

HOWE DUALITY OVER FINITE FIELDS III: FULL COMPUTATION AND THE GUREVICH-HOWE CONJECTURES

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ABSTRACT. In this third paper in a series on type I Howe duality for finite fields, we give a complete description of the restriction of the oscillator representation over a finite field to products of dual pairs of symplectic and orthogonal groups in all cases that occur. We also provide a dictionary with the notation of S.-Y. Pan, who identified which tensor products of irreducible representations occur with non-zero multiplicity. As an application, we give a recursive construction of all irreducible complex representations of finite symplectic and orthogonal groups and a recursive formula for the characters of unipotent cuspidal representations. We also give a proof of the Gurevich-Howe rank and exhaustion conjectures for type C groups.

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1. INTRODUCTION

The purpose of this paper is to complete the explicit description, started in the previous papers [30, 31], of the decomposition of the oscillator representation of a symplectic group over a finite field into irreducible representations of a product of a dual pair of type I, which consists of a symplectic and an orthogonal group.

This problem was suggested by R. Howe in [18]. The first approaches this question considered the behavior of *unipotent* representations appearing in a restricted oscillator representation. J. Adams and A. Moy [1] proved unipotent cuspidal representations paired with each other at their first occurrence in some oscillator representation. In combination with their calculation of the decomposition of the unipotent part of an oscillator representation restricted to type II dual pairs (consisting of general linear groups or unitary groups), A.-M. Aubert, J. Michel, and R. Rouquier [3] formed a conjecture on the decomposition of the unipotent part in the case of type I dual pairs.

Later, S. Gurevich and R. Howe [15, 16] approached the problem of finite field Howe duality from a different perspective, by considering whole oscillator representations, restricted to dual pairs in a certain *stable range*. For each such restriction, they constructed one-to-one correspondences sending each representation of the orthogonal subgroup in the pair to a newly occurring irreducible representation of the symplectic subgroup. This construction, which they called the *eta correspondence*, is related to a concept of *rank* which also carries significance for certain questions about the group dynamics of finite groups of Lie type.

Returning to the study of the unipotent part of a restricted oscillator representation, in [40], S.-Y. Pan proved the type I conjecture of Aubert, Michel, and Rouquier by computing the projection of its

character to the class functions spanned by virtual Deligne-Lusztig characters. In [41], Pan also proved a compatibility of this result with Lusztig's classification of irreducible representations to conclude a classification of the irreducible tensor products appearing with non-zero multiplicity in the restriction of an oscillator representation to a type I dual pair.

In the present paper and [30, 31], we approach the question of finite field Howe duality using a different method, which directly computes the decomposition of the restricted oscillator representation and verifies that each distinct tensor product appears with multiplicity one. In this paper, we use the structure of endomorphism algebras and interpolated category theory to extend results for stable ranges to every possible case of type I dual pair.

As a direct application of the methods used in this paper, we also give an inductive construction of all irreducible representations of symplectic and orthogonal groups over finite fields, which in particular gives a recursive formula for the characters of the unipotent cuspidal representations. Finally, we give a dictionary of our notation with the notation of S.-Y. Pan [41], who identified which tensor products of irreducible representations of type I dual subgroups occur with non-zero multiplicity in the restricted oscillator representation, and we prove that the multiplicity is always 1. As another application, we prove the Gurevich-Howe rank conjecture [15], Conjecture 0.3.8 on the coincidence of tensor rank and U -rank, and the exhaustion conjecture [15], Conjecture 0.4.12, which follows.

The part of this program completed in [30, 31] was to treat the so-called *stable ranges*, where the rank of one of the groups in the pair is much greater than the other. In [30], the general form of that case of the decomposition was established, in terms of certain correspondences between the sets of irreducible representations of symplectic and orthogonal groups, and, in [31], it was completely described in terms of Lusztig's classification of irreducible representations of finite groups of Lie type.

The strategy of this paper is to break up the remaining cases into two *metastable ranges*. In the metastable ranges, the stable picture breaks down in two ways, both of which are related to the occurrence of *generalized Lusztig symbols*, which relax some of the defining conditions of a Lusztig symbol. One type of generalized symbol predicts a 0-dimensional representation - those terms are simply omitted. However, one can also encounter alternating sums of induction terms coupled with generalized Lusztig symbols of the same dimension. These are

first shown to be genuine (as opposed to virtual) representations. This is done in Section 4. The other step is to compute them completely, which is done in Section 6.

To describe our results more concretely, we need some notation. Consider a finite field \mathbb{F}_q of characteristic not equal to 2, a symplectic \mathbb{F}_q -vector space V , and an \mathbb{F}_q -vector space W with a non-degenerate symmetric bilinear form B . In this finite field context, S. Gurevich and R. Howe (see e.g. [15]) proposed the problem of describing explicitly the restriction of the oscillator representation of $\mathrm{Sp}(V \otimes W)$ to the subgroup $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$. The previous two papers in this series [30, 31] described this decomposition explicitly in the *symplectic stable range* (i.e. where $\dim(V) \geq 2\dim(W)$) and the *orthogonal stable range* (where $\dim(V)$ is less than or equal to the dimension of the maximal isotropic subspace of W).

Denote by \widehat{G} the set of isomorphism classes of irreducible complex representations of a finite group G . Then in the symplectic resp. orthogonal stable range, there are correspondences

$$(1) \quad \eta_{W,B}^V : \widehat{\mathrm{O}(W, B)} \leftrightarrow \widehat{\mathrm{Sp}(V)}$$

$$(2) \quad \zeta_V^{W,B} : \widehat{\mathrm{Sp}(V)} \leftrightarrow \widehat{\mathrm{O}(W, B)}$$

(explicitly constructed in [30] and described in [31] in terms of G. Lusztig's classification of irreducible representations of finite groups of Lie type, see e.g. [37]) so that the Gurevich-Howe decomposition is a direct sum of terms of the form

$$(3) \quad \rho \otimes \eta_{W,B}^V(\rho)$$

resp.

$$(4) \quad \zeta_V^{W,B}(\pi) \otimes \pi$$

and “degenerate terms,” which can be explicitly described as tensor products analogous to (3), (4) involving parabolic Verma modules induced from smaller choices of V, W .

The purpose of the present paper is to describe the Gurevich-Howe decomposition in the remaining cases, broken up into two *metastable ranges* which, together with the stable ranges, cover all the cases of V, W . The precise definition of the metastable ranges is technical and will be given in Subsection 3.1 below (Definition 3.1.1).

Our main result can now be stated in broad terms as follows:

Theorem 1. *Consider a type I reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$. Then in the symplectic resp. orthogonal metastable ranges, there are*

correspondences

$$(5) \quad \eta_{W,B}^V : \widehat{O(W, B)} \rightarrow \widehat{Sp(V)} \cup \{0\}$$

$$(6) \quad \zeta_V^{W,B} : \widehat{Sp(V)} \rightarrow \widehat{O(W, B)} \cup \{0\},$$

explicitly described in terms of Lusztig's classification, such that the restriction of the oscillator representation of $Sp(V \otimes W)$ to $Sp(V) \times O(W, B)$ is a direct sum of terms of the form (3) resp. (4) and explicitly described as tensor products analogous to (3) resp. (4) involving alternating sums of parabolic Verma modules for smaller choices of V, W , each of which adds up to a linear combination of irreducible representations with positive integral coefficients. The alternating sums are explicitly resolved as sums of irreducible representations in terms of Lusztig's classification.

We restate this more precisely after we have introduced the necessary notation, in Theorems 3.5.1 and 3.5.2 below. The description of the alternating sum coefficients appearing with the eta and zeta correspondence is given in Theorem 6.1.1.

Remark: This result extends the results of S.-Y. Pan [40, 41]. We discuss this in Appendix A where we see how our decompositions of the restricted oscillator representation recover Pan's identification of the summands which occur with non-zero multiplicity. In fact, our proof explicitly shows that they all must appear with multiplicity 1 and that the sum of the products of the occurring dimensions does really add up to $q^{\dim(V) \cdot \dim(W)/2} = \dim(\omega[V \otimes W])$, using a somewhat subtle combinatorial argument. (The cases of rank ≤ 6 were verified on a computer using Maple and the GAP package CHEVIE.)

Further, our organization of these summands in terms of one-to-one correspondences between sets of irreducible representations of symplectic groups and those of orthogonal groups reconciles Pan's result with the original program of Howe [15, 16, 18, 20] (see Appendix A).

Every irreducible representation of a fixed symplectic group is obtained by applying the eta correspondence to an irreducible representation of an orthogonal group forming a symplectic metastable reductive dual pair. Similarly, every irreducible representation of an orthogonal group is obtained from a signed zeta correspondence. In particular, recursively applying the eta and zeta correspondences gives an inductive construction of every irreducible representation from certain "terminal representations" (see Subsection 4.5).

Even more concretely, by organizing the decomposition of the restricted oscillator representation into the eta and zeta correspondence, we can isolate describe the “top part” consisting of tensor products of irreducible representations of orthogonal or symplectic groups with their images under the eta or zeta correspondence, respectively. The character of this top part is approximable using the characters of the oscillator representations (which are computable from the Schrödinger model), and orthogonality of characters can then be used to obtain a formula for the character of the eta or zeta correspondence applied to a representation, in terms of the characters of the original input representation and the characters of the oscillator representation (see Subsection 4.6). The unipotent cuspidal representations are obtainable by recursively applying the eta and zeta correspondences to trivial and sign representations, meaning this process gives a concrete construction their characters.

One important application of our description of the eta correspondence is that it can be used to answer questions related the character theory of finite symplectic groups (see [15, 16]). In particular, in Section 5, we prove the Gurevich-Howe *rank conjecture*, which predict the equality of a two kinds of “ranks” for certain representations. The first notion of rank defined by Gurevich and Howe is called *U-rank* and is defined as the maximal “rank” of a character in the restriction of a representation to the Siegel unipotent subgroup of a symplectic group. We denote it by rk_U . The second notion of rank is called *tensor rank* and is defined to be the minimal natural number k such that every irreducible summand of the input representation is contained in a tensor product of less than or equal to k oscillator representations. We denote it by rk_{\otimes} . The main result of Section 5 is that Conjecture 0.3.8 of [15] holds:

Theorem 2 (The Gurevich-Howe Rank Conjecture, [15], Conjecture 0.3.8). *For an irreducible representation ρ of a finite symplectic group $Sp_{2N}(\mathbb{F}_q)$ (for q an odd prime power), if the U -rank of ρ is strictly less than N , then it agrees with the tensor rank of ρ :*

$$(7) \quad rk_U(\rho) = rk_{\otimes}(\rho).$$

We then also have:

Corollary 1 (The Gurevich-Howe Exhaustion Conjecture, [15], Conjecture 0.4.12). *For every choice of $0 < n < N$, every irreducible $Sp_{2N}(\mathbb{F}_q)$ -representation of U -rank n is produced in the image of an*

eta correspondence $\eta_{W,B}^{\mathbb{F}_q^{2N}}$ for one of the two choices of orthogonal spaces (W, B) of dimension n .

Proof. In [15], Theorem 0.4.13, it is proved that every irreducible representation of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ with tensor rank n is produced in the image of such an eta correspondence. Thus, since Theorem 2 states that, in this range, tensor rank is always equal to U -rank, the exhaustion conjecture follows immediately. \square

Remark: Similar results for groups of type A were proved by R. M. Guralnick, M. Larsen, and P. H. Tiep [14] and for groups of type B and D by M. Larsen and P. H. Tiep [34].

The correspondences (5), (6) are not formal extensions of the correspondences (1), (2) to the metastable range. If we extended the correspondences (1), (2) formally, we would obtain “generalized Lusztig symbols” which could either contain repeated terms (these translate to the term $\{0\}$ in (5), (6)) or non-monotone terms; those are replaced by different genuine Lusztig symbols. The reason for working in the two metastable ranges is to avoid the appearance of illegal Lusztig symbols with negative terms, which are more complicated to resolve.

Our method for proving Theorem 1 is interpolation of semisimple pre-Tannakian categories [6, 7, 17, 27, 28, 29]. There are other possible methods for approaching this problem. Notably, analogously as the general linear group is embedded into the multiplicative semigroup of all square matrices, the symplectic group is embedded into the *oscillator semigroup* discovered by Howe [18, 15, 16, 20]. Using this semigroup, one can also obtain results on the Gurevich-Howe decomposition beyond the stable ranges (see e.g. Proposition 8.2.1 of [15]).

Since (5) and (6) cover all the cases of V and W , it is possible to use Theorem 1 for an inductive construction of all irreducible representations of $\mathrm{Sp}(V)$, $\mathrm{O}(W, B)$ (see Subsection 4.5).

This paper is organized as follows: In Section 2, we discuss some background of the oscillator representations and the relevant cases of Lusztig’s classification of irreducible representations of finite groups of Lie type. In Section 3, we describe the metastable ranges, construct an eta or zeta correspondence for every case of type I reductive dual pair, and discuss the alternating sums replacing parabolic induction in the metastable range. This establishes the necessary notation to restate Theorem 1 in concrete terms. In Section 4, we discuss interpolated representation categories and the analogues of the results of [30, 31],

prove Theorem 1, and discuss the representation-theoretical implications. In Section 5, we discuss the Gurevich-Howe rank conjecture, proving Theorem 2. In Section 6, we resolve the alternating sums appearing as coefficients in the decomposition of the restricted oscillator representation.

In Appendix A, we discuss a dictionary between the notation and results of this series of papers with S.-Y. Pan's identification of the pairs of irreducible representations whose tensor products appear with non-zero multiplicity in the restricted oscillator representation.

2. BACKGROUND

We begin by recalling the results of [30, 31] more precisely.

First, we fix notation: For a symplectic group $\mathrm{Sp}(\mathbf{V})$, we write $\omega_a[\mathbf{V}]$ to denote the *oscillator representation* arising from the Weil-Shale representation of the Heisenberg group on \mathbf{V} with central (non-trivial, additive) character in \mathbb{F}_q corresponding to $a \in \mathbb{F}_q^\times$ under a fixed identification of \mathbb{F}_q with its Pontrjagin dual. In the case of $a = 1$, we omit the subscript and write $\omega[\mathbf{V}] = \omega_1[\mathbf{V}]$.

Now let us consider a type I reductive dual pair of subgroups of $\mathrm{Sp}(\mathbf{V})$, which must be of the form

$$(8) \quad (\mathrm{Sp}(V), \mathrm{O}(W, B)) \subseteq \mathrm{Sp}(\mathbf{V}),$$

for \mathbb{F}_q -spaces V and W , say with symplectic and symmetric bilinear forms S and B respectively, so that $\mathbf{V} = V \otimes W$, and we consider its symplectic form to be $S \otimes B$ (see, for example, [18]). Tensoring matrices gives an inclusion of the product $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$ into $\mathrm{Sp}(\mathbf{V})$. Note that if there exists a k -dimensional isotropic subspace of W (resp. V), then B (resp. S) can be expressed as a direct sum of k copies of a hyperbolic 2-dimensional symmetric bilinear (resp. symplectic) form with a form $B[-k]$ (resp. $S[-k]$) on a $\dim(W) - 2k$ -dimensional $W[-k]$ (resp. $\dim(V) - 2k$ -dimensional space $V[-k]$). If k is a dimension of an isotropic subspace, we write P_k^B (resp. P_k^V) for the maximal parabolic subgroup of $\mathrm{O}(W, B)$ (resp. $\mathrm{Sp}(V)$) with Levi factor $\mathrm{GL}_k(\mathbb{F}_q) \times \mathrm{O}(W[-k], B[-k])$ (resp. $\mathrm{GL}_k(\mathbb{F}_q) \times \mathrm{Sp}(V[-k])$).

In [30], we defined two stable ranges of such type I reductive dual pairs: We say a pair (8) is in the *symplectic stable range* if $\dim(W) \leq \dim(V)$, and similarly, we say it is in the *orthogonal stable range* if the dimension of V is less than or equal to the dimension of a maximal isotropic subspace of W with respect to B . Let us denote by h_W the dimension of a maximal isotropic subspace of W . We proved that, for $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the symplectic stable range, the restriction of

$\omega[V \otimes W]$ to a $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$ -representation is

$$(9) \quad \bigoplus_{k=0}^{h_W} \bigoplus_{\rho \in \widehat{\mathrm{O}(W[-k], B[-k])}} \eta^V(\rho) \otimes \mathrm{Ind}^{P_k^B}(\rho \otimes \epsilon(\det))$$

for a system of mutually disjoint injections

$$\eta_{W,B}^V : \widehat{\mathrm{O}(W, B)} \hookrightarrow \widehat{\mathrm{Sp}(V)}$$

called the *eta correspondence* (ommiting the subscript when the source is determined). See also, for this case, the original papers of S. Gurevich and R. Howe finding the eta correspondence [15, 16] and an approach. Similarly, for $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the orthogonal stable range, writing $\dim(V) = 2N$, the restriction of $\omega[V \otimes W]$ to $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$ decomposes as

$$(10) \quad \bigoplus_{k=0}^N \bigoplus_{\rho \in \widehat{\mathrm{Sp}(V[-k])}} \mathrm{Ind}^{P_k^V}(\rho \otimes \epsilon(\det)) \otimes \zeta^{W,B}(\rho)$$

for a system of mutually disjoint injections

$$\zeta_V^{W,B} : \widehat{\mathrm{Sp}(V)} \hookrightarrow \widehat{\mathrm{O}(W, B)}$$

called the *zeta correspondence* (again, ommiting the subscript when the source is determined).

Therefore, in either stable range, the problem of Howe duality reduces to explicitly computing the eta and zeta correspondences, which we was the main result [31], using Lusztig's classification of irreducible representations: Recall that, broadly, an irreducible representation of a finite group of Lie type is classified by data consisting of a conjugacy class of a semisimple element (s) in the dual group G^D (the “semisimple part”), a unipotent representation of (the dual of) the centralizer of s in G^D , and possible “central sign data” when $Z(G)$ is disconnected; we discuss this in more detail in Section 2 below. For $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the symplectic stable range, our computation of the eta correspondence

$$\eta_{W,B}^V : \widehat{\mathrm{O}(W, B)} \rightarrow \widehat{\mathrm{Sp}(V)}$$

can be summarized as transforming the Lusztig data of an irreducible representation of $\mathrm{O}(W, B)$ into Lusztig data specifying an irreducible representation of $\mathrm{Sp}(V)$ by

- Adding an appropriate number of -1 eigenvalues (and a single 1 eigenvalue, with position depending on the action of $\mathbb{Z}/2 \subseteq \mathrm{O}(W, B)$) to the semisimple part if $\dim(W)$ is odd, and 1 eigenvalues if $\dim(W)$ is even.

- Altering the unipotent part by considering the single changed factor of the centralizer of the new semisimple part (corresponding to -1 eigenvalues if $\dim(W)$ is odd and 1 eigenvalues of the semisimple part if $\dim(W)$ is even) and adding a single coordinate to the Lusztig symbol to get the appropriate new rank and defect.
- Central sign data is determined by the quadratic character applied to the semisimple part (as a torus element) multiplied by the discriminant of B when $\dim(W)$ is odd, and the central sign data of $SO(W, B)$ when $\dim(W)$ is even.

Similarly, for $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the orthogonal stable range, our construction of the zeta correspondence

$$\zeta_V^{W,B} : \widehat{\mathrm{Sp}(V)} \rightarrow \widehat{\mathrm{O}(W, B)}$$

can be summarized by altering the Lusztig data of an irreducible representation of $\widehat{\mathrm{Sp}(V)}$ by adding -1 eigenvalues to the semisimple part if $\dim(W)$ is odd, 1 eigenvalues if $\dim(W)$ is even, altering the affected factor of the unipotent part by adding a single appropriate coordinate, and assigning central sign data determined by the quadratic character of the original semisimple part and $\mathrm{disc}(B)$ or the original central sign data.

To be even more precise, we need to also recall Lusztig's classification. Consider a connected finite group of Lie type G . Consider a conjugacy class (s) of a semisimple element of the dual group $s \in G^D$. Consider also an irreducible unipotent representation u of the dual of s 's centralizer

$$u \in (\widehat{Z_{G^D}(s)})^D.$$

This data determines a G -representation we denote by $r^G[(s), u]$ of dimension

$$\dim(r^G[(s), u]) = \frac{|G|_{q'}}{|(Z_{G^D}(s))^D|_{q'}} \dim(u)$$

(recall that the order of a group of Lie type is equal to the order of its dual). In the case when G has connected center (e.g. $G = SO_{2m+1}(\mathbb{F}_q)$), the $r^G[(s), u]$ are precisely the irreducible G -representations.

We will also want to consider cases of G with non-trivial center (e.g. $G = \mathrm{Sp}_{2N}(\mathbb{F}_q)$). In the case of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, the representation $r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u]$ may split into two non-isomorphic irreducible summands

$$(11) \quad r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u] = r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, +1] \oplus r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, -1]$$

of dimensions

$$(12) \quad \dim(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, \pm 1]) = \frac{\dim(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u])}{2}$$

where the ± 1 corresponds to specifying an action of $Z(G) = \mathbb{Z}/2$ (with $+1$ corresponding to the trivial action and -1 corresponding to the sign). For the symplectic group, this occurs precisely when s has -1 as an eigenvalue and the tensor factor of u corresponding to the factor of s 's centralizer contributed by -1 eigenvalues is non-degenerate.

We also need to discuss the case of disconnected G , particularly the orthogonal groups $\mathrm{O}_{2m+1}(\mathbb{F}_q)$ and $\mathrm{O}_{2m}^{\pm}(\mathbb{F}_q)$. In these cases, we classify the irreducible representation according to data defining an irreducible representation of the corresponding special orthogonal group and certain *extension data* specifying a certain summand of the induction when it splits. In the case of odd orthogonal groups, the extension data always consists of a single sign specifying the action of the $\mathbb{Z}/2$ center on an irreducible representation, since we have a splitting $\mathrm{O}_{2m+1}(\mathbb{F}_q) = \mathbb{Z}/2 \times \mathrm{SO}_{2m+1}(\mathbb{F}_q)$. In the case of even orthogonal groups, the extension data is more complex and less canonical.

2.1. The semisimple data. The data of a conjugacy class of a semisimple element in a finite group of Lie type is equivalent to the data of its eigenvalues (under the action of the Weyl group). In any symplectic or special orthogonal group, every maximal torus is isomorphic to a product of SO_2^{\pm} factors, possibly on field extensions of \mathbb{F}_q

$$(13) \quad T \cong \prod_{i=1}^k \mathrm{SO}_2^{\pm}(\mathbb{F}_{q^{n_i}})$$

such that $n_1 + \cdots + n_k$ adds up to the total rank, and in the case of tori in even special orthogonal groups, the sign \pm denoting whether or not the full group is equal to the product of signs appearing in each SO_2^{\pm} factor. Note that each factor is cyclic $\mathrm{SO}_2^{\pm}(\mathbb{F}_{q^n}) \cong \mu_{q^n \mp 1}$, giving an identification of T with its Pontrjagin dual. Write $\chi_{(s)}$ for the character of T corresponding to a semisimple $(s) \in T$. Then if, for some G , we consider Lusztig classification data consisting of semisimple part $(s) \in G^D$ and unipotent part u , we have

$$(14) \quad r^G[(s), u] \subseteq \mathrm{Ind}_G^T(\chi_{(s)})$$

(noting on the right hand side that, writing (13), each T is self-dual and the maximal tori in G are identified with those in G^D).

Given a choice of semisimple part of Lusztig classification data consisting of a conjugacy class $(s) \in G^D$ before proceeding further with describing the rest of the Lusztig classification data, we must write

down its centralizer $Z_{G^D}(s)$. To compute this, consider s as an element of some maximal torus T as in (13), minimizing n_i where possible. Coordinates $\pm 1 \neq \lambda \in SO_2^\pm(\mathbb{F}_{q^n})$ of multiplicity j give a centralizer factor $U_j^\pm(\mathbb{F}_{q^n})$ (where we use the notation $U_j^+ = GL_j$). Eigenvalues $\pm 1 \in SO_2^\pm(\mathbb{F}_q)$ give centralizer factors depending on the specific choice of G^D and G .

