V_p = k\Sigma_p/\Sigma_{p-1}$. Then the basis of element of $\Lambda^{p-2}(D_{(p-1,1)})$ is

$$(v_1 - v_2) \wedge \cdots \wedge (v_1 - v_{p-1}).$$

We see that the permutation (12) acts by multiplication by $-1$, which proves that

$$\Lambda^{p-2}(D_{(p-1,1)}) = \sigma.$$

The proof of

$$\Lambda^{p-1} S_{(p-1,1)} = \sigma$$

is analogous (replacing $p-1$ with $p$). The condition (2) was already proved in the proof of Lemma 3.

$\square$

## 3. A Preliminary Result in Representation Theory

First, note that we may consider $\widetilde{V_p}\otimes\widetilde{V_p}$ as a $\Sigma_p\times\Sigma_p$-representation. Let us then consider the $\Sigma_{2p}$-representation obtained from inducing $\widetilde{V_p}\otimes\widetilde{V_p}$ to a $\Sigma_{2p}$-representation:

$$J := Ind_{\Sigma_{2p}}^{\Sigma_p\times\Sigma_p}(S_{(p-1,1)}\otimes S_{(p-1,1)}) =$$

$$= Ind_{\Sigma_{2p}}^{\mathbb{Z}/2\wr\Sigma_p}(Ind_{\mathbb{Z}/2\wr\Sigma_p}^{\Sigma_p\times\Sigma_p}(S_{(p-1,1)}\otimes S_{(p-1,1)})).$$

We have a decomposition

$$Ind_{\mathbb{Z}/2\wr\Sigma_p}^{\Sigma_p\times\Sigma_p}(S_{(p-1,1)}\otimes S_{(p-1,1)}) = I_+\oplus I_-$$

where $I_\pm$ is the $\pm1$ eigenspace of the $\mathbb{Z}/2$-action on $S_{(p-1,1)}\otimes S_{(p-1,1)}$ coming from switching the two tensor factors. Let

$$J_\pm := Ind_{\Sigma_{2p}}^{\mathbb{Z}/2\wr\Sigma_p}(I_\pm).$$

**Lemma 5.** *Over $\Sigma_p\wr\mathbb{Z}_2$, the only non-negligible summand of $\Lambda^2(\widetilde{V_{2p}})$ is $\Lambda^2(\widetilde{V_p})$ on which each copy of $\mathbb{Z}/2$ acts trivially.*

*Proof.* Write

$$H = \Sigma_p\wr\Sigma_2 = (\Sigma_2)^{\times p}\rtimes\Sigma_p.$$

We claim that as an $H$-representation, each copy of $\Sigma_2$ acts trivially on the only non-negligible $H$-summand of $\Lambda^2(\widetilde{V_{2p}})$.

First, let us write

$$V_{2p} = \mathbb{F}_p[\Sigma_{2p}/\Sigma_{2p-1}] = \mathbb{F}_p\{v_1,\dots,v_p,w_1,\dots,w_p\}$$

where the $\Sigma_2$-factors of $H$ act by switching a $v_i$ with $w_i$. Considering the map

$$\phi:\Lambda^2(V_{2p})\to\widetilde{V_{2p}}$$

given by sending, for generators $u,u'\in\{v_1,\dots,v_p,w_1,\dots,w_p\}$,

$$\phi(u\wedge u') = u-u'.$$

We obtain a short exact sequence

$$0\to\Lambda^2(\widetilde{V_{2p}})\to\Lambda^2(V_{2p})\to\widetilde{V_{2p}}\to 0.$$

Let us consider

$$U = \mathbb{F}_p\{v_1-w_1, v_2-w_2,\dots,v_p-w_p\}\cong V_p$$

$$W = \mathbb{F}_p\{v_1+w_1, v_2+w_2,\dots,v_p+w_p\}\cong V_p.$$

Then we can decompose

$$\Lambda^2(V_{2p}) = \Lambda^2(U)\oplus\Lambda^2(W)\oplus(U\otimes W)$$

and

$$\widetilde{V_{2p}} = U \oplus \widetilde{W}$$

where $\widetilde{W}$ denotes the kernel of the augmentation map on $W$.