If $G^D = \mathrm{Sp}_{2r}(\mathbb{F}_q)$ (to describe representations of $G = \mathrm{SO}_{2r+1}(\mathbb{F}_q)$), a semisimple $s \in G$ has centralizer

$$(15) \quad Z_{\mathrm{Sp}_{2r}(\mathbb{F}_q)}(s) = \prod_{i=1}^k U_{j_i}^\pm(\mathbb{F}_{q^{n_i}}) \times \mathrm{Sp}_{2p}(\mathbb{F}_q) \times \mathrm{Sp}_{2\ell}(\mathbb{F}_q)$$

where s has 1 as an eigenvalue of multiplicity $2p$ and -1 as an eigenvalue of multiplicity 2ℓ .

If $G^D = \mathrm{SO}_{2r+1}(\mathbb{F}_q)$ (to describe representations of $G = \mathrm{Sp}_{2r}(\mathbb{F}_q)$), a semisimple element $s \in G$ has centralizer

$$(16) \quad Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s) = \prod_{i=1}^k U_{j_i}^\pm(\mathbb{F}_{q^{n_i}}) \times \mathrm{SO}_{2p+1}(\mathbb{F}_q) \times \mathrm{SO}_{2\ell}^\pm(\mathbb{F}_q)$$

if s has 1 as an eigenvalue of multiplicity $2p+1$ and -1 as an eigenvalue of multiplicity 2ℓ (recall that a single 1 eigenvalue is automatic in this case, since it must be added to embed $T \subset \mathrm{SO}_{2r+1}(\mathbb{F}_q)$). The sign of the final factor $\mathrm{SO}_{2\ell}^\pm(\mathbb{F}_q)$ corresponds to whether we pick $T \subset \mathrm{SO}_{2r}^+(\mathbb{F}_q)$ or $\mathrm{SO}_{2r}^-(\mathbb{F}_q)$. We specifically will later consider the semisimple conjugacy classes (σ_r^+) and (σ_r^-) which both have -1 as an eigenvalue of multiplicity $2r$ and a single (automatic) 1 eigenvalue, placed so that σ_r^\pm is in a torus of the form (13) such that the product of all involved signs of SO_2 is equal to \pm . These elements satisfy $Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(\sigma_r^\pm) = \mathrm{SO}_{2r}^\pm(\mathbb{F}_q)$.

If $G^D = \mathrm{SO}_{2r}^\pm(\mathbb{F}_q)$ (to describe representations of $G = \mathrm{SO}_{2r}^\pm(\mathbb{F}_q)$), a semisimple $s \in G$ has centralizer

$$(17) \quad Z_{\mathrm{SO}_{2r}^\pm(\mathbb{F}_q)}(s) = \prod_{i=1}^k U_{j_i}^\pm(\mathbb{F}_{q^{n_i}}) \times \mathrm{SO}_{2p}^\pm(\mathbb{F}_q) \times \mathrm{SO}_{2\ell}^\pm(\mathbb{F}_q)$$

with signs chosen so that their product is the total sign of G , where s has 1 as an eigenvalue of multiplicity $2p$ and -1 as an eigenvalue of multiplicity 2ℓ . For more details, see [4].

2.2. The unipotent data. Given a choice of $(s) \in G^D$, the next part of Lusztig classification data is a unipotent representation u of the dual of the centralizer of s $(Z_{G^D}(s))^D$, which by (15), (16), (17), we can write down as a product of unitary groups and a pair of factors of B , C , D or 2D -type. We may consider u as a tensor product of

irreducible unipotent representations of each factor. It will turn out (recalling the constructions in [31]) that only the tensor factor of u corresponding to one of these final two factors will be “altered” in the description of η or ζ . Therefore, we describe in this subsection the classification of unipotent representations of $\mathrm{Sp}_{2r}(\mathbb{F}_q)$, $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$, and $\mathrm{SO}_{2r}^\pm(\mathbb{F}_q)$, using symbols.

The classification of irreducible unipotent representations of $\mathrm{Sp}_{2r}(\mathbb{F}_q)$ and $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$ are the same, since they are dual groups. A *symbol of B- or C-type* and rank r is defined to be a pair of increasing sequences

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

for $\lambda_i, \mu_i \in \mathbb{Z}_{\geq 0}$ such that $(\lambda_1, \mu_1) \neq (0, 0)$, $a - b$ is odd (the “defect condition”), and

$$\sum_{i=1}^a \lambda_i + \sum_{i=1}^b \mu_i = r + \frac{(a+b-1)^2}{4}$$

(the “rank condition”). We take switching rows to give the same symbol. The irreducible unipotents are in bijective correspondence with this combinatorial data (and we denote the representation corresponding to a symbol by the symbol itself).

Similarly, for the case of $\mathrm{SO}_{2r}^+(\mathbb{F}_q)$ (resp. $\mathrm{SO}_{2r}^-(\mathbb{F}_q)$), irreducible unipotent representations correspond to symbols of D - (resp. 2D -type) and rank r , which are defined to consist of pairs of increasing sequences $\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$ for $\lambda_i, \mu_i \in \mathbb{Z}_{\geq 0}$ such that $(\lambda_1, \mu_1) \neq (0, 0)$, the “defect condition” $a - b \equiv 0 \pmod{4}$ (resp. $\equiv 2 \pmod{4}$, for $\mathrm{SO}_{2r}^-(\mathbb{F}_q)$) and the “rank condition”

$$\sum_{i=1}^a \lambda_i + \sum_{i=1}^b \mu_i = r + \frac{(a+b)(a+b-2)}{4}$$

(the same rank condition is used for $\mathrm{SO}_{2r}^-(\mathbb{F}_q)$). Again, we denote a unipotent representation the same as its corresponding symbol.

Further, the dimension of a unipotent representation $\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$ of a symplectic or special orthogonal group G can be calculated as the following formula

$$\frac{\prod_{1 \leq i < j \leq a} (q^{\lambda_j} - q^{\lambda_i}) \cdot \prod_{1 \leq i < j \leq b} (q^{\mu_j} - q^{\mu_i}) \cdot \prod_{1 \leq i \leq a, 1 \leq j \leq b} (q^{\lambda_i} + q^{\mu_j}) \cdot |G|_q}{2^{c(a,b)} \cdot \prod_{1 \leq i \leq a} \prod_{j=1}^{\lambda_i} (q^{2j} - 1) \cdot \prod_{1 \leq i \leq b} \prod_{j=1}^{\mu_i} (q^{2j} - 1) \cdot q^{d(a,b)}}$$

where $c(a, b) = \lfloor (a+b-1)/2 \rfloor$, and we write $d(a, b) = \sum_{i=1}^{\lfloor (a+b)/2 \rfloor} \binom{a+b-2i}{2}$.

2.3. Central and extension sign data. As described in the beginning of the section, since the center of $\mathrm{Sp}_{2r}(\mathbb{F}_q)$ is $\mathbb{Z}/2$, the representation $r^{\mathrm{Sp}_{2r}(\mathbb{F}_q)}[(s), u]$ corresponding to semisimple $(s) \in \mathrm{SO}_{2r+1}(\mathbb{F}_q)$ and unipotent representation u of $Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)$ splits into two non-isomorphic summands (11) of half dimension (12) precisely when s has -1 eigenvalues and the corresponding factor of u is non-degenerate (and is otherwise irreducible). In the case of this splitting, we call the choice of sign ± 1 and call the collection of data $[(s), u, \pm 1]$ the corresponding $\mathrm{Sp}(V)$ -Lusztig classification data.

Next, consider the odd orthogonal groups $\mathrm{O}_{2r+1}(\mathbb{F}_q)$. In this case, the center splits $\mathrm{O}_{2r+1}(\mathbb{F}_q) = \mathbb{Z}/2 \times \mathrm{SO}_{2r+1}(\mathbb{F}_q)$. For $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$ the center is trivial, so the Lusztig classification data of a semisimple conjugacy class $(s) \in \mathrm{Sp}_{2r}(\mathbb{F}_q)$ and an irreducible unipotent representation of the centralizer $Z_{\mathrm{Sp}_{2r}(\mathbb{F}_q)}(s)$ determines all the irreducible representations $r^{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}[(s), u]$. The induction of such a representation always splits

$$\begin{aligned} \mathrm{Ind}_{\mathrm{O}_{2r+1}(\mathbb{F}_q)}^{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(r^{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}[(s), u]) = \\ (+1) \otimes r^{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}[(s), u] \oplus (-1) \otimes r^{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}[(s), u]. \end{aligned}$$

where (± 1) denote the trivial or sign representation of the $\mathbb{Z}/2$ central factor of $\mathrm{O}_{2r+1}(\mathbb{F}_q)$. Call this sign the $\mathrm{O}_{2m+1}(\mathbb{F}_q)$ -extension data. Therefore, writing

$$r^{\mathrm{O}_{2r+1}(\mathbb{F}_q)}[(s), u]^{\pm 1} := (\pm 1) \otimes r^{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}[(s), u],$$

we see that the irreducible representations of $\mathrm{O}_{2r+1}(\mathbb{F}_q)$ are classified by the extended $\mathrm{O}_{2r+1}(\mathbb{F}_q)$ -Lusztig classification data of a semisimple conjugacy class $(s) \in \mathrm{Sp}_{2r}(\mathbb{F}_q)$, an irreducible unipotent representation u of the dual of its centralizer, and extension sign data (± 1) .

Let us now consider $\widehat{O_{2r}^{\pm}(\mathbb{F}_q)}$ -representations. Suppose $\pi \in \widehat{O_{2r}^{\pm}(\mathbb{F}_q)}$. Then ρ is classified according to two effects

- (1) If ρ is a summand of a toral induction $\mathrm{Ind}_{O_{2r}^{\pm}(\mathbb{F}_q)}^T(\chi)$ for some character $\chi \neq \chi^{-1}$ in the Pontrjagin dual of T , then the restriction $\mathrm{Res}_{\mathrm{SO}_{2r}^{\pm}(\mathbb{F}_q)}(\rho)$ splits into two irreducible $\mathrm{SO}_{2r}^{\pm}(\mathbb{F}_q)$, with Lusztig classification data for each piece being the same, except semisimple parts are two distinct conjugacy classes of semisimple elements in $\mathrm{SO}_{2r}^{\pm}(\mathbb{F}_q)$ which are conjugate in $O_{2r}^{\pm}(\mathbb{F}_q)$.
- (2) If ρ is a summand of $\mathrm{Ind}_{O_{2r}^{\pm}(\mathbb{F}_q)}^T(\chi)$, then
 - If χ has any ± 1 eigenvalues (and no ∓ 1 eigenvalues), then there exists precisely one other irreducible $O_{2r}^{\pm}(\mathbb{F}_q)$ -representation ρ' such that

$$\mathrm{Res}_{\mathrm{SO}_{2r}^{\pm}(\mathbb{F}_q)}(\rho) = \mathrm{Res}_{\mathrm{SO}_{2r}^{\pm}(\mathbb{F}_q)}(\rho')$$

(i.e. this $SO_{2r}^{\pm}(\mathbb{F}_q)$ -representation corresponds to a pair of irreducible $O_{2r}^{\pm}(\mathbb{F}_q)$ -representations.

- If χ has both ± 1 eigenvalues, then there exists precisely three other distinct irreducible $O_{2r}^{\pm}(\mathbb{F}_q)$ -representations with the same restriction to $SO_{2r}^{\pm}(\mathbb{F}_q)$ as ρ (i.e. this $SO_{2r}^{\pm}(\mathbb{F}_q)$ -representation corresponds to four distinct irreducible $O_{2r}^{\pm}(\mathbb{F}_q)$ -representations.

In summary, an irreducible representation of $O_{2r}^{\pm}(\mathbb{F}_q)$ corresponds to the data of

- (1) a semisimple conjugacy class (s) in $O_{2r}^{\pm}(\mathbb{F}_q)$
- (2) a unipotent representation u of $(Z_{SO_{2r}^{\pm}}(s))^D$
- (3) a choice of $O_{2r}^{\pm}(\mathbb{F}_q)$ -extension data consisting of a choice of sign ± 1 if s has 1 eigenvalues and an (independently chosen) sign ± 1 if s has -1 eigenvalues.

Call this the Lusztig classification data of an irreducible representation of O_{2r}^{\pm} . Denote the corresponding irreducible representation by

$$r^{O_{2r}^{\pm}(\mathbb{F}_q)}[(s), u, (\pm 1, \pm 1)],$$

writing in $(\pm 1, \pm 1)$ the sign arising from $+1$ eigenvalues of s first and the sign arising from -1 eigenvalues of s second, and removing either if s has no such eigenvalues.

3. THE GENERAL STATEMENTS

Finally, in this section we precisely state the constructions of the extended eta and zeta correspondences, define the alternating sums of parabolic inductions, and state Theorem 1.

In Subsection 3.1, we precisely define the symplectic and orthogonal metastable ranges. In Subsections 3.2 and 3.3, we describe the constructions of the extended eta and zeta correspondences in terms of Lusztig classification data. The constructions are entirely the same as the constructions of the eta and zeta correspondences in the symplectic and orthogonal stable ranges, when they can be applied. For input representations where the construction cannot be applied (specifically, where the step of concatenating a new coordinate to a symbol factor in the unipotent part of the Lusztig classification fails to give a legal symbol), we set the extended eta and zeta correspondences to output 0. In Subsection 3.4, we define the alternating sums of Verma modules that play the role of coefficients for the eta and zeta correspondence terms in the decomposition of the restricted oscillator representation in Theorem 1. Finally, having prepared all the necessary notation, we

restate Theorem 1 precisely in Subsection 3.5 as Theorems 3.5.1 and 3.5.2.

3.1. The metastable ranges. First, for a general unstable choice of symplectic and orthogonal spaces V and (W, B) , we must still choose whether it is “closer” to the symplectic or orthogonal stable range.

We separate the pairs $(V, (W, B))$ which do not lie in the symplectic stable range or the orthogonal stable range into “metastable ranges” to indicate whether we intend to approach the decomposition of the restriction of $\omega[V \otimes W]$ by extending the eta or zeta correspondence. We consider the different cases of $O(W, B)$ individually.

Definition 3.1.1. *Consider a choice of symplectic and orthogonal spaces V and (W, B) . Write $\dim(V) = 2N$.*

- *If W is of odd dimension $\dim(W) = 2m + 1$, then we say $(V, (W, B))$ is in the symplectic metastable range if*

$$m < N < 2m + 1.$$

Say $(V, (W, B))$ is in the orthogonal metastable range if

$$m < 2N \leq 2m$$

- *If W is of even dimension $\dim(W) = 2m$ and B is not completely split, then we say $(V, (W, B))$ is in the symplectic metastable range if*

$$m \leq N < 2m.$$

Say $(V, (W, B))$ is in the orthogonal metastable range if

$$m - 1 < 2N < 2m.$$

- *If W is of even dimension $\dim(W) = 2m$ and B is completely split, then we say $(V, (W, B))$ is in the symplectic metastable range if*

$$m \leq N < 2m.$$

Say $(V, (W, B))$ is in the orthogonal metastable range if

$$m < 2N < 2m.$$

We see that under this definition, every unstable choice of symplectic and orthogonal spaces V and (W, B) is contained in precisely one metastable range. More specifically, the disjoint union of the symplectic stable and metastable ranges consists of all choices of symplectic spaces V and orthogonal spaces (W, B) such that

$$(18) \quad \frac{\dim(V)}{2} \geq \lfloor \frac{\dim(W)}{2} \rfloor,$$

while the disjoint union of the orthogonal stable and metastable ranges consist of V and (W, B) satisfying the complimentary condition

$$(19) \quad \frac{\dim(V)}{2} < \lfloor \frac{\dim(W)}{2} \rfloor.$$

Broadly, the conditions (18) and (19) should be thought of as detecting whether it is more computationally viable to decompose the oscillator representation in terms of the eta correspondence (i.e. as a sum of distinct irreducible $\mathrm{Sp}(V)$ -representations with potentially non-irreducible $O(W, B)$ -coefficients), or in terms of the zeta correspondence (i.e. as a sum of distinct irreducible $O(W, B)$ -representations with potentially non-irreducible $\mathrm{Sp}(V)$ -coefficients). Concretely, in our constructions of the eta and zeta correspondence, the conditions (18) and (19) ensure that, in either case, we never attempt to concatenate a negative coordinate to a Lusztig symbol. It is possible to further extend the eta and zeta correspondences to all ranges by interpreting them to output 0 when this occurs (Lusztig’s dimension formula for symbols indeed suggests that symbols with negative coordinates are 0-dimensional). However, approaching the decomposition of the oscillator representation from the “wrong side” is in general less computationally efficient, and we do not consider it for the purposes of this paper.

3.2. The extended eta correspondence. For a choice of symplectic and orthogonal spaces V , (W, B) which are in the disjoint union of the symplectic stable and metastable ranges, we define a map

$$\eta_{W,B}^V : \widehat{O(W, B)} \rightarrow \widehat{\mathrm{Sp}(V)} \cup \{0\}$$

selecting the “top” irreducible representation of $\mathrm{Sp}(V)$ whose tensor product with the input irreducible representation of $O(W, B)$ is a summand of the restricted oscillator representation (or outputting 0 when no such summand appears). More explicitly, for every $\rho \in \widehat{O(W, B)}$,

$$\eta_{W,B}^V(\rho) \otimes \rho \subset \mathrm{Res}_{\mathrm{Sp}(V) \times O(W, B)}(\omega[V \otimes W])$$

and there exists some $\pi \in \widehat{\mathrm{Sp}(V)}$ with $\pi \otimes \rho$ appearing as a summand of the restricted oscillator representation if and only if $\eta_{W,B}^V(\rho) \neq 0$.

With this formulation, we may treat $\eta_{W,B}^V$ for $(V, (W, B))$ in the symplectic stable and metastable range at the same time. In the case of symplectic stable choices of V and (W, B) , this is precisely a review of the construction of the eta correspondence given in [31]. To define the map $\eta_{W,B}^V$ we treat the different cases of (W, B) separately.

Case 1: Suppose W is of odd dimension $2m+1$, and suppose $(V, (W, B))$ is either in the symplectic stable or metastable range. In this case, writing $\dim(V) = 2N$, this means $m < N$.

For an irreducible representation $\pi \in \widehat{O(W, B)}$, we can then write it as

$$(20) \quad \pi \cong r^{O_{2m+1}(\mathbb{F}_q)}[(s), u]^{\pm 1}$$

where (s) is a conjugacy class of a semisimple element $s \in \mathrm{Sp}_{2m}(\mathbb{F}_q) = (SO_{2m+1}(\mathbb{F}_q))^D$, and u is an irreducible unipotent representation of $(Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s))^D$. Then to define $\eta_{W, B}^V(\pi)$, we must either specify Lusztig classification data for $\mathrm{Sp}_{2N}(\mathbb{F}_q) = \mathrm{Sp}(V)$, or put it to be 0.

The first part of Lusztig classification data for $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ is a conjugacy class of a semisimple element in $(\mathrm{Sp}_{2N}(\mathbb{F}_q))^D = \mathrm{SO}_{2N+1}(\mathbb{F}_q)$. Let us consider the semisimple part s of the input representation's Lusztig classification data as an element of a maximal torus

$$(21) \quad s \in T = \prod_{i=1}^k SO_2^\pm(\mathbb{F}_{q^{n_i}}) \subseteq (SO_{2m+1}(\mathbb{F}_q))^D = \mathrm{Sp}_{2m}(\mathbb{F}_q)$$

(such that $n_1 + \cdots + n_k = m$). Identifying each $SO_2^\pm(\mathbb{F}_{q^{n_i}})$ with the cyclic group $\mu_{q^{n_i} \mp 1}$, define $\epsilon(s)$ to be the product of the quadratic character $\epsilon : \mu_{q^{n_i} \mp 1} \rightarrow \{\pm 1\}$ applied to each coordinate of s in (21). Taking a product with $\sigma_{N-m}^\pm \in SO_{2(N-m)+1}(\mathbb{F}_q)$ (recalling that $N > m$ by the range conditions) gives two options of a conjugacy class of a semisimple element in $SO_{2N+1}(\mathbb{F}_q)$. Write

$$\phi^\pm(s) = s \oplus \sigma_{N-m}^\pm \in SO_{2N+1}(\mathbb{F}_q).$$

We choose the sign of \pm according to the sign of the (± 1) factor indicating the $\mathbb{Z}/2$ -action in (20). Suppose s has -1 as an eigenvalue of multiplicity 2ℓ , and write its centralizer as

$$(22) \quad Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s) = H \times \mathrm{Sp}_{2\ell}(\mathbb{F}_q).$$

Then $\phi^\pm(s)$ has -1 as an eigenvalue of multiplicity $2(N-m+\ell)$ and its centralizer can be factored $Z_{2O_{2N+1}(\mathbb{F}_q)}(\phi^\pm(s)) = H^D \times SO_{2(N-m+\ell)}^\pm(\mathbb{F}_q)$.

Next, writing (22), and then $Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s)^D = H^D \times SO_{2\ell+1}(\mathbb{F}_q)$, we may consider the unipotent part u of the Lusztig classification data of the input representation as a tensor product $u = u_{H^D} \otimes \left(\begin{smallmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{smallmatrix} \right)$ where, as the notation suggest, u_{H^D} is an irreducible unipotent representation of H^D , and $\left(\begin{smallmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{smallmatrix} \right)$ is a symbol specifying a unipotent representation of $SO_{2\ell+1}(\mathbb{F}_q)$. Since the defect $a - b$ of the symbol is odd, we may therefore permute the rows so that we may assume $a - b$

is 1 mod 4. Let us write

$$(23) \quad N'_\rho := N - m + \frac{a + b - 1}{2}.$$

Now if $N'_\rho < \lambda_a$, then $(\lambda_1 < \dots < \lambda_a < N'_\rho)$ describes a symbol of defect 2 mod 4 and rank precisely equal to $N - m + \ell$, and hence taking

$$\phi^-(u) = \widetilde{u_{HD}} \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a < N'_\rho \\ \mu_1 < \dots < \mu_b \end{pmatrix}$$

gives a unipotent representation of the dual of the centralizer of the new semisimple data $Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi^-(s))^D = H \times SO_{2(N-m+\ell)}^-(\mathbb{F}_q)$. In this case, say $\phi^-(u)$ is constructible. Say $\phi^-(u)$ is *inconstructible* if $N'_\rho \leq \lambda_a$.

Similarly, if $N'_\rho < \mu_b$, then $(\lambda_1 < \dots < \lambda_a)$ describes a symbol of defect 0 mod 4 and rank precisely equal to $N - m + \ell$, and hence taking

$$\phi^+(u) = \widetilde{u^{HD}} \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b < N'_\rho \end{pmatrix}$$

gives a unipotent representation of the dual of the centralizer of the new semisimple data $Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi^+(s))^D = H \times SO_{2(N-m+\ell)}^+(\mathbb{F}_q)$. In this case, say $\phi^+(u)$ is constructible. Say $\phi^+(u)$ is *inconstructible* if $N'_\rho \leq \mu_b$.

Definition 3.2.1. *Assume the above notation. We define $\eta_{W,B}^V(\pi)$ to be the irreducible $Sp(V)$ -representation with Lusztig classification data $[(\phi^\pm(s)), \phi^\pm(u), \epsilon(s) \cdot \text{disc}(B)]$*

$$(24) \quad \eta_{W,B}^V(\pi) := r^{Sp(V)}[(\phi^\pm(s)), \phi^\pm(u), \epsilon(s) \cdot \text{disc}(B)],$$

if $\phi^\pm(u)$ is constructible. We put $\eta_{W,B}^V(\rho) := 0$ if $\phi^\pm(u)$ is inconstructible.

Case 2: Suppose W is of even dimension $2m$, and write α for the sign so that $O(W, B) = O_{2m}^\pm(\mathbb{F}_q)$. Suppose also that $(V, (W, B))$ is either in the symplectic stable or metastable range, meaning that if we write $\dim(V) = 2N$, we have $m \leq N$.

Let us consider an input irreducible $O_{2m}^\pm(\mathbb{F}_q)$ -representation π . Let us suppose that its $O_{2m}^\pm(\mathbb{F}_q)$ -Lusztig classification data consists of a $O_{2m}^\pm(\mathbb{F}_q)$ -conjugacy class (s) of a semisimple element of $SO_{2m}^\pm(\mathbb{F}_q)$, an irreducible unipotent representation u of, say, $Z_{SO_{2m}^\pm(\mathbb{F}_q)}(s)^D$, and possible central sign data depending on which eigenvalues appear in s .

As in the previous case, consider s as an element of a torus $s \in T \cong \prod_{i=1}^k SO_2^\pm(\mathbb{F}_{q^{n_i}})$. We take a direct sum with the identity matrix

$I_{2(N-m)+1}$ to obtain a semisimple element

$$\phi(s) = s \oplus I_{2(N-m)+1} \in SO_{2N+1}(\mathbb{F}_q),$$

adding only 1 eigenvalues to s . If s originally does not have any 1 eigenvalues, then its centralizer is altered by taking a product with a special orthogonal group factor

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi(s)) = (Z_{SO_{2m}^\alpha(\mathbb{F}_q)}(s))^D \times SO_{2(N-m)+1}(\mathbb{F}_q).$$

We consider the irreducible unipotent representation

$$\phi(u) := \tilde{u} \otimes 1$$

of the dual of this group, tensoring \tilde{u} with the trivial representation of the new factor $(SO_{2(N-m)+1}(\mathbb{F}_q))^D$. On the other hand, if s has 1 as an eigenvalue of multiplicity $2p$ for $p > 0$, then, writing

$$Z_{SO_{2m}^\alpha(\mathbb{F}_q)}(s) = H \times SO_{2p}^\pm(\mathbb{F}_q),$$

we then have

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi(s)) = H^D \times SO_{2(N-m+p)+1}(\mathbb{F}_q)$$

(note that in this case $H = H^D$). Write $(Z_{SO_{2m}^\alpha(\mathbb{F}_q)}(s))^D = H^D \times SO_{2p}^\pm(\mathbb{F}_q)$, and $u = u_{H^D} \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}$ for an irreducible unipotent representation u_{H^D} of H^D , and a symbol $\begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}$ of $SO_{2p}^\pm(\mathbb{F}_q)$. Suppose first that $\begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}$ is non-degenerate. In the case where the 1 eigenvalues correspond to a non-split factor $SO_{2p}^-(\mathbb{F}_q)$ of s 's centralizer, then switch rows so that $a - b + 1$ is 1 mod 4. In the case where the 1 eigenvalues correspond to a split factor $SO_{2p}^+(\mathbb{F}_q)$ of s 's centralizer, then switch rows so that either $a > b$ or, if $a = b$, for the maximal i such that $\lambda_i \neq \mu_i$, we have $\lambda_i > \mu_i$. Write

$$N'_\rho = N - m + \frac{a+b}{2}.$$

Therefore, if $\lambda_a < N'_\rho$ or $\mu_b < N'_\rho$,

$$\phi^-(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a < N'_\rho \\ \mu_1 < \dots < \mu_b \end{pmatrix}, \text{ or}$$

$$\phi^+(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b < N'_\rho \end{pmatrix}$$

respectively define irreducible unipotent representations of the dual group $Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi(s))^D = H \times \text{Sp}_{2(N-m+p)}(\mathbb{F}_q)$. Say $\phi^+(u)$, $\phi^-(u)$ are *inconstructible* if $\lambda_a \geq N'_\rho$, or $\mu_b \geq N'_\rho$, respectively. We choose the sign of $\phi^\pm(u)$ according to the central sign data (± 1) chosen from s having 1 eigenvalues.

Finally, to define an output irreducible $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ -representation, we need to also choose output central sign data if $\phi(s)$ has -1 eigenvalues. By definition, $\phi(s)$ has the same number of -1 eigenvalues as s . Therefore, in this case, the original s has -1 eigenvalues also, so the $O(W, B)$ -Lusztig classification data supplies us with the data of one more central sign ± 1 , which we use as the output central sign data.

Definition 3.2.2. *Assume the above notation. We define $\eta_{W,B}^V(\pi)$ to be the irreducible $\mathrm{Sp}(V)$ -representation with Lusztig classification data $[\phi(s), \phi^\alpha(u), \beta]$ where the sign α denotes the central sign data from s 's 1 eigenvalues (and is omitted if no such data is given), the sign in β is the central sign data from s 's -1 eigenvalues (and is omitted if no such data is given):*

$$(25) \quad \eta_{W,B}^V(\pi) := r^{\mathrm{Sp}(V)}[(\phi(s)), \phi^\alpha(u), \beta],$$

if $\phi^\alpha(u)$ is constructible. We put $\eta_{W,B}^V(\rho) := 0$ if $\phi^\pm(u)$ is inconstructible.

3.3. The extended zeta correspondence. Similarly as in the previous subsection, for a choice of symplectic and orthogonal spaces V , (W, B) which are in the disjoint union of the orthogonal stable and metastable ranges, we define a map

$$\zeta_V^{W,B} : \widehat{\mathrm{Sp}(V)} \rightarrow \widehat{O(W, B)} \cup \{0\}$$

analogously selecting the “top” irreducible representation of $O(W, B)$ whose tensor product with the input irreducible representation of $\mathrm{Sp}(V)$ is a summand of the restricted oscillator representation. This construction proceeds entirely similarly, still considering the cases of $O(W, B)$ separately:

Case 1: Suppose W is of odd dimension $2m+1$, and suppose $(V, (W, B))$ is either in the orthogonal stable or metastable range. In this case, writing $\dim(V) = 2N$, this means $m \geq N$.

Now we fix an irreducible representation ρ of $\mathrm{Sp}(V) = \mathrm{Sp}_{2N}(\mathbb{F}_q)$. Our goal is to construct an irreducible representation of $O(W, B)$, which is equivalent to specifying a semisimple conjugacy class $(s) \in (\mathrm{Sp}_{2N}(\mathbb{F}_q))^D = \mathrm{SO}_{2N+1}(\mathbb{F}_q)$, an irreducible unipotent representation of the dual of its centralizer, and an extension sign. Say that s has -1 as an eigenvalue of multiplicity 2ℓ , for $0 \leq \ell \leq N$, and let us write

$$Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(s) = H \times \mathrm{SO}_{2\ell}^\pm(\mathbb{F}_q).$$

Recall, as a semisimple element of $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$, s must have at least one 1 eigenvalue, arising from the embedding of any maximal torus T of the form (13) into $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$. Therefore, considering s as an

element of the torus, let us write $\tilde{s} \in T$ (giving a $2N$ by $2N$ matrix), by removing the single “forced” eigenvalue 1 from s . Taking a direct sum with $-I_{2(m-N)}$,

$$\psi(s) := \tilde{s} \oplus (-I)_{2(m-N)}$$

we obtain a semisimple element of $\mathrm{Sp}_{2m}(\mathbb{F}_q) = (SO_{2m+1}(\mathbb{F}_q))^D$, which has -1 as an eigenvalue of multiplicity $2(m-N+\ell)$. Its centralizer is then

$$Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(\psi(s)) = H^D \times \mathrm{Sp}_{2(m-N+\ell)}(\mathbb{F}_q).$$

For the unipotent part of the Lusztig classification data of ρ , again write $u = u^{H^D} \otimes \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ for $u^{H^D} \in (\widehat{H^D})_u$ and a symbol $\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ specifying an irreducible unipotent representation of $SO_{2\ell}^\pm(\mathbb{F}_q)$. In the case where the 1 eigenvalues correspond to a non-split factor $SO_{2p}^-(\mathbb{F}_q)$ of s 's centralizer, then switch rows so that $a-b+1$ is 1 mod 4. In the case where the 1 eigenvalues correspond to a split factor $SO_{2p}^+(\mathbb{F}_q)$ of s 's centralizer, then switch rows so that either $a > b$ or, if $a = b$, for the maximal i such that $\lambda_i \neq \mu_i$, we have $\lambda_i > \mu_i$. Let us write

$$m'_\rho = m - N + \frac{a+b}{2}.$$

Then if $\lambda_a < m'_\rho$, $\mu_b < m'_\rho$, respectively, the symbols

$$(26) \quad \binom{\lambda_1 < \dots < \lambda_a < m'_\rho}{\mu_1 < \dots < \mu_b}, \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b < m'_\rho}$$

have odd defect and rank precisely equal to $m-N+\ell$, and therefore specify irreducible unipotent representations of $SO_{2(m-N+\ell)+1}(\mathbb{F}_q) = (\mathrm{Sp}_{2(m-N+\ell)}(\mathbb{F}_q))^D$.

In the case when $\ell = 0$, both of the symbols (26) specify the trivial representation, so let us put

$$\psi(u) = \widetilde{u_{H^D}} \otimes 1.$$

On the other hand, in the case when $\ell > 0$, we are given sign data ± 1 in the Lusztig classification data for ρ , which we can use to select which of the symbols (26) we attempt to use for the unipotent part of the Lusztig classification data of $\zeta_V^{W,B}(\rho)$. Specifically, if $\lambda_a < m'_\rho$, $\mu_b < m'_\rho$, respectively, we put

$$\begin{aligned} \psi^+(u) &= \widetilde{u_{H^D}} \otimes \binom{\lambda_1 < \dots < \lambda_a < m'_\rho}{\mu_1 < \dots < \mu_b} \\ \psi^-(u) &= \widetilde{u_{H^D}} \otimes \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b < m'_\rho}. \end{aligned}$$

If $\lambda_a \geq m'_\rho$, resp. $\mu_b \geq m'_\rho$, we say $\psi^+(u)$, resp. $\psi^-(u)$ is inconstructible.

Definition 3.3.1. Assume the above notation, writing $\rho = r^{\text{Sp}(V)}[(s), u]$. In the case where the semisimple part s of ρ 's Lusztig classification data has no -1 eigenvalues i.e. $\ell = 0$, we take $\zeta_V^{W,B}(\rho)$ to be the tensor product of the irreducible $SO(W, B)$ -representation corresponding to Lusztig classification data $[(\psi(s)), \psi(u)]$ with the sign $\epsilon(s) \cdot \text{disc}(B)$:

$$(27) \quad \zeta_V^{W,B}(\rho) = r^{O(W,B)}[(\psi(s)), \psi(u)]^{\epsilon(s) \cdot \text{disc}(B)}$$

In the case where s has -1 eigenvalues i.e., $\ell > 0$, writing $\rho = r^{\text{Sp}(V)}[(s), u, \pm 1]$, we take $\zeta_V^{W,B}(\rho)$ to be the tensor product of the irreducible $SO(W, B)$ -representation corresponding to Lusztig classification data $[(\psi(s)), \psi^\pm(u)]$ with the sign $\epsilon(s) \cdot \text{disc}(B)$

$$(28) \quad \zeta_V^{W,B}(\rho) = r^{O(W,B)}[(\psi(s)), \psi^\pm(u)]^{\epsilon(s) \cdot \text{disc}(B)},$$

if the involved $\psi^\pm(u)$ is constructible. Put $\zeta_V^{W,B}(\rho) = 0$ if $\psi^\pm(u)$ is inconstructible.

Case 2: Suppose W is of even dimension $2m$, and write α for the sign so that $O(W, B) = O_{2m}^\pm(\mathbb{F}_q)$. Suppose also that $(V, (W, B))$ is either in the orthogonal stable or metastable range, meaning that

Consider an irreducible representation ρ of $\text{Sp}(V) = \text{Sp}_{2N}(\mathbb{F}_q)$. We want to produce $O_{2m}^\pm(\mathbb{F}_q)$ -Lusztig classification data, which we recall consists of a semisimple conjugacy class $(\psi(s)) \in O_{2m}^\pm(\mathbb{F}_q)$, a unipotent representation $\psi(u)$ of the (dual) of the centralizer of s in $SO_{2m}^\alpha(\mathbb{F}_q)$, and central sign data depending on the eigenvalues of $\psi(s)$. We note that since $\text{Res}_{O(W,B)}(\omega[V \otimes W])$ is the permutation representation $\mathbb{C}W$ tensored with the representation $\epsilon(\det)$ (corresponding to the sign representation of $O(W, B)/SO(W, B)$), part of the central sign data is already forced. Specifically, as in the case of the symplectic group, we will only need to choose central sign data for the output representation corresponding to -1 -eigenvalues.

Write (s) with $s \in SO_{2N+1}(\mathbb{F}_q) = (\text{Sp}_{2N}(\mathbb{F}_q))^D$ for the semisimple part of the Lusztig classification data for the input representation ρ of $\text{Sp}_{2N}(\mathbb{F}_q)$. Say s has 1 as an eigenvalue of multiplicity $2p + 1$, and write $Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO_{2p+1}(\mathbb{F}_q)$. Again, we may remove a single "forced" 1 eigenvalue from s to view it as a $2N$ by $2N$ element of the maximal torus $\tilde{s} \in T$. Then consider the direct sum with the $2(m - N)$ by $2(m - N)$ identity matrix

$$\psi(s) = \tilde{s} \oplus I_{2(m-N)},$$

configured to give a $2m$ by $2m$ matrix that can be considered as an element of $SO(W, B) \subseteq O(W, B)$. As in Case 2 of the construction of the eta correspondence, each distinct $SO_{2N+1}(\mathbb{F}_q)$ -conjugacy class (s) gives a distinct $O_{2m}^\alpha(\mathbb{F}_q)$ -conjugacy class $\psi(s)$. We have

$$(29) \quad Z_{SO_{2m}^\pm(\mathbb{F}_q)}(\psi(s)) = H^D \times SO_{2(N-m+p)}^\sigma(\mathbb{F}_q),$$

for a single determined choice of sign σ (so that its product with the other signs appearing in H agrees with the fixed sign \pm of the orthogonal group $O(W, B)$).

For the unipotent part of the $O_{2m}^\alpha(\mathbb{F}_q)$ -Lusztig classification data we want to produce, write the unipotent representation u in ρ 's Lusztig classification data as $u = u_{H^D} \otimes \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ for $u_{H^D} \in \widehat{(H^D)}_u$ and a symbol $\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ specifying an irreducible unipotent representation of $SO_{2p+1}(\mathbb{F}_q)$. Switch rows so that the defect $a - b$ is 1 mod 4 (which is possible since this symbol has odd defect). Let us write

$$m'_\rho = m - N + \frac{a + b - 1}{2}.$$

Then, if $\sigma = +$, if $\mu_b < m'_\rho$, putting

$$\psi^+(u) = \widetilde{u_{H^D}} \otimes \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b < m'_\rho}$$

gives a unipotent representation of the group $H \times SO_{2(N-m+p)}^+(\mathbb{F}_q) = (Z_{SO_{2m}^\alpha(\mathbb{F}_q)}(\psi(s)))^D$. In this case, we put a sign $\alpha(\sigma)$ to be +1 unless $a = b + 1$, $m'_\rho \leq \lambda_a$ and for the maximal i such that $\lambda_{i+1} \neq \mu_i$, we have $\mu_i > \lambda_{i+1}$, in which case we put $\alpha(\sigma) = -1$. If $\sigma = +$, $\mu_b \geq m'_\rho$, then say $\psi(u)$ is inconstructible.

Similarly, if $\sigma = -$, if $\lambda_a < m'_\rho$, putting

$$\psi^-(u) = \widetilde{u_{H^D}} \otimes \binom{\lambda_1 < \dots < \lambda_a < m'_\rho}{\mu_1 < \dots < \mu_b}$$

gives a unipotent representation of the group $H \times SO_{2(N-m+p)}^-(\mathbb{F}_q) = (Z_{SO_{2m}^\alpha(\mathbb{F}_q)}(\psi(s)))^D$. In this case, we put a sign $\alpha(\sigma)$ to always be +1. If $\sigma = -$, $\lambda_a \geq m'_\rho$, then say $\psi(u)$ is inconstructible.

Now, as in Case 2 in the symplectic case, (s) and $(\psi(s))$ have the same multiplicity of -1 eigenvalues. When it does, we take the sign of the new $O_{2m}^\pm(\mathbb{F}_q)$ -extension data corresponding to the -1 eigenvalues of $\psi(s)$ to be the same as the input representation ρ 's corresponding $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ -central sign data.

Definition 3.3.2. *Suppose we are given the above notation. If s has no -1 eigenvalues, define $\zeta_V^{W,B}(\rho)$ to be the irreducible $O(W, B)$ -representation*

with $O(W, B)$ -Lusztig classification data $[(\psi(s)), \psi(u)]$ with the extension data (α^σ) corresponding to the 1 eigenvalues of $\psi(s)$

$$(30) \quad \zeta_V^{W,B}(\rho) := r^{O(W,B)}[(\psi(s)), \psi^\sigma(u)]^{(\alpha^\sigma)},$$

for σ denoting the sign determined by (29). If s has -1 eigenvalues, writing ± 1 for the central $\mathrm{Sp}(V)$ -data of the input representation ρ

$$(31) \quad \zeta_V^{W,B}(\rho) := r^{O(W,B)}[(\psi(s)), \psi^\sigma(u)]^{(\alpha^\sigma, \pm 1)},$$

again for σ denoting the sign determined by (29). If $\psi(u)$ is inconstructible, we put $\zeta_V^{W,B}(\rho) = 0$.

Recalling the results of [31], we found that in the symplectic (resp. orthogonal) stable range, every irreducible representation ρ of $O(W, B)$ (resp. $\mathrm{Sp}(V)$) appears in the restriction

$$\mathrm{Res}_{O(W,B)}(\mathrm{Res}_{\mathrm{Sp}(V) \times O(W,B)}(\omega[V \otimes W]))$$

(resp. $\mathrm{Res}_{\mathrm{Sp}(V)}(\mathrm{Res}_{\mathrm{Sp}(V) \times O(W,B)}(\omega[V \otimes W]))$). In the present paper's notation, for $(V, (W, B))$ in the symplectic stable range, for every $\rho \in \widehat{O(W, B)}$,

$$\eta_{W,B}^V(\rho) \neq 0.$$

Similarly, for $(V, (W, B))$ in the orthogonal stable range, for every $\rho \in \widehat{\mathrm{Sp}(V)}$,

$$\zeta_V^{W,B}(\rho) \neq 0.$$

Even in the metastable range, however, the range conditions ensure that we never need to add a negative coordinate to a Lusztig symbol: For $(V, (W, B))$ in the symplectic (stable or) metastable range, for every $\rho \in \widehat{O(W, B)}$, we have

$$N'_\rho > 0.$$

For $(V, (W, B))$ in the orthogonal (stable or) metastable range, for every $\rho \in \widehat{\mathrm{Sp}(V)}$, we have

$$m'_\rho > 0.$$

3.4. Alternating sums of parabolic inductions. Now we define the alternating sums of parabolic inductions (only needed in the metastable ranges) which form the coefficients of the extended eta or zeta correspondences in the restricted oscillator representation. The description given in this subsection is somewhat technical, but in each case, the principle is to preserve all of the Lusztig classification data, except for the symbol factor of the unipotent part that is altered in the construction of the extended eta or zeta correspondence. We take the sum of the representations obtained by replacing that symbol by those appearing in the alternating sum of parabolic inductions of the symbols

obtained by concatenating a final coordinate to one of the rows as in the construction of $\eta_{W,B}^V$ or $\zeta_V^{W,B}$, and removing another coordinate in the same row to recover the original symbol's row lengths (see (34) and (36) below).

Suppose we want to consider the decomposition of the restricted oscillator representation $Res_{Sp(V) \times O(W,B)}(\omega[V \otimes W])$. For this subsection, we now fix our choice of $(V, (W, B))$, specifically fixing a case of the parity of $\dim(W)$ and whether we consider the decomposition in terms of an extended eta correspondence (i.e. condition (18) is satisfied) or an extended zeta correspondence (i.e. condition (19) is satisfied).

Now consider a symbol

$$(32) \quad \theta = \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

of a group $K[-k]$, where K denotes a possible factor of a centralizer of a semisimple element s in the dual of the domain of whichever correspondence we have chosen to consider ($O(W, B)$ in the extended eta correspondence or $Sp(V)$ for the extended zeta correspondence), which would be altered in the fixed correspondence's construction (corresponding to -1 eigenvalues if $\dim(W)$ is fixed to be odd, and 1 eigenvalues if $\dim(W)$ is fixed to be even). We use the notation $[-k]$ to refer to the group of the same type and subtracting k from the rank (e.g. for $K = SO(W, B)$, we write $K[-k] = SO(W[-k], B[-k])$). Let us switch rows in (32) so that in the construction of the extended eta or zeta correspondence, if θ appears as the factor of a unipotent part u of the Lusztig classification data for an input representation, ϕ^+ or ψ^+ concatenates a new coordinate to the top row $\lambda_1 < \cdots < \lambda_a$ of (32) (this corresponds to a condition on the row lengths a, b , which varies depending on the case of the extended eta or zeta correspondence we consider). We do this in order to consolidate the notation and treat every case at once.

Fix k and fix some $N' \in \mathbb{N}_0$ such that

$$(33) \quad \lambda_1 < \lambda_2 < \cdots < \lambda_a < N'.$$

Consider the $1 \leq c \leq a$ such that

$$N' - \lambda_c \leq k < N' - \lambda_{c+1}.$$

Then, for $c \leq q \leq a + 1$, let us consider the symbol $\theta^+(N')_q$ with top row given by (33) with the q th coordinate removed, and unmodified

bottom row. For $c \leq q \leq a$ this gives

$$(34) \quad \theta^+(N')_q := \begin{pmatrix} \lambda_1 < \cdots < \widehat{\lambda}_q < \cdots < \lambda_a < N' \\ \mu_1 < \cdots < \mu_b \end{pmatrix}.$$

In the case of $q = a + 1$, we the $(a + 1)$ th coordinate of (33) is N' and thus we have $\theta(N')_q = \theta$. Each of these symbol has the same row lengths as θ , and $\theta(N')_q$ therefore describes a unipotent representation of $K[-(k - (N' - \lambda_q))]$. We may hence consider the alternating sum

$$(35) \quad A_k^+(\theta, N') := \bigoplus_{q=c}^{a+1} (-1)^{a+1-q} \cdot \text{Ind}^{P_{k-(N'-\lambda_q)}}(\theta^+(N')_q)$$

where here P_j denote the maximal parabolics in the full group K with Levi factors $K[-j] \times GL_j(\mathbb{F}_q)$, and we take trivial $GL_j(\mathbb{F}_q)$ -action in each induction term.

Similarly, for $N' \in \mathbb{N}_0$ such that $\mu_1 < \cdots < \mu_b < N'$, considering $1 \leq c \leq b$ such that

$$N' - \mu_c \leq k < N' - \mu_{c+1},$$

for $c \leq q \leq b$, we consider the symbol

$$(36) \quad \theta^-(N')_q := \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \widehat{\mu}_q < \cdots < \mu_b < N' \end{pmatrix}.$$

We also put $\theta^-(N')_{b+1} = \theta$. We can then define the alternating sum

$$(37) \quad A_k^-(\theta, N') := \bigoplus_{q=c}^{b+1} (-1)^{b+1-q} \cdot \text{Ind}^{P_{k-(N'-\mu_q)}}(\theta^-(N')_q).$$

We find that (35) and (37), in every case we consider, define genuine representations. In fact, we give a concrete description of which symbols appear in their decompositions in the Appendix.

Again, we approach the case of the extended eta correspondence first.

Definition 3.4.1. *Consider a choice of V and (W, B) in the symplectic (stable or) metastable range. Suppose we are given $0 \leq k \leq h_W$ and $m' > 0$. Consider an irreducible representation $\rho \in O(\widehat{W[-k]}, B[-k])$.*

- (1) *Suppose $\dim(W) = 2m + 1$ is odd. Write the extended Lusztig classification data of an irreducible $O(\widehat{W[-k]}, B[-k])$ -representation ρ as $\rho = r^{O(\widehat{W[-k]}, B[-k])}[(s), u]^\alpha$, and, writing 2ℓ for the multiplicity of -1 as an eigenvalue of s , write $Z_{Sp_{2(m-k)}(\mathbb{F}_q)}(s) = H \times Sp_{2\ell}(\mathbb{F}_q)$, and $u = u_{HD} \otimes \theta$. As in the construction of the*

eta correspondence, we can interpret $s \oplus (-I_{2k})$ as a semisimple element of $Sp_{2m}(\mathbb{F}_q)$ with centralizer $H \times Sp_{2(\ell+k)}(\mathbb{F}_q)$. Then define $\mathcal{A}_k(\rho, N')$ to be the $O(W, B)$ -representation

$$\mathcal{A}_k(\rho, N') = \bigoplus_{\chi \in A_k^\pm(\theta, N')} r^{O(W, B)}[(s \oplus (-I_{2k})), u_{H^D} \otimes \chi]^\alpha,$$

with the sum running over every distinct irreducible unipotent θ appearing in $A_k^\pm(\theta, N')$ where the superscript sign is chosen to agree with the sign of $\phi^\pm(u)$ we take when construction $\eta_{W, B}^V(\rho)$.