Now the terms $U$ (and hence $U \otimes W$) and $\Lambda^2(U)$ are negligible. Further, we have a short exact sequence

$$0 \to \Lambda^2(\widetilde{W}) \to \Lambda^2(W) \to \widetilde{W} \to 0.$$

Thus, the non-negligible summand of $\Lambda^2(\widetilde{V_{2p}})$ is $\Lambda^2(\widetilde{W}) \cong \Lambda^2(V_p) \cong S_{(p-2,1,1)}$, on which the $p$ copies of $\Sigma_2$ in $H$ act trivially. $\square$

## 4. Proof of Theorem 1

Suppose $\mathscr{C}$ is a Pre-Tannakian semi-simple category, with Frobenius type not contained in $Ver_p^+$. By Proposition 3.3 of [7], the only fusion subcategories of $Ver_p$ are the categories of vector spaces $Vec$, super vector spaces $sVec$, $Ver_p^+$, and $Ver_p$. Thus, we suppose $\mathscr{C}$ has Frobenius type $sVec$ or

$$Ver_p = sVec \boxtimes Ver_p^+.$$

Hence, since $sVec$ is generated by $L_{p-1}$, there exists some object $X \in Obj(\mathscr{C})$ such that $Fr_0(X) \in Obj(\mathscr{C} \boxtimes Ver_p)$ has a direct summand of the form $T \boxtimes L_{p-1}$ for a simple object $T \in Obj(\mathscr{C})$. Recall that the functor $Fr_0$ is given as the composition

$$\mathscr{C} \longrightarrow \mathscr{C} \boxtimes \overline{Rep(\Sigma_p)} \xrightarrow{Id_{\mathscr{C}} \boxtimes \Phi} \mathscr{C} \boxtimes Ver_p$$

where the first functor comes from considering the functor given by

$$\mathscr{C} \to \mathscr{C} \boxtimes Rep(\Sigma_p)$$

$$X \mapsto X^{\otimes p}$$

composed with the Deligne tensor product of $Id_{\mathscr{C}}$ with the natural semisimplification functor

$$Rep(\Sigma_p) \to \overline{Rep(\Sigma_p)}$$

from quotienting out negligible morphisms in $Rep(\Sigma_p)$, and the second functor is given as the Deligne tensor product of $Id_{\mathscr{C}}$ with the functor

$$\Phi : \overline{Rep(\Sigma_p)} \to Ver_p$$

given from Theorem 2 above by taking

$$\Phi(S_{(p-1,1)}) = L_{p-1}$$

$$\Phi(D_{(p-1,1)}) = L_{p-2}.$$

Hence, the $p$th tensor power $X^{\otimes p}$ of $X$ must have a direct summand of the form $T \boxtimes R$ where $R \in \underline{Rep(\Sigma_p)}$ and $\Phi(R) = L_{p-1}$. By Theorem 2, $R$ must be an odd tensor power of $S_{(p-1,1)}$ in $\underline{Rep(\Sigma_p)}$. Thus by replacing $X$ by an odd tensor power of itself, we may assume without loss of generality that

$$R = S_{(p-1,1)}.$$

Recall that, denoting $V_p = k[\Sigma_p/\Sigma_{p-1}]$, we have $S_{(p-1,1)} = \widetilde{V}_p$, the kernel of the augmentation map $V_p \to k$. Hence, in summary, we have obtained an object $X \in Obj(\mathscr{C})$ such that $X^{\otimes p}$ has a direct summand $T \boxtimes \widetilde{V}_p$ with $T$ a simple object of $\mathscr{C}$.

Now let us consider the $\Sigma_{2p}$-representation $X^{\otimes 2p}$. Recalling the splitting

$$X \otimes X = \Lambda^2(X) \oplus Sym^2(X)$$

(where $\Lambda^2(X)$, $Sym^2(X)$ denote the direct summands where switching the two copies of $X$ acts by $-$, $+$, respectively), we obtain

$$X^{\otimes 2p} = (\Lambda^2(X) \oplus Sym^2(X))^{\otimes p}.$$

Thus, $X^{\otimes 2p}$ can be expressed as a direct sum of $(\Lambda^2(X))^{\otimes p}$, $(Sym^2(X))^{\otimes p}$, and a direct sum of mixed terms $Y$. Every summand $Y$ occurs a number of times that is divisible by $p$ and hence, $Y$ is negligible as a $\mathbb{Z}/p$-representation (where we consider the copy of $\mathbb{Z}/p$ as the $p$-Sylow subgroup of the $\Sigma_p$ diagonally embedded in $\Sigma_p \times \Sigma_p$). Let us consider the identity morphism