- (2) Suppose $\dim(W) = 2m$ is even. Write the extended Lusztig classification data of an irreducible $O(W[-k], B[-k])$ -representation ρ as $\rho = r^{SO(W[-k], B[-k])}[(s), u]^\gamma$ (writing γ for the extension sign data depending on the eigenvalues of s). Say s has 1 as an eigenvalue of multiplicity $2p$ and write $Z_{Sp_{2(m-k)}(\mathbb{F}_q)}(s) = H \times Sp_{2p}(\mathbb{F}_q)$, and $u = u_{H^D} \otimes \theta$. As in the construction of the eta correspondence, we can interpret $s \oplus I_{2k}$ as a semisimple element of $Sp_{2m}(\mathbb{F}_q)$ with centralizer $H \times Sp_{2(p+k)}(\mathbb{F}_q)$. Then define $\mathcal{A}_k(\rho, N')$ to be the $O(W, B)$ -representation such that

$$\mathcal{A}_k(\rho, N') = \bigoplus_{\chi \in A_k^\pm(\theta, N')} r^{O(W, B)}[(s \oplus I_{2k}), u_{H^D} \otimes \chi]^\gamma$$

with the sum running over every distinct irreducible unipotent θ appearing in $A_k^\pm(\theta, N')$ where the superscript sign is chosen to agree with the sign of $\phi^\pm(u)$ we take when construction $\eta_{W, B}^V(\rho)$.

Similarly, consider the orthogonal metastable range. We note that we still need to separate the cases of the parity of the dimension of the orthogonal space W , though the role of W is somewhat hidden in the notation.

Definition 3.4.2. Consider a choice of V and (W, B) in the orthogonal (stable or) metastable range, and write $\dim(V) = 2N$. Suppose we are given $0 \leq k \leq N$ and $m' > 0$. Consider an irreducible representation $\rho \in \widehat{Sp(V[-k])}$.

- Suppose $\dim(W) = 2m + 1$ is odd. Write the extended Lusztig classification data of an irreducible $Sp(V[-k])$ -representation ρ as $\rho = r^{Sp(V[-k])}[(s), u, \pm 1]$, (omitting the central sign data ± 1 if s has no -1 eigenvalues). Writing 2ℓ for the multiplicity of -1 as an eigenvalue of s , say $Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO_{2\ell}^\pm(\mathbb{F}_q)$,

and $u = u_{HD} \otimes \theta$. Then define $\mathcal{A}_k(\rho, m')$ to be the $Sp(V)$ -representation

$$\mathcal{A}_k(\rho, m') = \bigoplus_{\chi \subset A_k^\pm(\theta, m')} r^{Sp(V)}[(s \oplus (-I)_{2k}), u_{HD} \otimes \chi, \pm 1]$$

with the sum running over every distinct irreducible unipotent θ appearing in $A_k^\pm(\theta, m')$, where the superscript sign is chosen to agree with the sign of $\psi^\pm(u)$ we take when construction $\zeta_{W,B}^V(\rho)$, and where the central sign in the Lusztig classification data of each summand is the same as ρ 's.

- Suppose $\dim(W) = 2m$ is even. Write the extended Lusztig classification data of an irreducible $Sp(V[-k])$ -representation ρ as $\rho = r^{Sp(V[-k])}[(s), u, \pm 1]$, (omitting the central sign data ± 1 if s has no -1 eigenvalues). Writing $2p$ for the multiplicity of 1 as an eigenvalue of s , write $Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO_{2p+1}(\mathbb{F}_q)$, and $u = u_{HD} \otimes \theta$. Then define $\mathcal{A}_k(\rho, m')$ to be the $Sp(V)$ -representation

$$\mathcal{A}_k(\rho, m') = \bigoplus_{\chi \subset A_k^\pm(\theta, m')} r^{Sp(V)}[(s \oplus I_{2k}), u_{HD} \otimes \chi, \pm 1]$$

with the sum running over every distinct irreducible unipotent θ appearing in $A_k^\pm(\theta, m')$, where the superscript sign is chosen to agree with the sign of $\psi^\pm(u)$ we take when construction $\zeta_{W,B}^V(\rho)$, where the central sign in the Lusztig classification data of each summand is the same as ρ 's.

3.5. The main statement. Now that we have established the necessary notation to describe the terms of the decomposition of a restricted oscillator representation claimed in Theorem 1, we may restate it concretely.

The first part of our main result Theorem 1, extending the eta correspondence, can be explicitly restated as:

Theorem 3.5.1. *Suppose $(V, (W, B))$ is in the symplectic metastable range. Then*

$$(38) \quad \text{Res}_{Sp(V) \times O(W, B)}(\omega[V \otimes W]) = \bigoplus_{k=0}^{h_B} \bigoplus_{\rho \in \widehat{O(W[-k], B[-k])}} \mathcal{A}_k(\rho, N'_\rho) \otimes \eta_{W, B}^V(\rho)$$

where $\mathcal{A}(\rho, N'_\rho)$ is considered as a representation of $O(W, B)$.

Similarly, the second part extending the zeta correspondence can be restated as

Theorem 3.5.2. *Suppose $(V, (W, B))$ is in the orthogonal metastable range. Then*

$$(39) \quad \text{Res}_{Sp(V) \times O(W, B)}(\omega[V \otimes W]) = \bigoplus_{k=0}^N \bigoplus_{\rho \in \widehat{Sp(V[-k])}} \zeta_V^{W, B}(\rho) \otimes \mathcal{A}_k(\rho, m'_\rho)$$

where $\mathcal{A}(\rho, m'_\rho)$ is considered as a representation of $Sp(V)$.

4. INTERPOLATION AND PROOF OF THE METASTABLE CORRESPONDENCES

In this section, we describe how the results of [30, 31] for stable range reductive dual pairs can be *interpolated*, which we can use to conclude Theorem 1. We focus on the case of the eta correspondence (interpolating the symplectic stable range result). The case of the zeta correspondence is similar.

In Subsection 4.1, we define interpolated representation categories $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q))$ and $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$, modelling symplectic groups of non-integer rank t . $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q))$ is defined to be tensor generated by a basic object corresponding to the standard permutation representation, and $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ is taken to be generated by basic objects corresponding to the oscillator representations, respectively. While $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q))$ is a more classical construction that can be studied, for example, using oligomorphic groups (see [17]), they do not actually contain objects corresponding to oscillator representations. For our purposes, the “finer” category $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ is necessary. These categories form semisimple abelian pre-Tannakian tensor categories for generic complex values of t . However, for certain values, including $t = N \in \mathbb{N}_0$, they are not themselves semisimple, though they are “semisimplifiable,” and quotienting out a certain class of “negligible” morphisms does give semisimple pre-Tannakian categories.

In Subsection 4.2, we define subcategories of $\mathcal{C}_B^{int}(t) \subseteq \overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ isolating the ranges of objects appearing in a certain tensor power of oscillator representations (a “partial pseudo-abelian envelope”) corresponding to a specific choice of orthogonal space (W, B) . For a fixed irreducible representation $\rho \in \widehat{O(W, B)}$, we further consider a subcategory $\mathcal{C}_\rho^{int}(t)$ detecting objects that interpolate representations of $Sp_{2N}(\mathbb{F}_q)$ symplectic stable with $O(W, B)$ whose tensor product with ρ appears in the restriction of $\omega[\mathbb{F}_q^{2N} \otimes W]$. We denote their images under

semisimplification by $\tilde{\mathcal{C}}_B^{int}(t)$, $\tilde{\mathcal{C}}_\rho^{int}(t)$. In Subsection 4.3, we enumerate the objects of this category in terms of “formal Lusztig symbols” and write down their dimensions. An interpolated decomposition of the restricted oscillator representation holds in these categories (Theorem 4.3.1 below).

In Subsection 4.4, we find that, at $t = N$ with $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ in the symplectic metastable range with $O(W, B)$, the relationship between an interpolated category $\tilde{\mathcal{C}}_\rho^{int}(N)$ and the subcategory of genuine $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ -representations π such that $\rho \otimes \pi$ appears in the restricted oscillator representation, gives that the decomposition of the restricted oscillator representation as a genuine representation can be derived from the interpolated statement after “cancelling terms.” Then we check that simplifying the cancelled terms gives the claimed decomposition as a genuine representation of the symplectic group.

In Subsection 4.5, we describe an application of the statement to give an inductive procedure to explicitly construct representations corresponding to Lusztig classification data.

4.1. Interpolated representation categories. First, to begin discussing interpolated representation theory, we briefly recall P. Deligne’s construction of the category of representations of a general linear group GL_c for $c \notin \mathbb{Z}$ (see [6, 7, 8]):

This interpolation is based on the fact that in $Rep(GL_N)$, denoting the permutation representation “basic object” of dimension N by X , we have a stable structure of the endomorphism algebras

$$(40) \quad \mathrm{End}_{GL_N}(X^{\otimes n}) \cong \mathbb{C}\Sigma_n$$

when $N \gg n$. However, in the genuine category $Rep(GL_N)$ of representations of $GL_N(\mathbb{C})$, when the tensor power degree becomes large compared to N , certain Schur functors predicted to occur in $X^{\otimes n}$ have dimension 0, causing (40) to fail. However, consider a formal basic generating object X of dimension $c \in \mathbb{C} \setminus \mathbb{Z}$, we may construct a diagrammatic category where the endomorphism algebra of $X^{\otimes n}$ is always the group algebra $\mathbb{C}\Sigma_n$ (since the polynomial dimensions of the “Schur functors” is never 0 when applied to $\dim(X) = c$). Formally adding direct sums and taking a pseudo-abelian envelope gives a category we denote here by $\mathfrak{Rep}(GL_c)$, which is semisimple for $c \in \mathbb{C} \setminus \mathbb{Z}$.

Attempting to apply this construction at $c = N$ gives a non-semisimple category $\mathfrak{Rep}(GL_N)$, which for example, has simple objects of dimension 0. However by applying a procedure of *semisimplification* (see [10]), which is designed to eliminate these simple objects, outputs a

semisimple category which is precisely the genuine category of representation $Rep(GL_N)$.

One may also attempt to define an interolated category $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q))$ with the standard permutation representation $\mathbb{C}V_t$ of dimension q^{2t} as a basic tensor-generating object, defined so that

$$(41) \quad \begin{aligned} Hom_{\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q))}((\mathbb{C}V_t)^{\otimes m}, (\mathbb{C}V_t)^{\otimes n}) = \\ Hom_{Rep(Sp_{2N}(\mathbb{F}_q))}((\mathbb{C}V_N)^{\otimes m}, (\mathbb{C}V_N)^{\otimes n}), \end{aligned}$$

for a large enough $N \gg m, n$ writing V_N for the q^{2N} -dimensional underlying symplectic space $Sp(V_N) = Sp_{2N}(\mathbb{F}_q)$. The semisimplification of this category at $t = N$ is known to give a semisimple pre-Tannakian category. However, it is not equivalent to the genuine representation category $Rep(Sp_{2N}(\mathbb{F}_q))$. For example, there is no object of dimension q^t corresponding to the oscillator representation.

Another commonly considered interpolated category $\mathfrak{Rep}(GL_t(\mathbb{F}_q))$ is defined similarly to be tensor-generated by the standard representation of dimension q^t written as $\mathbb{C}\mathbb{F}_q^t$, with

$$(42) \quad \begin{aligned} Hom_{\mathfrak{Rep}(GL_t(\mathbb{F}_q))}((\mathbb{C}\mathbb{F}_q^t)^{\otimes m}, (\mathbb{C}\mathbb{F}_q^t)^{\otimes n}) = \\ Hom_{Rep(GL_N(\mathbb{F}_q))}((\mathbb{C}\mathbb{F}_q^N)^{\otimes m}, (\mathbb{C}\mathbb{F}_q^N)^{\otimes n}), \end{aligned}$$

for a large enough $N \gg m, n$. Similarly, we may define an interpolated category $\mathfrak{Rep}(O_t(\mathbb{F}_q))$ with basic object corresponding to the signed $\mathbb{C}W \otimes \epsilon(det)$, writing $\epsilon(det)$ for the sign representation on the center $\mathbb{Z}/2$ corresponding to det . Writing the basic object $\mathbb{C}W_t \otimes \epsilon(det)$ of $\mathfrak{Rep}(O_t(\mathbb{F}_q))$, we set Hom -spaces between its tensor powers to be equal to what they would be in an orthogonal group of high enough rank (which may be chosen to be even or odd), as in (41), (42). Unlike in the symplectic case, modelling the (twisted) permutation representation as the basic generating object in $\overline{\mathfrak{Rep}}(O_t(\mathbb{F}_q))$ will be enough for our purposes, recalling that, in the case of 2-dimensional V with the standard symplectic form, we have

$$Res_{O(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]) = \mathbb{C}W^-$$

denoting the permutation representation $\mathbb{C}W$ tensored with $\epsilon(det)$ i.e. the sign representation of $O(W, B)/SO(W, B)$, (which can be seen from restricting first to $GL(W)$).

Now, both $\mathfrak{Rep}(GL_t(\mathbb{F}_q))$ and $\overline{\mathfrak{Rep}}(O_t(\mathbb{F}_q))$ are known to, after semisimplification when needed at natural number values of t , give semisimple categories. This can both be proved directly (see, for examples of this method, [6, 7, 27]), or be approached using the theory of oligomorphic groups of A. Snowden and N. Harman (see [17]).

However, we need to define a different interpolated tensor category $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ with the oscillator representations as basic objects (since the structure of endomorphism algebras and homomorphism modules between tensor products of oscillator representation is stable when the dimension of the underlying symplectic space is large enough compared to the tensor product degrees). For more details, see [28]. When attempting to put $t = N \in \mathbb{N}$, as in the case of the interpolated representation categories of the symmetric group, $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2N}(\mathbb{F}_q))$ is not a semisimple category (since, again, it has simple objects of dimension 0).

Consider a category $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))_0$ with objects $\omega_a^{\otimes m} \otimes \omega_b^{\otimes n}$, with Hom -spaces between objects defined to be, as \mathbb{C} -vector spaces

$$\begin{aligned} Hom_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))_0}(\omega_a^{\otimes m} \otimes \omega_b^{\otimes n}, \omega_a^{\otimes p} \otimes \omega_b^{\otimes l}) &:= \\ Hom_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(\omega_a^{\otimes m} \otimes \omega_b^{\otimes n}, \omega_a^{\otimes p} \otimes \omega_b^{\otimes l}) & \end{aligned}$$

for $N \gg m, n, p, l$. Tensor products of morphisms and the actions of the bijections on the coordinates of the different tensor factors are defined in the obvious way from $Rep(\mathrm{Sp}_{2N}(\mathbb{F}_q))$ for a large enough N . To construct a category (and involve the constant t), we also need to define a partial trace operation corresponding to matching factors of the generating objects ω_a . In this case, it suffices to define a trace operation for endomorphisms of each ω_a . We do this by considering

$$\begin{aligned} End_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))_0}(\omega_a) &= End_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(\omega_a) \\ &= Hom_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(1, \mathbb{C}V_N) \cong (\mathbb{C}V_N)^{\mathrm{Sp}_{2N}}, \end{aligned}$$

which has a basis consisting of (0) and $\sum_{v \neq 0 \in V_N} (v)$, where we put

$$tr((0)) = q^t, \quad tr\left(\sum_{v \neq 0 \in V_N} (v)\right) = 0.$$

Composition can be defined with a combination of tensor product, permutation, and partial trace (for more details, see [28]). This defines a \mathbb{C} -linear category $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))_0$.

We then define the category $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ by first formally adding direct sums to $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))_0$, and then applying a pseudo-abelian envelope, adding new objects defined as the images of idempotents in the endomorphism algebras of the objects $\omega_a^{\otimes m} \otimes \omega_b^{\otimes n}$.

In each case, the interpolated category is constructed from the data of a system of Hom -spaces between tensor powers of the basic object, with operations of permutation action, tensor product, and partial trace. This data can in fact be captured in the universal algebra structure of a T -algebra. To describe a \mathbb{C} -linear additive category with

strong duality and associative commutative unital tensor product generated by a basic object X , its corresponding T-algebra \mathcal{T} consists of vector spaces for every pair of finite sets S, T

$$\mathcal{T}_{S,T} = \text{Hom}(X^{\otimes S}, X^{\otimes T}),$$

with appropriate functoriality over the category of finite sets, linear partial trace operations corresponding to partial bijections between S and T and tensor product operations corresponding the set disjoint union, with appropriate axioms. Here, we denote the T-algebra corresponding to $\mathfrak{Rep}(GL_t(\mathbb{F}_q))$ with basic object corresponding to the standard representation $X = \mathbb{C}\mathbb{F}_q^t$ by $\mathcal{T}[GL_t(\mathbb{F}_q)]$ and the T-algebra corresponding to $\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))$ with basic object $X = \omega_a \otimes \omega_b$ by $\overline{\mathcal{T}}[\text{Sp}_{2t}(\mathbb{F}_q)]$. For more details, see [28].

Let us denote the *semisimplification* of a \mathbb{C} -linear additive category with strong duality and associative commutative unital tensor product \mathcal{C} by $\mathcal{S}(\mathcal{C})$. Recall that semisimplification refers to a construction quotienting out *negligible morphisms*, such as simple idempotents with trace 0, [10].

We recall a result of [28].

Proposition 4.1.1. *In every case of t , the semisimplification of the category $\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))$ is a semisimple (and, in particular, abelian) pre-Tannakian category. For values of $t \in \mathbb{C}$ such that $q^t \neq \pm q^n$ for $n \in \mathbb{N}_0$, the category $\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))$ itself is semisimple.*

Proof. First, there is an inclusion of T-algebras

$$\overline{\mathcal{T}}[\text{Sp}_{2t}(\mathbb{F}_q)] \hookrightarrow \mathcal{T}[GL_t(\mathbb{F}_q)],$$

since for every N , the restriction of an oscillator representation ω_a to $GL_N(\mathbb{F}_q) \subseteq \text{Sp}_{2N}(\mathbb{F}_q)$ is isomorphic to

$$\text{Res}_{GL_N(\mathbb{F}_q)}(\omega_a) \cong (\mathbb{C}V_N) \otimes \epsilon(\det).$$

In particular, then for finite sets S, T , restriction gives an inclusion from each Hom -space

$$\begin{aligned} \text{Hom}_{\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))}(\omega_a^{\otimes S_1} \otimes \omega_b^{\otimes S_2}, \omega_a^{\otimes T_1} \otimes \omega_b^{\otimes T_2}) = \\ \text{Hom}_{\text{Sp}_{2N}(\mathbb{F}_q)}(\omega_a^{\otimes S_1} \otimes \omega_b^{\otimes S_2}, \omega_a^{\otimes T_1} \otimes \omega_b^{\otimes T_2}) \end{aligned}$$

making up $\overline{\mathcal{T}}[\text{Sp}_{2t}(\mathbb{F}_q)]_{S,T}$ for $S_1 \amalg S_2 = S, T_1 \amalg T_2 = T$, into the $GL_N(\mathbb{F}_q)$ -equivariant Hom -space on the restrictions

$$\text{Hom}_{GL_N(\mathbb{F}_q)}((\mathbb{C}V_N \otimes \epsilon(\det))^{S_1 \amalg S_2}, (\mathbb{C}V_N \otimes \epsilon(\det))^{T_1 \amalg T_2}),$$

which is isomorphic to $\mathcal{T}[GL_t(\mathbb{F}_q)]_{S,T}$. Partial trace, tensor product, and functoriality (and therefore composition) are all compatible.

We then apply

Lemma 4.1.2. *The semisimplification of a \mathbb{C} -linear additive category with strong duality and an associative, commutative, unital tensor product generated by a basic object X is semisimple if and only if for every endomorphism $f \in \text{End}(X^{\otimes n})$, if the trace of f is non-zero, then for every n , there exists an $m > n$ such that*

$$\text{tr}(f^{\circ m}) \neq 0.$$

Proof of Lemma 4.1.2. To prove sufficiency, consider an endomorphism f of some tensor power $X^{\otimes n}$ is non-negligible, i.e. there exists some morphism $g \in \text{End}(X^{\otimes n})$ such that the trace of $f \circ g$ is non-zero. The trace condition then gives that for every n , there exists a $m > n$ such that

$$\text{tr}((f \circ g)^{\circ m}) \neq 0,$$

and hence, f is not an element of the Jacobian ideal of the endomorphism algebra $\text{End}(X^{\otimes n})$, and in particular, is not nilpotent. Therefore, the semisimplification of the category is semisimple.

Necessity follows from the general result that in a semisimple \mathbb{C} -algebra (e.g. the endomorphism algebra of $X^{\otimes n}$ in the semisimplification), if some general trace operation (i.e. a linear combination of trace on each factor, consider the endomorphism algebra as a product of matrix algebras) is non-zero on an element f , then for every n , there exists an $m > n$ such that the trace operation is non-zero on $f^{\circ m}$. (This follows, for example, by considering the Vanermonde determinant.) \square

In particular, if the semisimplification of a category defined by a T-algebra \mathcal{T} is semisimple, then the semisimplifications of any category defined by a sub-T-algebra $\mathcal{T}' \subseteq \mathcal{T}$ is semisimple as well. Therefore, since the semisimplification of $\mathfrak{Rep}(GL_t(\mathbb{F}_q))$ is semisimple, so is the semisimplification of $\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))$.

The second claim follows, for example, by examining the polynomial order of $\text{Sp}_{2N}(\mathbb{F}_q)$, and replacing N by t , to conclude that every indecomposable object is non-vanishing in the semisimplification. \square

For our purposes here, we will want to consider the case of $t = N$, making the first part of this statement more relevant. The second part of this statement for generic values of t can be used to conclude an interpolated version of our decomposition statement, for example, describing an interpolated eta correspondence

$$\eta_{W,B}^t : \widehat{O(W,B)} \hookrightarrow \overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q)),$$

sending the irreducible representations of $O(W, B)$ to simple objects of $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$.

4.2. Partial pseudo-abelian envelopes and “isotypical” subcategories. In this subsection, we give an argument using interpolation for our full statement of Howe duality. From here on, we will focus on the case of interpolating the eta correspondence with target in odd orthogonal group representations, for simplicity. In this part of the argument, every other case is entirely similar. Fix a choice of orthogonal space and bilinear form (W, B) .

Recall that in this case, for a choice of V lying with (W, B) in the symplectic stable range, the eta correspondence is an injective map $\eta_{W,B}^V$ sending the irreducible representations of $O(W, B)$ to irreducible representations of $\mathrm{Sp}(V)$. The methods of [30, 31] ultimately only rely on information carried by the endomorphism algebras of (tensor powers of) oscillator representations, whose structure is stable under enlarging the dimension of V , in the stable symplectic range, and therefore, by definition, they pass to the interpolated categories.

Write $\dim(W) = n$, and suppose B is, as a symmetric bilinear form, equivalent to a diagonal matrix with entries $a_1, \dots, a_n \in \mathbb{F}_q^\times$. We may then consider the object

$$\omega^{\otimes B} = \omega_{a_1} \otimes \cdots \otimes \omega_{a_n}$$

in $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$. Let us consider the “partial pseudo-abelian envelope” $\mathcal{C}_B^{\mathrm{int}}(t)$ defined as the subcategory of $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ consisting of images of idempotents of $\mathrm{End}_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))}(\omega^{\otimes B})$. We do not consider a tensor product on $\mathcal{C}_B^{\mathrm{int}}(t)$, only working with its structure as an additive \mathbb{C} -linear category.