(12)

$$(\Lambda^2(X))^{\otimes p} \oplus (Sym^2(X))^{\otimes p} \oplus Y = X^{\otimes 2p} \xrightarrow{Id} X^{\otimes 2p} = X^{\otimes p} \otimes X^{\otimes p}$$

Note that $X^{\otimes p} \otimes X^{\otimes p}$ also has a direct summand

$$(T \boxtimes \widetilde{V}_p) \otimes (T \boxtimes \widetilde{V}_p) = (T \otimes T) \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p).$$

Again, we may write

$$T \otimes T = \Lambda^2(T) \oplus Sym^2(T).$$

Without loss of generality, we may assume that $\Lambda^2(X)$ is non-zero since otherwise we may choose an object $Y \in Obj(\mathscr{C})$ with $\Lambda^2(Y) \neq 0$ (e.g. we could even take $Y = X^{\otimes(p-1)}$) and replace $X$ by $X \otimes Y$. Hence, we can take $Z$ to be a simple non-trivial summand of $\Lambda^2(T)$ (so switching he order of the two copies of $T$ acts by $-$ on $Z$). Then

$$Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$$

forms a $\mathbb{Z}/2 \wr \Sigma_p$-direct summand of $X^{\otimes p} \otimes X^{\otimes p}$.

We can consider $\widetilde{V}_p \otimes \widetilde{V}_p$ as a representation of $\mathbb{Z}/2 \wr \Sigma_p \subseteq \Sigma_{2p}$. Hence, (12) together with a $\mathbb{Z}/2 \wr \Sigma_p$-equivariant splitting of $Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p) \subseteq X^{\otimes 2p}$ gives an epimorphism over $\mathbb{Z}/2 \times \Sigma_p$

$$f : (\Lambda^2(X))^{\otimes p} \oplus (Sym^2(X))^{\otimes p} \oplus Y \twoheadrightarrow Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$$

and thus, morphisms

$$f_1 : (\Lambda^2(X))^{\otimes p} \to Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$$
$$f_2 : (Sym^2(X))^{\otimes p} \to Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$$
$$f_3 : Y \to Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p).$$

Since a switch of the two tensor factors $X^{\otimes p}$ in $X^{\otimes 2p}$ acts by $-$ on $Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$ while it acts by $+$ on $(Sym^2(X))^{\otimes p}$ (since each individual switch of two factors $X$ in a copy of $X \otimes X$ in $X^{\otimes 2p}$ acts by $+$ on $Sym^2(X)$), considering the action of $\mathbb{Z}/2 \subset \mathbb{Z}/2 \wr \Sigma_p$ switching the two copies of $\Sigma_p$, we must have

$$f_2 = 0.$$

Hence, if $f_1$ were negligible over $\mathbb{Z}/p$, $f$ would given an epimorphism

$$Y' \oplus Y \twoheadrightarrow Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$$

for $Y'$ a negligible direct summand of $(\Lambda^2(X))^{\otimes p}$. Thus, since $Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$ was assumed to be non-negligible over $\mathbb{Z}/p$,

$$f_1 : (\Lambda^2(X))^{\otimes p} \to Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$$

is also non-negligible.

Now $Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$ forms a direct summand of

$$Z \boxtimes Ind_{\Sigma_{2p}}^{\mathbb{Z}/2 \wr \Sigma_p}(Res_{\mathbb{Z}/2 \wr \Sigma_p}^{\Sigma_{2p}}(\widetilde{V}_p \otimes \widetilde{V}_p))$$

over $\mathbb{Z}/2 \wr \Sigma_p$. By the double coset formula, we may choose an indecomposable non-negligible $\Sigma_{2p}$-representation $W$ with a map

$$W \to Ind_{\Sigma_{2p}}^{\mathbb{Z}/2 \wr \Sigma_p}(\widetilde{V}_p \otimes \widetilde{V}_p) = J.$$

In particular, we then obtain a map of $\Sigma_p \times \Sigma_p$-representations

$$Res_{\mathbb{Z}/2 \wr \Sigma_p}^{\Sigma_{2p}}(W) \to \widetilde{V}_p \otimes \widetilde{V}_p.$$

Hence, we may consider $Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p)$ as a direct summand of $Z \boxtimes W$.