Further, considering

$$\begin{aligned} \mathrm{End}_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))}(\omega^{\otimes B}) &\cong \mathrm{End}_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(\omega_{a_1}[V_N] \otimes \cdots \otimes \omega_{a_n}[V_N]) = \\ &\mathrm{End}_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(\omega[V_N \otimes W]), \end{aligned}$$

for a large enough rank N , where on the right hand side, we consider the restriction of $\omega[V_N \otimes W]$ along the inclusion

$$\mathrm{Sp}_{2N}(\mathbb{F}_q) = \mathrm{Sp}(V_N) \subseteq \mathrm{Sp}(V_N) \times O(W, B) \subseteq \mathrm{Sp}(V_N \otimes W).$$

Therefore, there is a built in action of $O(W, B)$ on the endomorphism algebras $\mathrm{End}_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))}(\omega^{\otimes B})$.

In particular, for a fixed irreducible representation ρ of $O(W, B)$, we may consider the full ρ -isotypical subcategory $\mathcal{C}_\rho^{\mathrm{int}}(t) \subseteq \mathcal{C}_B^{\mathrm{int}}(t)$, with objects consisting of images of idempotents ι in $\mathrm{End}_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))}(\omega^{\otimes B})$

where, considering the $O(W, B)$ -fixed point algebra

$$\begin{aligned} \text{End}_{O(W, B) \times \overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))}(\omega^{\otimes B}) = \\ (\text{End}_{\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))}(\omega^{\otimes B}))^{O(W, B)} \quad , \end{aligned}$$

there is a simple idempotent of the form $\kappa \otimes \iota$, for κ an idempotent in $\text{Rep}(O(W, B))$ with image isomorphic to ρ . We may also describe these objects as the images of ι lying in the ρ -isotypical part of $\text{End}_{\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))}(\omega^{\otimes B})$ as an $O(W, B)$ -representation. We cannot take a semisimplification of $\mathcal{C}_\rho^{\text{int}}(t)$, since we have given up its tensor structure. However, we may consider the images of $\mathcal{C}_B^{\text{int}}(t)$, $\mathcal{C}_\rho^{\text{int}}(t)$ under the quotient semisimplification functor

$$\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q)) \rightarrow \mathcal{S}(\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))).$$

Writing $\tilde{\mathcal{C}}_B^{\text{int}}(t)$, $\tilde{\mathcal{C}}_\rho^{\text{int}}(t)$ for these images, they form semisimple abelian subcategories of $\mathcal{S}(\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q)))$. Note that this is only non-trivial for a choice of $t = N$ a natural number.

On the other hand, we may also consider the full subcategories $\mathcal{C}_\rho^{\text{gen}}(N)$ of $\text{Rep}(\text{Sp}_{2N}(\mathbb{F}_q))$ consisting of direct sums of all genuine irreducible representations $\pi \in \text{Rep}(\text{Sp}_{2N}(\mathbb{F}_q))$ such that

$$\pi \otimes \rho \subseteq \text{Res}_{O(W, B) \times \text{Sp}(V)}(\omega[V \otimes W]).$$

4.3. Interpolating correspondences. The purpose of this subsection is to describe how the objects of the subcategories $\mathcal{C}_B^{\text{int}}(t)$ of the interpolation $\overline{\mathfrak{Rep}}(\text{Sp}_{2t}(\mathbb{F}_q))$ constructed in tensor powers of oscillator representations of odd degree such that the total product of central characters has a certain quadratic character α (i.e. corresponding to odd orthogonal spaces and symmetric bilinear forms of a certain prescribed discriminant α), can be written down according to an ‘‘formally interpolated Lusztig classification.’’ We discuss this in detail for the case of $\dim(W)$ odd. All other cases are similar.

More specifically, an object of $\mathcal{C}_B^{\text{int}}(t)$ is of the form $\eta_{W, B}^t(\rho)$, for W of dimension $2m + 1$ for $m \in \mathbb{N}$, with a form B of discriminant $\text{disc}(B) = \alpha$, and an irreducible representation ρ of $O(W, B)$. Say that as a representation of $O(W, B)$, ρ is of the form $r^{O(W, B)}[(s), u]^{\pm 1}$ corresponding to Lusztig classification data with semisimple part $(s) \in \text{Sp}_{2m}(\mathbb{F}_q) = (SO_{2m+1}(\mathbb{F}_q))^D$, unipotent part $u \in (Z_{\widehat{\text{Sp}_{2m}(\mathbb{F}_q)}}(s))^D_u$, and central sign data ± 1 . More concretely, further say that s has -1 as an

eigenvalue of multiplicity 2ℓ , writing $Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s) = H \times \mathrm{Sp}_{2\ell}(\mathbb{F}_q)$, and

$$u = u_{H^D} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

for $u_{H^D} \in \widehat{H^D}_u$ and $\begin{pmatrix} \lambda_a < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$ (switching rows so that $a - b$ is 1 mod 4) specifying a unipotent representation of $SO_{2\ell+1}(\mathbb{F}_q) = (\mathrm{Sp}_{2\ell}(\mathbb{F}_q))^D$. Then, for the sign $+$, we say that $\eta_{W,B}^t(r^{O(W,B)}[(s), u]^{+1})$ corresponds to “interpolated Lusztig classification data”

$$(43) \quad [\phi^+(s), \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a & \\ \mu_1 < \cdots < \mu_b & t'_\rho \end{pmatrix}],$$

writing $t'_\rho = t - m + \frac{a+b-1}{2}$. This is exactly the Lusztig classification data of a stable range eta correspondence $\eta_{W,B}^t(\rho)$ for $\dim(V) = 2N$, with N replaced by t (we omit the final $<$ symbol in the second row of the symbol notation, since at an interpolated value of t , writing $\mu_b < t'_\rho$ may be false or incomparable). Again, $\mathrm{Sp}_{2t}(\mathbb{F}_q)$ is not a genuine group, and writing $\phi^+(s)$ indicates an element with finitely many eigenvalues not equal to -1 (which would contribute genuine factors in its “centralizer”) and has -1 as an eigenvalue of “multiplicity $2(t - m + \ell)$.” Interpolating the stable formula one would obtain for $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, replacing N by t , its dimension is

$$(44) \quad \frac{\dim(\eta_{W,B}^t(r^{O(W,B)}[(s), u]^{+1})) = \dim(\rho) \cdot \prod_{i=t'+1}^t (q^{2i} - 1) \cdot \prod_{i=1}^a (q^{t'_\rho} + q^{\lambda_i}) \cdot \prod_{i=1}^b (q^{t'_\rho} - q^{\mu_i})}{2 \cdot q^{(a+b-1)(a+b+1)/4} \cdot |SO_{2m+1}(\mathbb{F}_q)|_{q'}}.$$

Similarly, at the sign -1 , we say that $\eta_{W,B}^t(r^{O(W,B)}[(s), u]^{-1})$ corresponds to “interpolated Lusztig classification data”

$$(45) \quad [\phi^-(s), \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a & t'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}],$$

writing $t'_\rho = t - m + (a + b - 1)/2$. Interpolating the stable formula one would obtain for $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, replacing N by t , its dimension is

$$(46) \quad \frac{\dim(\eta_{W,B}^t(r^{O(W,B)}[(s), u]^{-1})) = \dim(\rho) \cdot \prod_{i=t'+1}^t (q^{2i} - 1) \cdot \prod_{i=1}^a (q^{t'_\rho} - q^{\lambda_i}) \cdot \prod_{i=1}^b (q^{t'_\rho} + q^{\mu_i})}{2 \cdot q^{(a+b-1)(a+b+1)/4} \cdot |SO_{2m+1}(\mathbb{F}_q)|_{q'}}.$$

Now we may consider the “restriction” functor

$$\mathbf{Res} : \mathcal{S}(\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2nt}(\mathbb{F}_q))) \rightarrow \mathcal{S}(\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))) \boxtimes \mathrm{Rep}(O(W, B)).$$

The interpolated Howe duality statement then is

Theorem 4.3.1. *In the semisimplification $\mathcal{S}(\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q)))$, the original decomposition of $\mathbf{Res}(\omega_1)$ as*

$$(47) \quad \bigoplus_{k=0}^{m_W} \bigoplus_{\rho \in O(\widehat{W[-k]}, \widehat{B[-k]})} \eta_{W,B}^t(\rho) \boxtimes \mathrm{Ind}^{P_k}(\rho \otimes \epsilon(\det))$$

holds, as objects of $\mathcal{S}(\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))) \boxtimes \mathrm{Rep}(O(W, B))$.

In summary, the objects of $\mathcal{C}_B^{\mathrm{int}}(t)$ are precisely direct sums of all

$$(48) \quad \eta_{\widehat{W[-k]}, \widehat{B[-k]}}^t(\rho)$$

for irreducible representations $\rho \in O(\widehat{W[-k]}, \widehat{B[-k]})$. For a fixed irreducible representation ρ of $O(W, B)$, the objects of $\mathcal{C}_\rho^{\mathrm{int}}(t)$ consist of direct sums of objects $\eta_{\widehat{W[-k]}, \widehat{B[-k]}}^t(\rho')$ corresponding to irreducible representations $\rho' \in O(\widehat{W[-k]}, \widehat{B[-k]})$ such that ρ is a summand of the parabolic induction

$$\rho \subseteq \mathrm{Ind}^{P_k}(\rho' \otimes \epsilon(\det))$$

writing $P_k \subseteq O(W, B)$ for the maximal parabolic subgroup with Levi factor $O(W[-k], B[-k]) \times \mathrm{GL}_k(\mathbb{F}_q)$, considering $\epsilon(\det)$ as a representation of the factor $\mathrm{GL}_k(\mathbb{F}_q)$.

At $t = N$ corresponding to a reductive dual pair $(\mathrm{Sp}_{2N}(\mathbb{F}_q), O(W, B))$ in the symplectic metastable range, the semisimplification images $\widetilde{\mathcal{C}}_B(N)$, $\widetilde{\mathcal{C}}_\rho(N)$ are semisimple categories with objects consisting of direct sums of simple objects corresponding to all formal interpolated Lusztig classification, eliminating 0-dimensional objects, which occur precisely when $N'_\rho = \lambda_i$ or μ_i for some i in (43) or (45). Note that the remaining formal interpolated Lusztig classification objects, say where $\lambda_i < N'_\rho < \lambda_{i+1}$ in (43) or $\mu_i < N'_\rho < \mu_{i+1}$ in (45), have dimension equal to a genuine irreducible $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ -representation where N'_ρ is inserted in the appropriate place, multiplied by $(-1)^{a-i}$ or $(-1)^{b-i}$. Call this the *true permutation* of the formal interpolated Lusztig data as $t = N$.

4.4. The proof of the metastable eta correspondence. Now, to approach a choice of $(V, (W, B))$ in the symplectic metastable range, we attempt to apply Theorem 4.3.1 to $t = N$, giving a decomposition in the semisimplification $\mathcal{S}(\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q)))$ in terms of objects of $\widetilde{\mathcal{C}}_B(N)$. We must relate this category to genuine $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ -representations. In fact, we claim that replacing the formal Lusztig classification data by its true permutation, with the corresponding sign, gives a genuine decomposition of the restricted oscillator representation in the Grothendieck

group $K(\text{Rep}(\text{Sp}_{2N}(\mathbb{F}_q)))$. Simplifying will precisely give the claimed decomposition in Theorem 3.5.1.

To see this, for each $\rho \in \widehat{O(W, B)}$, consider functors

$$\Phi : \mathcal{C}_\rho^{\text{gen}}(N) \rightarrow \tilde{\mathcal{C}}_\rho^{\text{int}}(N)$$

defined as follows: Consider a simple object π of the source, such that

$$\pi \otimes \rho \subseteq \text{Res}_{\text{Sp}(V) \times O(W, B)}(\omega[V \otimes W]).$$

We may consider an idempotent ι_π in the $\text{Sp}(V)$ -equivariant endomorphism algebra of $\text{Res}_{\text{Sp}(V)}(\omega[V \otimes W]) \cong \omega^{\otimes B}$ whose image is one of these copies of π . By duality, since each oscillator representation has $\omega_a \otimes (\omega_a)^\vee \cong \mathbb{C}V$, we may consider

$$\begin{aligned} \iota_\pi \in \text{End}_{\text{Sp}(V)}(\omega^{\otimes B}) &\cong \text{Hom}_{\text{Sp}(V)}(1, (\mathbb{C}V)^{\otimes B}) = \\ &(\mathbb{C}(V \otimes W))^{\text{Sp}(V)} \end{aligned}$$

as a linear combination of $\text{Sp}(V)$ -orbits on $V \otimes W = V^{\oplus n}$ (recall that an $\text{Sp}(V)$ -orbit consists of a set of n -tuples of vectors (v_1, \dots, v_n) satisfying some linear (in)dependence conditions, and equations for the values of the symplectic form on pairs of them). These orbits can also be considered as orbits of any $\text{Sp}_{2M}(\mathbb{F}_q)$ acting on $(\mathbb{F}_q^{2M})^{\oplus n}$ for any higher M , and therefore ι_π corresponds to an interpolated endomorphism

$$\text{End}_{\overline{\mathfrak{Rep}}(\text{Sp}_{2N}(\mathbb{F}_q))}(\omega^{\otimes B}) \cong ((\mathbb{C}(\mathbb{F}_q^{2M} \otimes W))^{\text{Sp}_{2M}(\mathbb{F}_q)})$$

for $M \gg n$ (by the definition of $\overline{\mathfrak{Rep}}(\text{Sp}_{2N}(\mathbb{F}_q))$). Since partial trace operations (and therefore compositions) are computed the same in $\text{Rep}(\text{Sp}_{2N}(\mathbb{F}_q))$ and $\overline{\mathfrak{Rep}}(\text{Sp}_{2N}(\mathbb{F}_q))$ (the difference between them arising instead from certain morphisms in the interpolated category not existing in the genuine category), this new endomorphism is still an idempotent, with image equal to an object in $\tilde{\mathcal{C}}_\rho^{\text{int}}(N)$ of the same dimension as π . We put $\Phi(\pi)$ to be this object.

On the other hand, define

$$(49) \quad \Psi : K(\tilde{\mathcal{C}}_\rho^{\text{int}}(N)) \rightarrow K(\mathcal{C}_\rho^{\text{gen}}(N))$$

by sending a simple object in $\tilde{\mathcal{C}}_\rho^{\text{int}}(N)$ which must be of the form (43) or (45) at $t = N$ for some choice of s, u , $\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ to $(-1)^{b-i}$, where i is the index so that $\mu_i < t'_\rho < \mu_{i+1}$ or $\lambda_i < t'_\rho < \lambda_{i+1}$, times the genuine irreducible representation of $\text{Sp}(V)$ whose Lusztig classification data is the same as (43) or (45) with the formal interpolated symbol replaced

by

$$\left(\begin{array}{c} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_i < t'_\rho < \mu_{i+1} < \cdots < \mu_b \end{array} \right)$$

or

$$\left(\begin{array}{c} \lambda_1 < \cdots < \lambda_i < t'_\rho < \lambda_{i+1} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{array} \right),$$

respectively. In other words, we them precisely to their signed true permutations.

Proposition 4.4.1. *The composition of*

$$K(\mathcal{C}_\rho^{\text{gen}}(N)) \xrightarrow{K(\Phi)} K(\widetilde{\mathcal{C}}_\rho^{\text{int}}(N)) \xrightarrow{\Psi} K(\mathcal{C}_\rho^{\text{gen}}(N))$$

is $\text{Id}_{K(\mathcal{C}_\rho^{\text{gen}}(N))}$.

Proof. Note that this holds immediately when formal Lusztig classification data is actually genuine Lusztig classification data defining a $\text{Sp}_{2N}(\mathbb{F}_q)$ -representation, since both $K(\Phi)$ and Ψ act as the identity on these objects, considered in either categories. The general statement follows since dimensions are preserved by Ψ and Φ , and it is not possible for $\Psi \circ K(\Phi)$ when applied to an irreducible representation of $\mathcal{C}_\rho^{\text{gen}}(N)$ to output a linear combination of multiple different irreducible representations in $\text{Rep}(\text{Sp}_{2N}(\mathbb{F}_q))$ with integer coefficients, both by dimension and the fact that it would violate the decomposition (47). \square

Therefore, the decomposition of the restricted oscillator representation as a genuine representation of $\text{Sp}_{2N}(\mathbb{F}_q)$ can be obtained from (47) by applying Ψ , and cancelling terms as in $K(\mathcal{C}_B(N))$. It remains to reconcile this cancellation with the claimed answer. Consider a term arising from $\rho \in O(\widetilde{W[-j]}, B[-j])$. Write $\rho = r^{O(W[-j], B[-j])}[(s), u]^{\pm 1}$. Suppose $s \in \text{Sp}_{2(m-j)}(\mathbb{F}_q) = (SO_{2(m-j)+1}(\mathbb{F}_q))^D$ has -1 as an eigenvalue of multiplicity 2ℓ , and write $Z_{\text{Sp}_{2(m-j)}(\mathbb{F}_q)}(s) = H \times \text{Sp}_{2\ell}(\mathbb{F}_q)$. Then $\phi^\pm(s)$ remains as defined in the stable range, so that $Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi^\pm(s)) = H^D \times SO_{2(N-m+\ell)}^\pm(\mathbb{F}_q)$. If $\phi^\pm(u)$ is constructible, then the interpolated eta correspondence outputs a genuine $\text{Sp}_{2N}(\mathbb{F}_q)$ -representation, which is our proposed metastable construction of $\eta_{W,B}^V$. In the case corresponding to when $\phi^\pm(u)$ is “inconstructible,” however, the interpolated eta correspondence outputs a representation involving an illegal symbol

$$(50) \quad \phi(u) = \widetilde{u_{H^D}} \otimes \left(\begin{array}{c} \lambda_1 < \cdots < \lambda_a \quad N'_\rho \\ \mu_1 < \cdots < \mu_b \end{array} \right).$$

This term

$$r^{\mathrm{Sp}(V)}[(\phi^\pm(s)), \phi^\pm(u), \epsilon(s) \cdot \mathrm{disc}(B)] \otimes \mathrm{Ind}^{P_i}(r^{O(W[-j], B[-j])}[(s), u]^{\pm 1})$$

must be eliminated when we semisimplify.

To see where this term cancels, suppose that

$$\lambda_1 < \cdots < \lambda_c < N'_\rho < \lambda_{c+1} < \cdots < \lambda_a,$$

then, for $c+1 \leq k \leq a$, we have that the multiplicity $2p$ of -1 as an eigenvalue of s must be greater than or equal to $2(\lambda_k - N'_\rho)$. first let us write

$$j^{(k)} = j - \lambda_k + N'_\rho.$$

Note that by the rank conditions of Lusztig symbols, the multiplicity $2p$ of -1 as an eigenvalue must be greater than or equal to $2(\lambda_k - N'_\rho)$. Then write $s^{(k)}$ for the semisimple element of

$$\mathrm{Sp}_{2(m-j+\lambda_k-N'_\rho)}(\mathbb{F}_q) = \mathrm{Sp}_{2(m-j^{(k)})}(\mathbb{F}_q) = (SO_{2(m-j^{(k)})+1}(\mathbb{F}_q))^D$$

obtained by removing $2(\lambda_k - N'_\rho)$ eigenvalues -1 . Write

$$u^{(k)} = u_{HD} \otimes$$

$$\left(\begin{array}{c} \lambda_1 < \cdots < \lambda_c < N'_\rho < \lambda_{c+1} < \cdots < \lambda_{k-1} < \lambda_{k+1} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{array} \right)$$

Consider the representation corresponding to the Lusztig data

$$r^{O(W[-j^{(k)}], B[-j^{(k)}])}[(s^{(k)}), u^{(k)}]^{\pm 1}$$

giving an irreducible representation of $O_{2(m-j^{(k)})+1}(\mathbb{F}_q)$. Then $\psi^\pm(s) = \psi^\pm(s^{(k)})$, and we have

$$N'_{r^{O(W[-j^{(k)}], B[-j^{(k)}])}[(s^{(k)}), u^{(k)}]^{\pm 1}} = \lambda_k.$$

Therefore,

$$\phi^\pm(u^{(k)}) = \widetilde{u_{HD}} \otimes$$

$$\left(\begin{array}{c} \lambda_1 < \cdots < \lambda_c < N'_\rho < \lambda_{c+1} < \cdots < \lambda_{k-1} < \lambda_{k+1} < \cdots < \lambda_a & \lambda_k \\ \mu_1 < \cdots < \mu_b \end{array} \right).$$

Note that the interpolated symbol part has dimension $(-1)^{a-k}$, multiplied by the dimension of the genuine symbol

$$(51) \quad \left(\begin{array}{c} \lambda_1 < \cdots < \lambda_c < N'_\rho < \lambda_{c+1} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{array} \right).$$

Therefore, at $j^{(k)}$, this contributes a term

$$(52) \quad \begin{aligned} & r^{\mathrm{Sp}(V)}[(\phi^\pm(s)), \psi^\pm(u^{(k)}), \epsilon(s) \mathit{disc}(B)] \otimes \\ & \mathrm{Ind}^{P_{j^{(k)}}} (r^{O_{2(m-j^{(k)})+1}(\mathbb{F}^a)}[(s^{(k)}), u^{(k)}]^\pm), \end{aligned}$$

which has terms that can cancel recursively (e.g. $k = 1$ completely cancels (50), though it introduces new summands that are illegal if $a > 1$), since switching coordinates in interpolated Lusztig symbols gives a change in the sign of dimension. At $k = a$, we have the legal symbol (51).

Putting together the terms (50) and (52) for $c+1 \leq k \leq a$, this gives the final genuine term, which is $\eta_{W,B}^V(r^{O(W[-j^{(a)}], B[-j^{(a)}])}[(s^{(a)}), u^{(a)}]^\pm)$, tensored with a coefficient of the $O(W, B)$ -representation given by the alternating sum

$$\bigoplus_{k=c}^a (-1)^{a-k} \mathrm{Ind}^{j^{(k)}} (r^{O(W[-j^{(k)}], B[-j^{(k)}])}[(s^{(k)}), u^{(k)}]^\pm)$$

(writing $j^{(0)} = j$, $s^{(0)} = s$, $u^{(0)} = u$). By definition, this $O(W, B)$ -representation is precisely the proposed alternating sum

$$\mathcal{A}_{j^{(a)}} (r^{O(W[-j^{(a)}], B[-j^{(a)}])}[(s^{(a)}), u^{(a)}]^\pm, N'_{r^{O(W[-j^{(a)}], B[-j^{(a)}])}[(s^{(a)}), u^{(a)}]^\pm}).$$

Therefore, in the final decomposition of $\mathit{Res}_{\mathrm{Sp}(V) \times O(W, B)}(\omega[V \otimes W])$ in the genuine representation categories $\mathit{Rep}(\mathrm{Sp}(V))$, $\mathit{Res}(O(W, B))$, we obtain that the illegal constructions should be taken to be 0, in exchange for replacing $\mathrm{Ind}^{P_{j^{(a)}}}$ in the corresponding final legal level by $\mathcal{A}_{j^{(a)}}$, exactly as claimed.

We also note that Theorems 3.5.1, 3.5.2 can be checked elementarily, using the global dimension formula calculated in [30, 31] and observations about how the dimensions of endomorphism algebras of tensor powers of oscillator representations degenerate in the degrees corresponding to metastable reductive dual pairs.

4.5. An “inductive” construction. We now describe how the eta and zeta correspondence can be used to construct families of irreducible representations of the finite symplectic and orthogonal groups recursively. This process re-expresses any irreducible representation to a chain of eta and (possibly signed) zeta correspondences, applied to a certain “terminal” representations which can not be expressed as the eta or zeta correspondence applied to a representation of a group of lower rank. A terminal representations is precisely one such that the semisimple component (s) of its Lusztig classification data has a centralizer only involving factors of type A or 2A .

For a fixed choice of symplectic space V , let us consider the eta correspondences

$$\eta_V^{W,B} : \widehat{O(W,B)} \rightarrow \widehat{\mathrm{Sp}(V)} \cup \{0\},$$

for every choice of orthogonal space (W, B) such that $(\mathrm{Sp}(V), O(W, B))$ forms a reductive dual pair in the symplectic metastable range. Every irreducible representation $\rho \in \mathrm{Sp}_{2N}(\mathbb{F}_q)$ is contained as a summand of a tensor product of oscillator representations

$$(53) \quad \omega_{a_1}[V] \otimes \omega_{a_2}[V] \otimes \cdots \otimes \omega_{a_n}[V],$$

of degree $n \leq 2N$, which we can further interpret as the restriction of an oscillator representation $\omega[V \otimes W]$ for an n -dimensional orthogonal space W with respect to a form B given by the diagonal matrix with entries a_1, \dots, a_n , along the inclusion $\mathrm{Sp}(V) \subseteq \mathrm{Sp}(V) \times O(W, B) \subseteq \mathrm{Sp}(V \otimes W)$. In particular, we find that the union of the non-zero images of the eta correspondences makes up the whole set of irreducible representation

$$\widehat{\mathrm{Sp}(V)} = \coprod_{\substack{(W,B), \\ \dim(W) \leq 2N}} \mathrm{Im}(\eta_V^{W,B}) \setminus \{0\}.$$

Therefore, knowing $\pi \in \widehat{O(W, B)}$ satisfying $\eta_{W,B}^V(\pi) = \rho$, we may consider the idempotent

$$(54) \quad \frac{\dim(\pi)}{|O(W, B)|} \sum_{g \in O(W, B)} \chi_\pi(g)^{-1} \cdot g \in \mathbb{C}O(W, B)$$

in

$$\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W]) = \mathrm{End}_{\mathrm{Sp}(V)}\left(\bigotimes_{i=1}^n \omega[V]_{a_i}\right),$$

where $\chi_\pi : O(W, B) \rightarrow \mathbb{C}^\times$ denotes the character corresponding to π . The image of (54) in (53) recovers ρ . Outside of the stable range, the constructions of the reflection elements are readily interpolated. In the metastable range, if $\rho \otimes \pi$ is in the top part of the restricted oscillator representation, then the corresponding idempotent (54) survives and is unaltered by semisimplification.

On the other hand, we note that for every irreducible representation π of $SO(W, B)$, at least one irreducible representation obtained as a summand of the induction of π from $SO(W, B)$ to $O(W, B)$ occurs in a tensor product

$$(\epsilon(\det) \otimes \mathbb{C}W)^{\otimes N}$$

of degree $N \leq h_W$. Again, for any irreducible representation $\rho \in \widehat{\mathrm{Sp}(V)}$, the summand $\zeta_{W,B}^V(\rho)$ can be obtained as the image of the idempotent

$$(55) \quad \frac{\dim(\rho)}{|\mathrm{Sp}(V)|} \sum_{g \in \mathrm{Sp}(V)} \chi_\rho(g)^{-1} \cdot g \in \mathbb{C}\mathrm{Sp}(V)$$

in

$$\mathrm{End}_{O(W,B)}(\omega[V \otimes W]) = \mathrm{End}_{O(W,B)}((\epsilon(\det) \otimes \mathbb{C}W)^{\otimes N}),$$

again where $\chi_\rho : \mathrm{Sp}(V) \rightarrow \mathbb{C}^\times$ denotes the character corresponding to ρ . All representations of $O(W, B)$ can then be obtained by tensoring the representations appearing in the image of the zeta correspondences by a sign representation.

In particular, one can attempt to produce a given representation of a symplectic group $\mathrm{Sp}(V)$ or an orthogonal group $O(W, B)$ as a chain of alternating eta and zeta correspondences (and tensors with sign representations) applied to a simpler representation of a smaller symplectic or orthogonal group. However, we note that both the eta and zeta correspondences can never alter any input Lusztig classification to introduce eigenvalues other than 1 or -1 to the semisimple component of the data. Therefore, we find that any irreducible representation

Definition 4.5.1. *Consider an irreducible representation ρ of a symplectic group or orthogonal group.*

- (1) For $\rho \in \widehat{Sp_{2N}(\mathbb{F}_q)}$, we call ρ terminal if in its Lusztig classification data, the semisimple component $(s) \in SO_{2N+1}(\mathbb{F}_q) = (Sp_{2N}(\mathbb{F}_q))^D$ has 1 as an eigenvalue of multiplicity one and no -1 eigenvalues.
- (2) For $\rho \in \widehat{O_{2m+1}(\mathbb{F}_q)}$, we call ρ terminal if in the Lusztig classification data of the restriction of ρ to the special orthogonal group $SO_{2m+1}(\mathbb{F}_q)$, the semisimple component of the data $(s) \in Sp_{2m}(\mathbb{F}_q) = (SO_{2m+1}(\mathbb{F}_q))^D$ has no 1 or -1 eigenvalues.
- (3) For $\rho \in \widehat{O_{2m}^\pm(\mathbb{F}_q)}$, we call ρ terminal if in the Lusztig classification data of the restriction of ρ to the special orthogonal group $SO_{2m}^\pm(\mathbb{F}_q)$, the semisimple component $(s) \in SO_{2m}^\pm(\mathbb{F}_q) = (SO_{2m}^\pm(\mathbb{F}_q))^D$ has no 1 or -1 eigenvalues.

We may then produce any irreducible representation of a symplectic or orthogonal group by applying alternating eta and zeta correspondences and possibly tensor products with sign representations during

steps of the construction in $O(\widehat{W}, B)$. Suppose we are given a representation whose Lusztig data has semisimple component involving 1 as an eigenvalue contributing a factor of the centralizer of rank p and -1 as an eigenvalue contributing a factor of the centralizer of rank ℓ . Say the corresponding factors of the unipotent component of the Lusztig classification data are symbols of the form

$$\left(\lambda_1 < \cdots < \lambda_a \right) \otimes \left(\lambda'_1 < \cdots < \lambda'_{a'} \right) \\ \left(\mu_1 < \cdots < \mu_b \right) \otimes \left(\mu'_1 < \cdots < \mu'_{b'} \right).$$

Then (disregarding steps where we tensor with sign representations to obtain $O(W, B)$ -representations which are only obtained up to sign in the zeta correspondence), $a + b + a' + b'$ steps of applying eta and zeta correspondences are needed.

4.6. Character computations. In this subsection, we use the “inductive” construction to describe an algorithm for computing characters of an arbitrary given irreducible representation. There are three major steps to be considered: The calculation of the character of the top part of an oscillator representation, how to use this top part’s character to obtain the character of the representation obtained from applying the eta or zeta correspondence to a given input representation, and the calculation of the characters of the terminal representations.

The action of the symplectic group generators on our model of the oscillator representation are as follows:

$$(56) \quad \begin{aligned} \omega_a \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} (v) &= \psi\left(\frac{a}{2}A(v, v)\right) \cdot (v) \\ \omega_a \begin{pmatrix} 0 & B \\ -B^{-1} & 0 \end{pmatrix} (v) &= \sum_{w \in V_-} \frac{\psi(aB(v, w))}{\sum_{u \in V_-} \psi\left(\frac{a}{2}B(u, u)\right)} \cdot (w) \\ \omega_a \begin{pmatrix} (C^T)^{-1} & 0 \\ 0 & C \end{pmatrix} (v) &= \epsilon_{\mathbb{F}_q}(-1) \cdot (Cv). \end{aligned}$$

The action of any symplectic group element can then be deduced by expressing it as a product of these standard generators. In particular, in principle, we can form a closed expression for the character of the oscillator representation at any symplectic group element. For example, in the case of the standard generators, we find character values

$$\begin{aligned} \chi_{\omega_a} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} &= (-1)^{n(\ell+1)} \text{disc}\left(\frac{A}{2}\right) (\epsilon_{\mathbb{F}_q}(-1)q)^{n/2} \\ \chi_{\omega_a} \begin{pmatrix} 0 & B \\ -B^{-1} & 0 \end{pmatrix} &= (\epsilon_{\mathbb{F}_q}(-2))^n \end{aligned}$$

$$\chi_{\omega_a} \begin{pmatrix} (C^T)^{-1} & 0 \\ 0 & C \end{pmatrix} = \epsilon_{\mathbb{F}_q}(\det(C)) \cdot q^{\dim(\ker(C-I))}.$$

Now, while the expression for the restricted oscillator representation in terms of the extended eta correspondence and the alternating sums given in Theorems 3.5.1 and 3.5.2 are more ideal from the point of view of a genuine decomposition statement, it is inefficient to organize the characters of the alternating sums when doing concrete computations.

Instead, for the purpose of computing characters, let us introduce the *virtual eta* and *virtual zeta correspondences*: For any reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the symplectic metastable range, we define the virtual eta correspondence as the function from the set of irreducible $\mathrm{O}(W, B)$ -representations to the Grothendieck group on the category of representations of $\mathrm{Sp}(V)$

$$\boldsymbol{\eta}_V^{W,B} : \widehat{\mathrm{O}(W, B)} \rightarrow K(\mathrm{Rep}(\mathrm{Sp}(V)))$$

so that the restriction of the oscillator representation to $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$ decomposes as

$$(57) \quad \bigoplus_{k=0}^{h_W} \bigoplus_{\pi \in \widehat{\mathrm{O}(W[-k], B[-k])}} \boldsymbol{\eta}_V^{W[-k], B[-k]}(\pi) \otimes \mathrm{Ind}^{P^k_{\mathrm{O}(W, B)}}(\pi^-).s$$

Similarly, for any reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the orthogonal metastable range, we define the virtual zeta correspondence as the function from the set of irreducible $\mathrm{Sp}(V)$ -representations to the Grothendieck group on the category of $\mathrm{O}(W, B)$ -representations

$$\boldsymbol{\zeta}_{W,B}^V : \widehat{\mathrm{Sp}(V)} \rightarrow K(\mathrm{Rep}(\mathrm{O}(W, B)))$$

so that the restriction of the oscillator representation to $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$ decomposes as

$$(58) \quad \bigoplus_{k=0}^{\dim(V)/2} \bigoplus_{\rho \in \widehat{\mathrm{Sp}(V[-k])}} \mathrm{Ind}^{P^k_{\mathrm{Sp}(V)}}(\rho^-) \otimes \boldsymbol{\zeta}_{W,B}^{V[-k]}(\rho).$$

Specifically, if $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the symplectic (resp. orthogonal) stable range, we take $\boldsymbol{\eta}_V^{W,B}$ (resp. $\boldsymbol{\zeta}_{W,B}^V$) to be equal to $\eta_V^{W,B}$ (resp. $\zeta_{W,B}^V$). If $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the symplectic metastable range, we take $\boldsymbol{\eta}_V^{W,B} = \Psi \circ \eta_{W,B}^{t=N}$, where $\eta_{W,B}^{t=N}$ denotes the interpolated eta correspondence applied at $t = N$, and Ψ is defined as in (49). Similarly, if $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the orthogonal metastable range, we take $\boldsymbol{\zeta}_{W,B}^V = \Psi \circ \zeta_{W,B}^{t=N}$, where $\zeta_{W,B}^{t=N}$ denotes the interpolated eta correspondence applied at $t = N$.

Concretely, for an input representation ρ of $\boldsymbol{\eta}_V^{W,B}$ or $\boldsymbol{\zeta}_{W,B}^V$, we compute the output by attempting the appropriate case of the construction described in Section 3. Say the symbol $\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ corresponds to the alterable factor in the unipotent component of ρ 's Lusztig classification data. If the output is ‘‘constructible,’’ meaning that the genuine eta or zeta correspondence $\eta_V^{W,B}(\rho)$ or $\zeta_{W,B}^V(\rho)$ is a non-zero representation, we put

$$\boldsymbol{\eta}_V^{W,B}(\rho) := \eta_V^{W,B}(\rho) \text{ or } \boldsymbol{\zeta}_{W,B}^V(\rho) := \zeta_{W,B}^V(\rho).$$

Suppose now that it is not constructible, meaning that (possibly after switching rows so that the original construction proposes adding N'_ρ to the first row of the alterable symbol) for some $i \leq a$, we have $\lambda_{i-1} < N'_\rho \leq \lambda_i$ (considering $\lambda_0 = 0$). If $N'_\rho = \lambda_i$, then we put $\boldsymbol{\eta}_V^{W,B}(\rho)$ or $\boldsymbol{\zeta}_{W,B}^V(\rho)$ to be the zero representation. Then we complete the construction with the factor

$$\binom{\lambda_1 < \dots < \lambda_{i-1} < N'_\rho < \lambda_i < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$$

in place of where we would have used the symbol with N'_ρ concatenated at the end of the first row of the alterable symbol. This constructs a genuine representation, which we multiply by the coefficient $(-1)^{a-i}$ in $K(\text{Rep}(\text{Sp}(V)))$ or $K(\text{Rep}(O(W, B)))$. We set the resulting (possibly virtual) representation to be $\boldsymbol{\eta}_V^{W,B}(\rho)$ or $\boldsymbol{\zeta}_{W,B}^V(\rho)$. In particular, the virtual eta and zeta correspondences always output a sign times an irreducible representation. We note that for a representation which is sent to zero by the original eta or zeta correspondence, the virtual eta or zeta correspondence outputs a sign of an irreducible representation already in the image of the genuine eta or zeta correspondence for a smaller orthogonal or symplectic group (respectively). We consider the character of a virtual representation to be the linear combination (with possible negative coefficients) of the characters of its genuine terms.

Now let us consider the virtual representations

$$(59) \quad \widetilde{\omega}^{\text{top}}[V \otimes W] = \bigoplus_{\pi \in \widehat{O(W, B)}} \boldsymbol{\eta}_V^{W,B}(\pi) \otimes \pi$$

of $\text{Sp}(V) \times O(W, B)$ for every reductive dual pair in the symplectic metastable range, and the virtual representations

$$(60) \quad \widetilde{\omega}^{\text{top}'}[V \otimes W] = \bigoplus_{\rho \in \widehat{\text{Sp}(V)}} \rho \otimes \boldsymbol{\zeta}_{W,B}^V(\rho)$$

of $\mathrm{Sp}(V) \times O(W, B)$ for every reductive dual pair in the orthogonal metastable range. These virtual ‘‘top parts’’ are key to understanding the relationship of an irreducible representation’s character and the character of the representation obtained by applying the (virtual) eta or zeta correspondence.

First, we must compute the character values of (59) and (60). This can be done recursively by comparing the top parts to the full oscillator representations at each level, similarly to how the dimensions of the stable top parts were computed in [30]. In the symplectic metastable case, we may re-write the decomposition of the restriction of $\omega[V \otimes W]$ to $\mathrm{Sp}(V) \times O(W, B)$ described in (57) as

$$\bigoplus_{k=0}^{h_W} \mathrm{Ind}_{\mathrm{Sp}(V) \times O(W, B)}^{\mathrm{Sp}(V) \times P_k^{O(W, B)}} (\widetilde{\omega^{\mathrm{top}}}[V \otimes W[-k]]^-)$$

and similarly, in the orthogonal metastable case, we may re-write the decomposition restriction of $\omega[V \otimes W]$ to $\mathrm{Sp}(V) \times O(W, B)$ described in (58) as

$$\bigoplus_{k=0}^{\dim(V)/2} \mathrm{Ind}_{\mathrm{Sp}(V) \times O(W, B)}^{P_k^{\mathrm{Sp}(V)} \times O(W, B)} (\widetilde{\omega^{\mathrm{top}'}}[V[-k] \otimes W]^-).$$

Now we recall that to compute the character of an induction $\mathrm{Ind}_G^H(\rho)$ for a subgroup $H \subseteq G$ and an H -representation, for a given $g \in G$, we have

$$\chi_{\mathrm{Ind}_G^H(\rho)}(s) = \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \chi_\rho(xgx^{-1}).$$

Hence, we obtain that for every choice of $g \in \mathrm{Sp}(V)$, $h \in O(W, B)$ for $(\mathrm{Sp}(V), O(W, B))$ in the symplectic metastable range, the sum

$$\sum_{k=0}^{h_W} \frac{1}{|\mathrm{Sp}(V) \times P_k^{O(W, B)}|} \sum_{\substack{x \in \mathrm{Sp}(V) \times O(W, B) \\ x(g \otimes h)x^{-1} \in \mathrm{Sp}(V) \times P_k^{O(W, B)}}} \chi_{\widetilde{\omega^{\mathrm{top}}}[V \otimes W[-k]]^-}(x(g \otimes h)x^{-1})$$

is equal to the character value $\chi_{\omega[V \otimes W]}(g \otimes h)$, which for any explicit choice of g and h could be calculable using (56). Similarly, for every $g \in \mathrm{Sp}(V)$, $h \in O(W, B)$, for $(\mathrm{Sp}(V), O(W, B))$ in the orthogonal metastable range, the sum

$$\sum_{k=0}^{\dim(V)/2} \frac{1}{|O(W, B) \times P_k^{\mathrm{Sp}(V)}|} \sum_{\substack{x \in \mathrm{Sp}(V) \times O(W, B) \\ x(g \otimes h)x^{-1} \in P_k^{\mathrm{Sp}(V)} \times O(W, B)}} \chi_{\widetilde{\omega^{\mathrm{top}'}}[V[-k] \otimes W]^-}(x(g \otimes h)x^{-1})$$

is equal to the character value $\chi_{\omega[V \otimes W]}(g \otimes h)$. This gives a linear system of equations that can then be used to recursively calculate the virtual characters of the representations $\widetilde{\omega^{\text{top}}[V \otimes W]}$ and $\widetilde{\omega^{\text{top}'[V \otimes W]}$.

Suppose we have calculated the virtual characters of $\widetilde{\omega^{\text{top}}[V \otimes W]}$ (resp. $\widetilde{\omega^{\text{top}'[V \otimes W]}$) for pairs $(\text{Sp}(V), O(W, B))$ in the symplectic (resp. orthogonal) metastable range. We then use their values to deduce the effect of the extended eta (resp. zeta) correspondence on irreducible representations:

Proposition 4.6.1. *Fix a reductive dual pair $(\text{Sp}(V), O(W, B))$.*

- (1) *Suppose the pair is in the symplectic metastable range, and fix an irreducible representation π of $O(W, B)$ such that $\eta_V^{W, B}(\pi)$ is non-zero. Then, for any $g \in \text{Sp}(V)$, the value of the character associate to $\eta_V^{W, B}(\pi)$ as g can be computed as*

$$(61) \quad \chi_{\eta_V^{W, B}(\pi)}(g) = \sum_{(h) \in O(W, B)} \frac{|(h)|}{|O(W, B)|} \cdot \overline{\chi_\pi(h)} \cdot \chi_{\widetilde{\omega^{\text{top}}[V \otimes W]}}(g \otimes h).$$

- (2) *Suppose the pair is in the orthogonal metastable range, and fix an irreducible representation ρ of $\text{Sp}(V)$ such that $\zeta_{W, B}^V(\rho)$ is non-zero. Then, for any $h \in O(W, B)$, the value of the character associate to $\zeta_{W, B}^V(\rho)$ as h can be computed as*

$$(62) \quad \chi_{\zeta_{W, B}^V(\rho)}(h) = \sum_{(g) \in \text{Sp}(V)} \frac{|(g)|}{|\text{Sp}(V)|} \cdot \overline{\chi_\rho(g)} \cdot \chi_{\widetilde{\omega^{\text{top}'[V \otimes W]}}}(g \otimes h).$$

In fact, our proof of this result also shows that (61) is equal to

$$(63) \quad \sum_{(h) \in O(W, B)} \frac{|(h)|}{|O(W, B)|} \cdot \overline{\chi_\pi(h)} \cdot \chi_{\omega^{\text{top}}[V \otimes W]}(g \otimes h)$$

for choices of π which survive the extended eta correspondence. The expressions (61) and (63) are different only for irreducible representations $\pi \in \widehat{O(W, B)}$ with $\eta_{W, B}^V(\pi) = 0$, in which case (61) gives the virtual character $\chi_{\eta_V^{W, B}}(g)$ (which will be a sign times a genuine irreducible character), while the sum (63) gives 0.

Proof of Proposition 4.6.1. This statement follows from elementary manipulations of character theory. Let us fix an order of the conjugacy classes of $O(W, B)$ and an order of its irreducible characters. We may then consider the square matrix obtained from the character table of $O(W, B)$ written according to these orderings. Denote this matrix by $\text{ct}(O(W, B))$. By orthogonality of characters, recall that the inverse

of $\text{ct}(O(W, B))$ can be expressed as a diagonal matrix consisting of the fractions $|(h)|/|O(W, B)|$ of the order of a conjugacy class (h) of $O(W, B)$ divided by the group order, multiplied on the right by the conjugate of the transpose of $\text{ct}O(W, B)$

$$(64) \quad \text{ct}(O(W, B))^{-1} = \begin{pmatrix} \frac{|(h_1)|}{|O(W, B)|} & 0 & \cdots \\ 0 & \frac{|(h_2)|}{|O(W, B)|} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \cdot \overline{\text{ct}(O(W, B))}^T.$$

Now fix a group element $g \in \text{Sp}(V)$. Then, for any $h \in O(W, B)$, by definition (59), we find that the virtual character of the representation $\widetilde{\omega}^{\text{top}}$ applied to $g \otimes h \in \text{Sp}(V) \times O(W, B)$ is

$$\chi_{\widetilde{\omega}^{\text{top}}[V \otimes W]}(g \otimes h) = \sum_{\pi \in \widehat{O(W, B)}} \chi_{\eta_V^{W, B}(\pi)}(g) \cdot \chi_{\pi}(h).$$

In terms of matrices, this can be interpreted as the statement that the column of character values $\chi_{\widetilde{\omega}^{\text{top}}[V \otimes W]}(g \otimes h)$ (varying conjugacy classes (h)) is obtained by multiplying the transpose of the character table $\text{ct}O(W, B)$ by the column of character values $\chi_{\eta_V^{W, B}(\pi)}(g)$ (varying $\pi \in \widehat{O(W, B)}$):

$$\begin{pmatrix} \chi_{\widetilde{\omega}^{\text{top}}[V \otimes W]}(g \otimes h_1) \\ \chi_{\widetilde{\omega}^{\text{top}}[V \otimes W]}(g \otimes h_2) \\ \vdots \end{pmatrix} = \text{ct}(O(W, B))^T \cdot \begin{pmatrix} \chi_{\eta_V^{W, B}(\pi_1)}(g) \\ \chi_{\eta_V^{W, B}(\pi_2)}(g) \\ \vdots \end{pmatrix}.$$

Hence, by applying (64), we can calculate the column of character values $\chi_{\eta_V^{W, B}(\pi)}(g)$ as

$$\overline{\text{ct}(O(W, B))} \cdot \begin{pmatrix} \frac{|(h_1)|}{|O(W, B)|} & 0 & \cdots \\ 0 & \frac{|(h_2)|}{|O(W, B)|} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} \chi_{\widetilde{\omega}^{\text{top}}[V \otimes W]}(g \otimes h_1) \\ \chi_{\widetilde{\omega}^{\text{top}}[V \otimes W]}(g \otimes h_2) \\ \vdots \end{pmatrix}$$

Multiplying this out, we find that each character value $\chi_{\eta_V^{W, B}(\pi)}(g)$ can be calculated as the sum

$$\chi_{\eta_V^{W, B}(\pi)}(g) = \sum_{(h) \in O(W, B)} \frac{|(h)|}{|O(W, B)|} \cdot \overline{\chi_{\pi}(h)} \cdot \chi_{\widetilde{\omega}^{\text{top}}[V \otimes W]}(g \otimes h),$$

matching the right hand side of (61). The claim then follows, since for $\pi \in \widehat{O(W, B)}$ such that $\eta_V^{W, B}(\pi)$ is non-zero, we have $\eta_V^{W, B}(\pi) = \eta_V^{W, B}(\pi)$. \square

5. THE GUREVICH-HOWE RANK CONJECTURE

Finally, as advertised in the introduction, the purpose of this section is to apply our calculation of the eta correspondence to prove Theorem 2, verifying the rank conjecture of S. Gurevich and R. Howe, which predicts the equality of U -rank and tensor-rank for every representation of a symplectic group $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ not attaining top possible U -rank N .