Now there is also an injective morphism of $\mathbb{Z}/2 \wr \Sigma_p$-representations

$$\widetilde{V}_p \otimes \widetilde{V}_p \to \Lambda^2(\widetilde{V_{2p}}),$$

(i.e. a map

$$\widetilde{V}_p \otimes \widetilde{V}_p \to Res^{\Sigma_{2p}}_{\mathbb{Z}/2\wr\Sigma_p} \Lambda^2(\widetilde{V_{2p}}),$$

where $\mathbb{Z}/2$ acts by $-$ the switch on the left hand side) and thus, by the adjunction between restriction and induction, we obtain a map of $\Sigma_{2p}$-representations

$$W \to \Lambda^2(\widetilde{V_{2p}})$$

On the other hand, note that we have a short exact sequence

$$0 \longrightarrow \Lambda^2(\widetilde{V_{2p}}) \longrightarrow \Lambda^2(V_{2p}) \xrightarrow{g} \widetilde{V_{2p}} \to 0$$

Now $\Lambda^2(V_{2p}) = \Lambda^2(V_p \oplus V_p)$ has terms $\Lambda^2(V_p) \otimes 1$, $V_p \otimes V_p$, and $1 \otimes V_p$. Note that as $\mathbb{Z}/p$-representations, $V_p$ is free and hence, the summand $\Lambda^2(V_p)$ of $V_p \otimes V_p$ is projective. Considering we have another short exact sequence

$$0 \longrightarrow \widetilde{V}_p \otimes \widetilde{V}_p \longrightarrow V_p \otimes V_p \longrightarrow \widetilde{V_{2p}} \longrightarrow 0$$

the term $\widetilde{V}_p \otimes \widetilde{V}_p = Ker(h)$ must $\mathbb{Z}/p$-split in $Ker(g)$. Hence, $\widetilde{V}_p \otimes \widetilde{V}_p$ is a $\mathbb{Z}/p$-summand of $\Lambda^2(\widetilde{V_{2p}})$ (with $\mathbb{Z}/p$ acting diagonally), and we can write an inclusion

$$\widetilde{V}_p \otimes \widetilde{V}_p \hookrightarrow \Lambda^2(\widetilde{V_{2p}}).$$

Hence, we obtain an inclusion of a direct summand as $\mathbb{Z}/p$-representations

$$\begin{tikzcd} \widetilde{V}_p \otimes \widetilde{V}_p \arrow[d] \arrow[dr, "\subseteq"] & \\ W \arrow[r] & \Lambda^2(\widetilde{V_{2p}}). \end{tikzcd}$$

Thus, we obtain a diagram after taking the Deligne tensor product with $Z$:

$$\begin{tikzcd} Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p) \arrow[d] \arrow[dr, "\subseteq"] & \\ Z \boxtimes W \arrow[r] & Z \boxtimes \Lambda^2(\widetilde{V_{2p}}) \end{tikzcd}$$

where the diagonal morphism is an inclusion of a direct summand over $\mathbb{Z}/p$. On the other hand, we also have, over $\mathbb{Z}/2 \wr \Sigma_p$,

$$\begin{tikzcd} & Z \boxtimes (\widetilde{V}_p \otimes \widetilde{V}_p) \\ (\Lambda^2(X))^{\otimes p} \arrow[ur, "f_1"] \arrow[r] & Z \boxtimes W \arrow[u] \end{tikzcd}$$

Hence, by composing these diagrams, over $\mathbb{Z}/p$,

$$(\Lambda^2(X))^{\otimes p} \hookrightarrow Z \boxtimes \Lambda^2(\widetilde{V_{2p}})$$

forms a non-negligible direct summand. Since the Sylow subgroup of $\Sigma_p \wr \mathbb{Z}/2$ is isomorphic to $\mathbb{Z}/p$, $(\Lambda^2(X))^{\otimes p}$ must still form a non-negligible direct summand of $Z \boxtimes \Lambda^2(\widetilde{V_{2p}})$ over $\Sigma_p \wr \mathbb{Z}/2$. By Lemma 5, this gives a contradiciton, since each copy of $\mathbb{Z}/2$ acts by $-$ on $(\Lambda^2(X))^{\otimes p}$.

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