First, in Subsection 5.1, we recall the definitions of U -rank and tensor rank in more detail and recall the results of Gurevich and Howe in [15, 16] which reduce Theorem 2 to a statement that all irreducible representations with tensor rank larger than N have U -rank equal to N (see Proposition 5.1.1). Next, in Subsection 5.2, we use our explicit description of the extended eta correspondence to obtain an induction relation (see Proposition 5.2.1) between tensor ranks from an analogue of the Pieri rule, which reduces the claimed statement to lower tensor rank, again.

5.1. U -rank and the eta correspondence. First we recall the definitions of U -rank and tensor rank given in [15, 16]. We begin with U -rank. Consider a symplectic group $\mathrm{Sp}_{2N}(\mathbb{F}_q)$. The Siegel unipotent subgroup is defined as

$$U_N = \left\{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \mid A \in M_{N \times N}(\mathbb{F}_q) \text{ symmetric} \right\} \subseteq \mathrm{Sp}_{2N}(\mathbb{F}_q).$$

Note that, as a group, U_N is isomorphic to the abelian group of symmetric $N \times N$ matrices, with respect to addition. In particular, we may fix an identification of U_N with its Pontrjagin dual U_N^* in the standard way, i.e. by fixing a non-trivial additive character $\chi_0 : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ and identifying each element of U_N corresponding to a symmetric matrix A with the character

$$U_N \rightarrow \mathbb{C}^\times \\ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \mapsto \chi_0(\mathrm{tr}(AB)).$$

Gurevich and Howe [15, 16] then define the U -rank of an $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ -representation ρ as the maximal rank of a character appearing in its restriction to U_N :

$$(65) \quad rk_U(\rho) := \max\{rk(\chi) \mid \chi \in U_N^* \text{ and } \chi \subseteq \mathrm{Res}_{U_N}(\rho)\},$$

where for a character $\chi \in U_N^*$, its rank is defined to be the matrix rank of the symmetric $N \times N$ -matrix specifying its corresponding element of U_N .

On the other hand, the tensor rank of a $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ -representation is defined according to the oscillator representations. Recall that each oscillator representation of a symplectic group $\mathrm{Sp}(V)$ decomposes into two irreducible summands

$$\omega_a[V] = \omega_a^+[V] \oplus \omega_a^-[V]$$

(from the perspective of the eta correspondence, $\omega_a^\pm[V]$ is obtained by applying the eta correspondence to the representations (± 1) of $\mathbb{Z}/2 = O_1(\mathbb{F}_q)$). These pieces of the oscillator representation are the smallest non-trivial irreducible representations of $\mathrm{Sp}(V)$, and they each have U -rank 1. The *tensor rank* of a representation ρ is then defined as the minimal degree n such that every irreducible component of ρ appears in a tensor product of less than or equal to n oscillator representations:

$$rk_\otimes(\rho) := \min\{n \mid \text{for } \pi \in \widehat{\mathrm{Sp}(V)}, \pi \subseteq \rho, \text{ there exists} \\ m \leq n, a_1, \dots, a_m \in \mathbb{F}_q^\times \text{ with } \pi \subseteq \omega_{a_1}[V] \otimes \cdots \otimes \omega_{a_m}[V]\}$$

Now recalling again that a degree n tensor product

$$\omega_{a_1}[V] \otimes \cdots \otimes \omega_{a_n}[V]$$

can be considered as the restriction of the oscillator representation of a larger symplectic group $\mathrm{Sp}(V \otimes W)$ along the inclusion

$$\mathrm{Sp}(V) \hookrightarrow \mathrm{Sp}(V) \times O(W, B) \hookrightarrow \mathrm{Sp}(V \otimes W),$$

where (W, B) denotes an n -dimensional \mathbb{F}_q -space with a non-degenerate symplectic form B corresponding to the diagonal matrix with entries a_1, \dots, a_n , we see that understanding the restricted oscillator representation $Res_{\mathrm{Sp}(V) \times O(W, B)}(\omega[V \otimes W])$ is key to. In particular, the results of this paper for pairs $(\mathrm{Sp}(V), O(W, B))$ in the symplectic stable or metastable range explicitly classify the irreducible representations of each tensor rank $0 \leq rk_\otimes \leq 2N$.

In [15, 16], Gurevich and Howe also found a connection between the restricted oscillator representations $Res_{\mathrm{Sp}(V) \times O(W, B)}(\omega[V \otimes W])$ and U -rank, which was one of their original motivation for defining the eta correspondence:

The original statement describing the eta correspondence given in, say, Theorem 4.3.3 of [16], is that for choices of V and (W, B) in the symplectic stable range (which we recall means $\dim(W) \leq \dim(V)/2$, there is a system of injections

$$\eta_{W, B}^V : O(\widehat{W, B}) \hookrightarrow \widehat{\mathrm{Sp}(V)}$$

(we omit the subscript when the source is determined) such that for every irreducible representations $\rho \in O(\widehat{W, B})$, the tensor product $\rho \otimes$

$\eta_{W,B}^V(\rho)$ is a summand of $\text{Res}_{\text{Sp}(V) \times O(W,B)}(\omega[V \otimes W])$, and

$$(66) \quad rk_U(\eta_{W,B}^V(\rho)) = \dim(W).$$

Further, every other $\pi \in \widehat{\text{Sp}(V)}$ such that $\rho \otimes \pi$ appears in the restricted oscillator representation has strictly lower U -rank. (Note that though Theorem 4.3.3 of [16] does not include the case of $\dim(W) = \dim(V)/2$, the result still applies to this case as described in Remark 4.3.6.)

First, we note that the results of [15, 16] immediately imply the agreement of tensor- and U -rank in cases covered by the symplectic stable range:

Corollary 1. *For irreducible $Sp_{2N}(\mathbb{F}_q)$ -representations ρ of tensor rank $\leq N$, the notions of rank coincide:*

$$rk_{\otimes}(\rho) = rk_U(\rho).$$

Therefore, it only remains to prove the following

Proposition 5.1.1. *Consider an irreducible representation ρ of a symplectic group $Sp_{2N}(\mathbb{F}_q)$ obtained first in the restriction of an oscillator representation to an unstable reductive dual pair, meaning*

$$N < rk_{\otimes}(\rho) \leq 2N.$$

Then ρ attains top U -rank

$$rk_U(\rho) = N.$$

5.2. An induction rule and concluding Theorem 2. As described in the previous subsection, to prove Proposition 5.1.1, we need more explicit information about the symplectic group representations of each tensor rank, which can be obtained from Theorem 3.5.1. In particular, in the decomposition (38), we may further restrict to $\text{Sp}(V)$ -representations by treating the coefficient $O(W, B)$ -representations as multiplicity spaces, obtaining a classification of the irreducible $\text{Sp}(V)$ -representations of tensor rank $rk_{\otimes} = r$ for each $0 \leq r \leq 2N$ as precisely those constructed in the image of an eta correspondence

$$\eta_{W,B}^V : \widehat{O(W, B)} \rightarrow \widehat{\text{Sp}(V)} \cup \{0\},$$

for one of the two non-equivalent choices of (W, B) with dimension r .

The key step we use to conclude Proposition 5.1.1 and Theorem 2 is the following result, which gives a relationship between the different rank layers of the eta correspondence, according to parabolic induction:

Proposition 5.2.1. *Fix a \mathbb{F}_q -vector space W with symmetric bilinear form B . Consider symplectic spaces V, U of dimension $2N \leq 2M$ respectively, such that both reductive dual pairs $(Sp(V), O(W, B))$ and $(Sp(U), O(W, B))$ are in the symplectic stable or metastable ranges. Then we may consider the eta correspondences*

$$\eta^V : \widehat{O(W, B)} \rightarrow \widehat{Sp(V)} \cup \{0\}$$

$$\eta^U : \widehat{O(W, B)} \rightarrow \widehat{Sp(U)} \cup \{0\}.$$

For an irreducible representation $\pi \in \widehat{O(W, B)}$ such that $\eta^V(W) \neq 0$, we have

$$(67) \quad \eta^U(\pi) \subseteq \text{Ind}^{P_{M-N}^U}(\eta^V(\pi)^\pm),$$

where the \pm denotes whether we consider a sign character on the factor $GL_{M-N}(\mathbb{F}_q)$ of the Levi subgroup before inflating to P_{M-N}^U and applying the induction. The sign is $+$ when W is even dimensional and is $-$ when W is odd dimensional.

To prove this, we now recall briefly the analogue of the Pieri rule for symbols. We give a more concrete statement in Subsection 6.2 below with a more detailed explanation on how it can be derived from the results of [36]. For the purposes of proving Theorem 2, we only need one case of it, so we do not give the full general statement in this subsection.

Consider a unipotent representation of $Sp_{2N}(\mathbb{F}_q)$ corresponding to a symbol $(\lambda_1 < \dots < \lambda_a)$. Write P_1 for the maximal parabolic subgroup of $Sp_{2(N+1)}(\mathbb{F}_q)$ with Levi factor $Sp_{2N}(\mathbb{F}_q) \times GL_1(\mathbb{F}_q)$, and consider $(\lambda_1 < \dots < \lambda_a)$ as a its representation by letting the $GL_1(\mathbb{F}_q)$ factor of the Levi subgroup act trivially and inflating on the unipotent radical trivially. Then its parabolic induction to a $Sp_{2(N+1)}(\mathbb{F}_q)$ -representation $\text{Ind}_{Sp_{2(N+1)}(\mathbb{F}_q)}^{P_1}((\lambda_1 < \dots < \lambda_a))$ is a direct sum of unipotent representations corresponding to symbols

$$(68) \quad \begin{pmatrix} \lambda_1 < \dots < \lambda_{i-1} < \lambda_i + 1 < \lambda_{i+1} < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix},$$

$$\begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_{i-1} < \mu_i + 1 < \mu_{j+1} < \dots < \mu_a \end{pmatrix}$$

when possible, i.e. for $1 \leq i \leq a$ or $1 \leq j \leq b$ where $\lambda_i + 1 < \lambda_{i+1}$ or $\mu_j + 1 < \mu_{j+1}$, respectively, and the unipotent representations

$$(69) \quad \begin{pmatrix} 1 < \lambda_1 + 1 < \lambda_2 + 1 < \cdots < \lambda_a + 1 \\ 0 < \mu_1 + 1 < \cdots < \mu_b + 1 \end{pmatrix} \\ \begin{pmatrix} 0 < \lambda_1 + 1 < \cdots < \lambda_a + 1 \\ 1 < \mu_1 + 1 < \mu_2 + 1 < \cdots < \mu_b + 1 \end{pmatrix}$$

when possible, i.e. when $\lambda_1 > 0$ or $\mu_1 > 0$, respectively. This is a full description of the “one step” Pieri rule.

More generally, for the “ r step” Pieri rules, describing the parabolic induction from a maximal parabolic P_r with Levi subgroup $\mathrm{Sp}_{2N}(\mathbb{F}_q) \times \mathrm{GL}_r(\mathbb{F}_q)$ to $\mathrm{Sp}_{2(N+r)}(\mathbb{F}_q)$ (still taking $\mathrm{GL}_r(\mathbb{F}_q)$ and the unipotent radical to act trivially on the input representation), instead of adding a single “box” to the underlying Young diagrams corresponding to a symbol $(\lambda_1 < \cdots < \lambda_a, \mu_1 < \cdots < \mu_b)$, we must add a “row of r boxes.” More specifically, one must undo the procedures described in Proposition 3.2 and Subsection 4.6 of [36], then apply the classical Pieri rule adding a “row of r boxes” as a Weyl group representation, before re-applying the procedures of [36] to recover the original defect and the new rank $N + r$. This rule can be derived directly from the definition of the symbols (see [36], Subsection 4.8).

In particular, summands that always appear in $\mathrm{Ind}^{P_r}((\lambda_1 < \cdots < \lambda_a, \mu_1 < \cdots < \mu_b))$ are symbols obtained by adding r to the final coordinate in a row:

$$(70) \quad \begin{pmatrix} \lambda_1 < \cdots < \lambda_{a-1} < \lambda_a + r \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \\ \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_{b-1} < \mu_b + r \end{pmatrix}.$$

To apply such a parabolic induction Ind^{P_r} to a general representation ρ of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, the resulting $\mathrm{Sp}_{2(N+r)}(\mathbb{F}_q)$ representation consists of summands which add 1’s to the semisimple part of ρ ’s Lusztig classification data and have unipotent part consisting of the input unipotent part with the factor corresponding to the centralizer of 1 eigenvalues replaced by the possible pieces of its r step parabolic induction. We also consider the “signed parabolic induction” $\mathrm{Ind}^{P_r}(\rho^-)$, by which we denote the $\mathrm{Sp}_{2(N+r)}(\mathbb{F}_q)$ -representation obtained by tensoring ρ with the sign character of the $\mathrm{GL}_r(\mathbb{F}_q)$ factor of the Levi subgroup of P_r before inflating and inducing. The procedure on Lusztig classification data giving the signed parabolic induction is completely similar to the unsigned case, except that -1 ’s are added to the semisimple part of the

data corresponding to the input representation (instead of 1's) and the symbol corresponding to this factor of its centralizer is altered, instead.

In particular, by combining the symbol Pieri rule with our description of the eta correspondence given in Definitions 3.2.1 and 3.2.2 we are able to conclude Proposition 5.2.1:

Proof of Proposition 5.2.1. The choice of sign in (67) precisely specifies whether the induction operation will add 1's or -1 's to the Lusztig classification data of the input representation. Since it is chosen according to the parity of $\dim(W)$, the semisimple part of the Lusztig classification data of $\eta^U(\pi)$ agrees with that of the irreducible summands of $\text{Ind}^{P_{M-n}^U}(\eta^V(\pi)^\pm)$, reducing the claim to the fact that in the "altered factor" of the unipotent parts of $\eta^U(\rho)$ and $\eta^V(\rho)$,

$$\left(\begin{array}{c} \lambda_1 < \cdots < \lambda_a < M'_\rho \\ \mu_1 < \cdots < \mu_b \end{array} \right) \subseteq \text{Ind}^{P_{M-N}} \left(\begin{array}{c} \lambda_1 < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{array} \right),$$

which follows from the symbol Pieri rule. \square

Considering the effect of parabolic induction on U -rank then allows us to reduce Proposition 5.1.1 to Corollary 1, and conclude Theorem 2:

Proof of Proposition 5.1.1 and Theorem 2. Consider a representation ρ of $\text{Sp}(V)$ of tensor rank

$$N < rk_\otimes(\rho) \leq 2N.$$

Then by Theorem 3.5.1, there exists a choice of (W, B) with $\dim(W) = rk_\otimes(\rho) > N$, and an irreducible representation $\pi \in \widehat{O(W, B)}$ such that $\eta_{W, B}^V(\pi) = \rho$.

Let us denote by V' the symplectic space of dimension $\dim(V') = 2 \cdot \dim(W)$, i.e. the maximal dimensional symplectic space such that $(\text{Sp}(V'), O(W, B))$ is a reductive dual pair in the symplectic stable range. Consider the eta correspondence

$$\eta^{V'} : \widehat{O(W, B)} \leftrightarrow \widehat{\text{Sp}(V')},$$

and its image of π . Let us write $\rho' = \eta^{V'}(\pi) \in \widehat{\text{Sp}(V')}$. Applying Corollary 1, we know that as a representation of $\text{Sp}(V') = \text{Sp}_{2 \cdot \dim(W)}(\mathbb{F}_q)$, its U -rank is $rk_U(\rho') = rk_\otimes(\rho') = \dim(W)$.

Now applying Proposition 5.2.1 gives that ρ' appears as a summand of a (possibly signed) parabolic induction of ρ from a parabolic subgroup with Levi factor $\text{Sp}(V) \times GL_{\dim(W)-N}(\mathbb{F}_q) \subseteq \text{Sp}(V')$, which is

an operation that can only increase U -rank by at most the difference $\dim(W) - N$. In other words, the U -rank of ρ is at least

$$\begin{aligned} rk_U(\rho) &\geq rk_U(\rho') - (\dim(W) - N) = \\ &rk_{\otimes}(\rho') - (\dim(W) - N) = N, \end{aligned}$$

and therefore we must have equality $rk_U(\rho) = N$, obtaining (7). \square

6. RESOLVING THE ALTERNATING SUMS

In Theorem 3.5.1 and 3.5.2, we decompose the restricted oscillator representation $Res_{Sp(V) \times O(W, B)}(\omega[V \otimes W])$ in terms of the eta correspondence (in the case of the symplectic stable or metastable range, corresponding to the condition (18)) or the zeta correspondence (in the orthogonal metastable or stable ranges, corresponding to the complementary condition (19)). Further, we described the eta and zeta correspondences in terms of Lusztig classification data, allowing us to directly compute the $Sp(V)$ -representation and $O(W, B)$ -representation summands occurring in the restriction of $\omega[V \otimes W]$.

However, in (38) and (39), the eta and zeta correspondence terms appear with coefficients $\mathcal{A}_k(\rho, N'_\rho)$ (giving an $O(W, B)$ -representation, for ρ an irreducible $O(W[-k], B[-k])$ -representation) or $\mathcal{A}_i(\rho, m'_\rho)$ (giving a $Sp(V)$ -representations, for ρ an irreducible $Sp(V[-k])$, respectively. Our description of these terms for the purpose of proving the Theorem was as certain alternating sums of parabolic inductions. The purpose of this section is to simplify these sums $\mathcal{A}_k(\rho, N'_\rho)$ (resp. $\mathcal{A}_i(\rho, m'_\rho)$) and describe their irreducible $O(W, B)$ - (resp. $Sp(V)$ -) representation summands in a way that can be used for concrete computations.

6.1. The main statement. Our main result is

Theorem 6.1.1. *Fix a reductive dual pair $(Sp(V), O(W, B))$ in the symplectic stable or metastable range. For an irreducible representation ρ of $O(W[-k], B[-k])$, consider the factor of the unipotent part of its Lusztig classification data writable as a symbol*

$$(71) \quad \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix},$$

such that it is replaced by a symbol

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

in the construction of $\eta_{W[-k], B[-k]}^V(\rho)$ (switch rows if necessary). Then the $O(W, B)$ -representation $\mathcal{A}_k(\rho, N'_\rho)$ consists of the irreducible summands appearing in the parabolic induction $\text{Ind}^{P_k}(\rho \otimes \epsilon(\det))$ such that, when performing the Pieri rule (see Proposition 6.2.1) on the row $\lambda_1 < \dots < \lambda_a$ of (71), the highest coordinate $\lambda'_{a'}$ of the corresponding row of the new symbol satisfies

$$\lambda'_{a'} < N'_\rho + (a' - a).$$

There is a similar description in the case of $(Sp(V), O(W, B))$ in the orthogonal stable or metastable range, of the $Sp(V)$ -representations $\mathcal{A}_i(\rho, m'_\rho)$ for $Sp(V[-i])$ -representations ρ .

To prove this, we use the Pieri rule for Lusztig symbols, which we state in Proposition 6.2.1 below. Recall that the combinatorial data of a symbol (classifying the irreducible unipotent representations) is equivalent to the data of its defect (which gives the information of the underlying cuspidal representation) and a pair of Young diagrams corresponding to the irreducible representation of the remaining Weyl group specifying which piece of the induced cuspidal representation the symbol corresponds to (this data comes from undoing the procedures given in Proposition 3.2 and Subsection 4.6 or 4.7 of [36]; we give more details below).

First, as we discussed in Subsection 3.4, in each case, in every summand of the alternating sums of parabolic inductions all factor through precisely to the symbol altered by the eta or zeta correspondence (the symbol of the factor of the unipotent part corresponding to $(-1)^{\dim(W)}$ eigenvalues), with the other Lusztig classification data being preserved. Therefore, the problem of simplifying $\mathcal{A}_k(\rho, N'_\rho)$ (resp. $\mathcal{A}_k(\rho, m'_\rho)$) reduces to simplifying an alternating sum of parabolic inductions of symbols. As in Definitions 3.4.1 and 3.4.2, write $\theta = \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}$ for the symbol corresponding to the factor of the unipotent part of ρ 's Lusztig classification data which is altered in the extended eta correspondence (resp. the extended zeta correspondence). Arrange the rows so that the eta correspondence alters the top row. Write

$$(72) \quad k^{(i)} = k - N'_\rho + \lambda_i,$$

so that we have $k^{(a)} \geq k^{(a-1)} \geq \dots \geq k^{(1)}$. Pick the minimal i_0 such that $k^{(i_0)} \geq 0$. We then need to find the symbols χ appearing in the alternating sum $A_k^\pm(\theta, N'_\rho)$ (with superscript sign agreeing with the sign of $\phi^\pm(u)$ or $\psi^\pm(u)$ appearing in the construction of $\eta_{W, B}^V(\rho)$ or

$\zeta_V^{W,B}(\rho)$), which can be written out as

$$(73) \quad \bigoplus_{i=i_0}^{a+1} (-1)^{a+1-i} \cdot \text{Ind}^{P_k(i)} \left(\begin{array}{c} \lambda_1 < \cdots < \widehat{\lambda}_i < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{array} \right)$$

(for the case of the extended zeta correspondence, replace N'_ρ by m'_ρ). The irreducible components of $\mathcal{A}_k(\rho, N'_\rho)$ will then consist of $O(W, B)$ -representations obtained with Lusztig classification data obtained by adding $(-I)_{2k}$ to ρ 's original semisimple part, and taking the unipotent part obtained by tensoring a symbol in (73) (and choosing the same central sign data as ρ).

In other words, Theorem 6.1.1 can be more precisely stated as

Theorem 6.1.2. *Assume the above notation. The alternating sum (73) simplifies as the sum of all symbols*

$$\left(\begin{array}{c} \lambda'_1 < \cdots < \lambda'_{a'} \\ \mu_1 < \cdots < \mu'_b \end{array} \right) \subseteq \text{Ind}^{P_k} \left(\begin{array}{c} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{array} \right)$$

such that

$$\lambda_{a'} < N'_\rho + (a' - a).$$

This statement can now be proved directly by considering the induction rule on symbols, which can be done by translating the symbols into Young diagram data.

Consider a symbol of type B , C , D or 2D -type

$$(74) \quad \left(\begin{array}{c} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{array} \right).$$

We want to consider the “ k -step induction”

$$(75) \quad \text{Ind}^{P_k} \left(\begin{array}{c} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{array} \right),$$

where P_k denotes the standard maximal parabolic $P_k^{W,B}$ if we take (74) to correspond to a unipotent representation of $O(W[-k], B[-k])$ for some orthogonal space (W, B) , or P_k^V if we take (74) to correspond to a unipotent representation of $\text{Sp}(V[-k])$ for some symplectic space V . To consider a symbol (74) as a representation of P_k in either of these cases, let the factor $GL_k(\mathbb{F}_q)$ of the Levi subgroup of P_k act trivially, and inflate by letting. Then (74) denotes the induction of the resulting representation to $O(W, B)$ or $\text{Sp}(V)$. The decomposition of (75) is according to a *Pieri rule*, which we concretely state in Proposition 6.2.1 below.

6.2. The Pieri rule. First, let us briefly recall the role of symbols as representations. In (4.6.2), (4.7.1) of [36], Lusztig described how the combinatorial data of a pair of increasing sequences (74) corresponds to an irreducible representation of a certain Hecke algebra $\mathcal{H}_n(q, q^d)$ defined according to certain relations (see Subsection 4.1 of [36]) which are equivalent to the classical Iwahori relations and recover the group algebra of the Weyl group (see [5], §68A). In Subsection 4.8 of [36], Lusztig also describes how induction is preserved by these correspondences. For the Weyl groups of the groups we consider here, the irreducible representations in each case are classified by pairs of Young diagrams whose total numbers of boxes add up to the rank. Therefore, the induction (75) can be computed by applying the Pieri rule to these Young diagrams i.e., by considering all choices of $k_1 + k_2 = k$, and adding a row of length k_1 to the top row's corresponding Young diagram and a row of length k_2 to the bottom row's corresponding Young diagram.

To find the Weyl group representation corresponding to a symbol (74), without loss of generality, switch rows so that $a \geq b$, and denote the defect by $d = a - b$.

First suppose d is strictly positive. The symbol notation then indicates that the unipotent representation $\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ is constructed in an induction of the cuspidal unipotent representation corresponding to the symbol $\binom{0 < 1 < 2 < \dots < d-1}{\emptyset}$ (the minimal rank symbol of defect d). The first step of Lusztig's procedure is to "remove" this cuspidal representation from the symbol (i.e. by undoing the bijection j of Proposition 3.2 of [36]), to obtain a defect one symbol

$$(76) \quad \binom{\lambda_1 < \dots < \lambda_a}{0 < 1 < \dots < d-2 < \mu_1 + (d-1) < \dots < \mu_b + (d-1)},$$

(using the convention of [36] describing how to reduce a symbol if the coordinate of its first two rows is 0).

The next step is to undo the procedure described in Subsection 4.6 of [36] to obtain Young diagrams. In the case of (76), we obtain a Young diagram

$$(77) \quad (\lambda_1 \leq \lambda_2 - 1 \leq \dots \leq \lambda_a - (a-1))$$

where the i th row has length $\lambda_{a-i+1} - (a-i)$ corresponding to the top row, and a Young diagram

$$(78) \quad (\mu_1 \leq \mu_2 - 1 \leq \dots \leq \mu_b - (b-1))$$

where the i th row has length $\mu_{b-i+1} - (b-i)$ corresponding to the bottom row (not writing the rows with length 0 corresponding to the

coordinates $0 < 1 < \dots < d - 2$ concatenated onto the bottom row in (76)).

In the case of defect $d = 0$, we undo the procedure in Subsection 4.7 of [36] to obtain this same answer, of a Weyl group representation corresponding to Young diagrams (77), (78).

We will denote the Young diagrams (77), (78), by α, β , denoting the i th row lengths by

$$(79) \quad \begin{aligned} \alpha_i &:= \lambda_{a-i+1} - (a - i) \\ \beta_i &:= \mu_{b-i+1} - (b - i). \end{aligned}$$

(We use the convention that the first row of the Young diagram is the longest.)

We use the terminology that, for a Young diagram $(\gamma_n \leq \dots \leq \gamma_1)$ and for a natural number k , we say Young diagrams of the form

$$(k_{n+1} \leq \gamma_n + k_n \leq \gamma_n + k_{n-1} \leq \dots \leq \gamma_1 + k_1)$$

where k_i are natural numbers satisfying $k_1 + \dots + k_{n+1} = k$ and, for every $i = 1, \dots, n$, we have $k_{i+1} \leq \gamma_i - \gamma_{i+1}$ (putting $\gamma_{n+1} = 0$ are the Young diagrams *obtained by adding a row of length k to γ*).

The Pieri rule for symbols can then be stated as follows:

Proposition 6.2.1. *Fix an orthogonal space (W, B) (resp. a symplectic space V) and consider a symbol $(\lambda_1 < \dots < \lambda_a)$ defining a unipotent representation of $O(W[-k], B[-k])$ (resp. $Sp(V[-k])$). Recall that we denote by P_k the standard maximal parabolic with Levi subgroup $O(W[-k], B[-k]) \times GL_k(\mathbb{F}_q)$ (resp. $Sp(V[-k]) \times GL_k(\mathbb{F}_q)$), and consider the symbol as a P_k -representation by letting $GL_k(\mathbb{F}_q)$ act trivially and inflating by letting the unipotent part of the parabolic act trivially. Then its parabolic induction to an $O(W, B)$ -representation decomposes as a sum of symbols*

$$Ind^{P_k} \left(\begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix} \right) = \bigoplus_{\mathcal{S}_k[(\lambda_1 < \dots < \lambda_a)]} \begin{pmatrix} \lambda'_1 < \dots < \lambda'_{a'} \\ \mu'_1 < \dots < \mu'_{b'} \end{pmatrix}$$

where the sum runs over the set of symbols $\mathcal{S}_k[(\lambda_1 < \dots < \lambda_a)]$ consisting of $(\lambda'_1 < \dots < \lambda'_{a'})$ where, for some $k_1 + k_2 = k$, the Young diagram

$$(\lambda'_1 \leq \lambda'_2 - 1 \leq \dots \leq \lambda'_{a'} - (a' - 1))$$

is obtained by adding a row of length k_1 to

$$(\lambda_1 \leq \lambda_2 - 1 \leq \dots \leq \lambda_a - (a - 1)),$$

and the Young diagram

$$(\mu'_1 \leq \mu'_2 - 1 \leq \cdots \leq \mu'_{b'} - (b' - 1))$$

is obtained by adding a row of length k_2 to

$$(\mu_1 \leq \mu_2 - 1 \leq \cdots \leq \mu_b - (b - 1)).$$

(The awkwardness of this statement is in order to accomodate all cases of input symbols, including those where one of the rows of the original symbol begins with a 0 coordinate, and to properly address the case when a' and b' are larger than a and b .)

6.3. The proof of the main statement. We then conclude Theorem 6.1.2 by considering it in terms of Young diagrams, and applying this Pieri rule.

Proof of Theorem 6.1.2. Let us use the above notation (79) for an alterable unipotent part of a representation ρ for which we want to find the coefficient $\mathcal{A}_k(\rho, N'_\rho)$ appearing with $\eta_{W,B}^V(\rho)$. Since the resulting Young diagrams (77), (78) ultimately do not depend on which row was on top or longer, we may say without loss of generality that the top row is the one where we add the coordinate N'_ρ when constructing $\eta_{W,B}^V(\rho)$.

Applying this to the context of Theorem 6.1.2, the symbol

$$(80) \quad \left(\begin{array}{c} \lambda_1 < \cdots < \lambda_{i-1} < \lambda_{i+1} < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{array} \right)$$

appearing in the i th term of the alternating sum (73) corresponds to the same cuspidal representation $\binom{0 < 1 < \cdots < d-1}{\emptyset}$ as the original symbol (since it has the same row lengths) and the Young diagrams

$$(81) \quad \begin{array}{c} (\alpha_a \leq \cdots \leq \alpha_{a-i+2} \leq \alpha_{a-i} + 1 \leq \cdots \leq \alpha_1 + 1 \leq N'_\rho - a) \\ (\beta_b \leq \cdots \leq \beta_1). \end{array}$$

(See Figure 1 for an example.)

The claimed reduction of the alternating sum consists of symbols corresponding to the same underlying cuspidal representation and a pair of Young diagrams (α', β') obtained from applying the k -step Pieri rule to (α, β) such that the first row length of α' is strictly bounded

$$(82) \quad \alpha'_1 < N'_\rho - a.$$

First we note that the summands of the initial parabolic induction satisfying this condition must survive in the alternating sum, since the condition (82) guarantees that such an (α', β') cannot appear in the parabolic induction of any of the other terms for $i \leq a$, since the first row is shorter than the first row of any top Young diagram in (81).

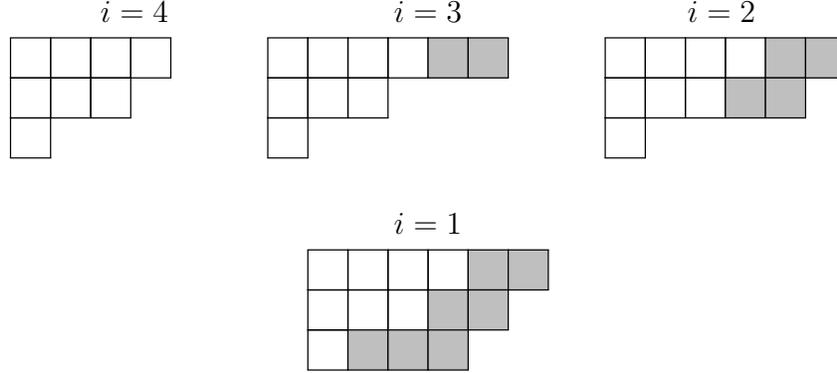


FIGURE 1. These are the top row Young diagrams corresponding to the symbols (80), in the case of $\alpha = (1 \leq 3 \leq 4)$, $N'_\rho - a = 6$. The boxes highlighted gray show the skew semistandard tableau which, after removal from each Young diagram recovers the original α . At each i , just enough boxes are added to a row of the $(i+1)$ th Young diagram to not be obtainable from the $(i+2)$ th.

It remains to see that every other term in the alternating sum vanishes. First, for summands of the initial parabolic induction term $Ind^{P_k}((\lambda_1 < \dots < \lambda_a)_{\mu_1 < \dots < \mu_b})$ which fail the condition (82) are cancelled in the next term at $i = a$, since its corresponding top Young diagram ($\alpha_a \leq \dots \leq \alpha_2 \leq N'_\rho - a$). Similarly, we may proceed inductively to see that every pair of Young diagrams appearing from the i th term of the alternating sum which was not used to cancel the $(i+1)$ th term is cancelled at the $(i-1)$ th term. This follows since, at each step i , the condition on a pair of Young diagrams to have already appeared in the $(i+1)$ th term (and therefore be cancelled in the previous step) is that the length of the $(a-i+2)$ th row does not exceed what can be attained from the previous step without adding boxes directly above each other, i.e.

$$(83) \quad \alpha'_{a-i+2} \leq \alpha_{a-i+1}.$$

On the other hand, the condition on α' to appear in the $(i-1)$ th term is complementary

$$\alpha'_{a-i+2} \geq \alpha_{a-i+1} + 1,$$

since the condition is that the $(a-i+2)$ th row is at least as long as the corresponding row of the top Young diagram in the $(i-1)$ th term, so all the cases are covered.

It remains therefore to check that every term from the final term (corresponding to the lowest i) in the alternating sum cancels. Recall that the sum (73) has final term at $i = i_0$, which is the smallest. First,

suppose $i_0 > 1$. Then by definition

$$(84) \quad k^{(i_0-1)} = k - N'_\rho + \lambda_{i_0-1} < 0,$$

and thus, when considering the final induction

$$\text{Ind}^{k^{(i_0)}} \left(\begin{array}{c} \lambda_1 < \cdots < \widehat{\lambda_{i_0}} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{array} \right),$$

on the Young diagram level, every top Young diagram appearing satisfies the condition (83), since it we would need to add at least $\lambda_{i_0} - \lambda_{i_0-1}$ boxes, which is strictly more than $k^{(i_0)}$ by (84). Therefore, all terms are used up in the previous step, giving the claim. In the case when $i_0 = 1$, we again know that (83) must also follow, by the metastable dimension condition, since the final number of boxes to be added is

$$k^{(1)} = k - N'_\rho + \lambda_1 < \lambda_1$$

(since $k \leq m$ and $N'_\rho > m$ because we assumed that V and (W, B) lie in the symplectic stable or metastable range). \square

Example: We emphasize the point that this alternating sum, even in extreme cases, need not be irreducible. The largest symbol (giving the unipotent irreducible representation of maximal dimension) for $SO_{2m+1}(\mathbb{F}_q)$ is

$$\begin{pmatrix} 0 < 1 < 2 < \cdots < m \\ 1 < 2 < \cdots < m \end{pmatrix},$$

which gives the Steinberg representation St_m , of dimension q^{m^2} . In this case, its corresponding pair of Young diagrams consists of a column of m boxes (corresponding to the top row) and the empty Young diagram (corresponding to the bottom row). In this case, the Pieri rule adding a single box gives

$$\begin{aligned} \text{Ind}^{P_1}(St_m) &= \begin{pmatrix} 0 < 1 < 2 < \cdots < m-1 < m+1 \\ 1 < 2 < \cdots < m \end{pmatrix} \oplus \\ &\quad \begin{pmatrix} 0 < 1 < 2 < \cdots < m \\ 1 < 2 < \cdots < m-1 < m+1 \end{pmatrix} \oplus St_{m+1} \end{aligned}$$

Then, for example, the alternating sum of symbols $A_1^\pm(St_m, m+1)$ with sign chosen to alter the top row outputs

$$\begin{aligned} \text{Ind}^{P_1}(St_m) - \begin{pmatrix} 0 < 1 < 2 < \cdots < m-1 < m+1 \\ 1 < 2 < \cdots < m \end{pmatrix} &= \\ \begin{pmatrix} 0 < 1 < 2 < \cdots < m \\ 1 < 2 < \cdots < m-1 < m+1 \end{pmatrix} \oplus St_{m+1} & \end{aligned}$$

APPENDIX A. DICTIONARY WITH THE NOTATION OF S.-Y. PAN

The purpose of this Appendix is to provide a dictionary between the notation used in this paper, and the notation used by S.-Y. Pan in [40, 41].

First note that Pan uses a modified description of the unipotent irreducible representations of the symplectic and orthogonal groups. We consider the classical description of unipotent irreducible representations of a connected group G , say $SO_{2m+1}(\mathbb{F}_q)$, $Sp_{2N}(\mathbb{F}_q)$, or $SO_{2m}^\pm(\mathbb{F}_q)$ in terms of Lusztig symbols with unordered rows, so that

$$(85) \quad \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} = \begin{pmatrix} \mu_1 < \cdots < \mu_b \\ \lambda_1 < \cdots < \lambda_a \end{pmatrix}$$

(requiring, in the case when G is of B - or C -type, simply that the defect is odd). This is sufficient for our discussion of the irreducible unipotent representations of symplectic groups and the odd orthogonal groups (where the center splits off $O_{2m+1}(\mathbb{F}_q) = \mathbb{Z}/2 \times SO_{2m+1}(\mathbb{F}_q)$). In the case of D -type, the combinatorial data of symbols (85), requiring defect to be 0 or 2 mod 4, correspond to the irreducible unipotent representations of $SO_{2m}^+(\mathbb{F}_q)$ and $SO_{2m}^-(\mathbb{F}_q)$.

However, performing an induction from $SO_{2m}^\pm(\mathbb{F}_q)$ to $O_{2m}^\pm(\mathbb{F}_q)$ (for $m > 0$), the resulting unipotent representation of $O_{2m}^\pm(\mathbb{F}_q)$ of twice the dimension of $\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$ splits into two non-isomorphic equidimensional pieces. In our notation, we identify these pieces by labelling these unipotent irreducible $O_{2m}^\pm(\mathbb{F}_q)$ -representations according to their underlying symbol (85) and the data of a central sign \pm indicating the action of $\mathbb{Z}/2 = O_{2m}^\pm(\mathbb{F}_q)/SO_{2m}^\pm(\mathbb{F}_q)$, determining the relevant piece of the underlying symbol's induction:

$$\begin{aligned} & \text{Ind}_{O_{2m}^\pm(\mathbb{F}_q)}^{SO_{2m}^\pm(\mathbb{F}_q)} \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} = \\ & r^{O_{2m}^\pm(\mathbb{F}_q)}[(1), \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}]^{(+1)} \oplus r^{O_{2m}^\pm(\mathbb{F}_q)}[(1), \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}]^{(-1)}. \end{aligned}$$

On the other hand, Pan labels these two unipotent irreducible $O_{2m}^\pm(\mathbb{F}_q)$ -representations by enforcing an ordering on the rows of a symbol (in this paper, we distinguish this notation by using square brackets), so that

$$\text{Ind}_{O_{2m}^\pm(\mathbb{F}_q)}^{SO_{2m}^\pm(\mathbb{F}_q)} \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} = \left[\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \right] \oplus \left[\begin{pmatrix} \mu_1 < \cdots < \mu_b \\ \lambda_1 < \cdots < \lambda_a \end{pmatrix} \right]$$

for non-isomorphic

$$\begin{bmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{bmatrix} \neq \begin{bmatrix} \mu_1 < \cdots < \mu_b \\ \lambda_1 < \cdots < \lambda_a \end{bmatrix}.$$

(We note Pan also changes the order of a symbol's entries in his notation in [40], writing each row in strictly decreasing order). Pan also imposes an ordering on the symbols corresponding to the unipotent irreducible representations of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ or $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$, demanding then that

$$\begin{bmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{bmatrix}$$

has defect $a - b$ exactly 1 mod 4 (so that there is exactly one ordered symbol corresponding to an odd defect unordered symbol (85)).

Now, Pan describes in [41] the decomposition of the unipotent part of

$$(86) \quad \mathrm{Res}_{\mathrm{Sp}(V) \times O(W, B)}(\omega[V \otimes W])$$

for even-dimensional W as a direct sum of tensor products $\rho \otimes \pi$ for certain pairs of irreducible unipotent representations $\rho \in \widehat{\mathrm{Sp}(V)}$ and $\pi \in \widehat{O(W, B)}$. Specifically, Pan describes that in the split case $O(W, B) = O_{2m}^+(\mathbb{F}_q)$, a tensor product of an ordered symbol of $\mathrm{Sp}(V)$ with an ordered symbol of $O(W, B)$

$$(87) \quad \begin{bmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{bmatrix} \otimes \begin{bmatrix} \lambda'_1 < \cdots < \lambda'_{a'} \\ \mu'_1 < \cdots < \mu'_{b'} \end{bmatrix}$$

is a summand of the unipotent part of (86) precisely when

- The Young diagram with row lengths

$$(\lambda'_{a'} - (a' - 1), \lambda'_{a'-1} - (a' - 2), \dots, \lambda'_1)$$

can be obtained by adding a row (we discuss this notion in more detail in Subsection 6.2) to the Young diagram with row lengths

$$(\mu_b - (b - 1), \mu_{b-1} - (b - 2), \dots, \mu_1).$$

- The Young diagram with row lengths

$$(\lambda_a - (a - 1), \lambda_{a-1} - (a - 2), \dots, \lambda_1)$$

can be obtained by adding a row to the Young diagram with row lengths

$$(\mu'_{b'} - (b' - 1), \mu'_{b'-1} - (b' - 2), \dots, \mu'_1).$$

- The defects precisely satisfy $a' - b' = -(a - b) + 1$.

Similarly, Pan proves that in the non-split case $O(W, B) = O_{2m}^-(\mathbb{F}_q)$, a tensor product (87) of an ordered symbol of $\mathrm{Sp}(V)$ with an ordered symbol of $O(W, B)$ is a summand of the unipotent part of (86) precisely when

- The Young diagram with row lengths

$$(\mu'_{b'} - (b' - 1), \mu'_{b'-1} - (b' - 2), \dots, \mu'_1)$$

can be obtained by adding a row to the Young diagram with row lengths

$$(\lambda_a - (a - 1), \lambda_{a-1} - (a - 2), \dots, \lambda_1).$$

- The Young diagram with row lengths

$$(\mu_b - (b - 1), \mu_{b-1} - (b - 2), \dots, \mu_1)$$

can be obtained by adding a row to the Young diagram with row lengths

$$(\lambda'_{a'} - (a' - 1), \lambda'_{a'-1} - (a' - 2), \dots, \lambda'_1).$$

- The defects precisely satisfy $a' - b' = -(a - b) - 1$.

Therefore, Pan decomposes the unipotent part of the restriction of an oscillator representation to $\mathrm{Sp}(V) \times O(W, B)$, for even-dimensional W . (Of course, in the case of odd-dimensional W , there is no unipotent part of the restriction of $\omega[V \otimes W]$.)

In [41], Pan approaches the question of which pairs $\rho \otimes \pi$ appear with non-zero multiplicity in the restricted oscillator representation for general irreducible representations $\rho \in \widehat{\mathrm{Sp}(V)}$ and $\pi \in \widehat{O(W, B)}$ by claiming a “commutation with the Lusztig correspondence.” In essence, this means that a pair of irreducible representations appears with non-zero multiplicity precisely when a pair of factors (corresponding to a certain eigenvalue of the semisimple part of the Lusztig classification data) of the unipotent part of their Lusztig classification data appears in the unipotent part of the appropriate restricted oscillator representation.

We now explain how our decomposition recovers Pan’s classification of the occurring summands.

First, we note that our constructions of the pieces of the restricted oscillator representation always only involve “altering” the Lusztig data of an input representation associated to a certain specific eigenvalue of the semisimple data (the eigenvalue 1 for even orthogonal groups and the eigenvalue -1 for odd orthogonal groups). In particular, we can see the effect of “commuting with the Lusztig correspondence” in the

sense which Pan uses to pass from his decomposition of the unipotent part of the restricted oscillator representation to information about the irreducible pairs which could appear in the full representation. This reduces us to needing to see that the unipotent part of our decomposition in each case matches the description of Pan's unipotent summands.

We begin with considering $(V, (W, B))$ in the symplectic stable or metastable range. We use the eta correspondence and its extension in this case. From the decomposition given in Theorem 3.5.1, we find the unipotent part of the restriction of $\omega[V \otimes W]$ to $\mathrm{Sp}(V) \times O(W, B)$ is the sum over $k = 0, \dots, h_B$ and over every unipotent irreducible representation π of $O(W[-k], B[-k])$ of summands of the form

$$\eta_{W[-k], B[-k]}^V(\pi) \otimes \mathcal{A}_k(\pi, N'_\pi).$$

Say the restriction of π to $SO(W[-k], B[-k])$ corresponds to the symbol $\left(\begin{smallmatrix} \alpha_1 < \dots < \alpha_a \\ \beta_1 < \dots < \beta_b \end{smallmatrix} \right)$ (we use different notation for the entries here to avoid confusion with Pan's notation), so that $N'_\pi = N - m + \frac{a+b}{2}$. Say $\eta_{W, B}^V(\pi)$ is constructible. Then, depending on the central sign, $\eta_{W, B}^V(\pi)$ is the unipotent irreducible representation of $\mathrm{Sp}(V)$ corresponding to one of the symbols

$$(88) \quad \left(\begin{smallmatrix} \alpha_1 < \dots < \alpha_a \\ \beta_1 < \dots < \beta_b < N'_\pi \end{smallmatrix} \right) \text{ or } \left(\begin{smallmatrix} \alpha_1 < \dots < \alpha_a < N'_\pi \\ \beta_1 < \dots < \beta_b \end{smallmatrix} \right).$$

Now, for the orthogonal groups representation factor, the Harish-Chandra induction of this symbol is a sum of symbols

$$(89) \quad \left(\begin{smallmatrix} \alpha'_1 < \dots < \alpha'_{a'} \\ \beta'_1 < \dots < \beta'_{b'} \end{smallmatrix} \right)$$

such that the Young diagram with row lengths

$$(\alpha'_{a'} - (a' - 1), \alpha'_{a'-1} - (a' - 2), \dots, \alpha'_1)$$

can be obtained by adding a row to the Young diagram with row lengths

$$(\alpha_a - (a - 1), \alpha_{a-1} - (a - 2), \dots, \alpha_1),$$

and similarly, the Young diagram with row lengths

$$(\beta'_{b'} - (b' - 1), \beta'_{b'-1} - (b' - 2), \dots, \beta'_1)$$

can be obtained by adding a row to the Young diagram with row lengths

$$(90) \quad (\beta_b - (b - 1), \beta_{b-1} - (b - 2), \dots, \beta_1),$$

according to the Pieri rule. The summands which contribute and survive in the alternating sum $\mathcal{A}_k(\pi, N'_\pi)$ are these symbols (89) such that

$$\beta'_{b'} - (b' - 1) \leq N'_\pi - b \text{ or } \alpha'_{a'} - (a' - 1) \leq N'_\pi - a,$$

respectively (corresponding to (88)). To understand the relation with Pan's description, we can rephrase the conditions (90) as demanding that the Young diagram

$$(N'_\pi - a, \alpha_a - (a - 1), \alpha_{a-1} - (a - 2), \dots, \alpha_1)$$

can be obtained by adding a row to $(\alpha'_{a'} - (a' - 1), \alpha'_{a'-1} - (a' - 2), \dots, \alpha'_1)$ or, respectively, that the Young diagram

$$(N'_\pi - b, \beta_b - (b - 1), \beta_{b-1} - (b - 2), \dots, \beta_1)$$

can be obtained by adding a row to $(\beta'_{b'} - (b' - 1), \beta'_{b'-1} - (b' - 2), \dots, \beta'_1)$.

Let us now suppose we are specifically working in the split case, i.e. $SO(W, B) = SO_{2m}^+(\mathbb{F}_q)$, so that $a - b$ is 2 mod 4. The summand we have identified, in the ordered symbol notation, is

$$\begin{bmatrix} \beta_1 < \dots < \beta_b < N'_\pi \\ \alpha_1 < \dots < \alpha_a \end{bmatrix} \otimes \begin{bmatrix} \alpha'_1 < \dots < \alpha'_{a'} \\ \beta'_1 < \dots < \beta'_{b'} \end{bmatrix}$$

or

$$\begin{bmatrix} \alpha_1 < \dots < \alpha_a < N'_\pi \\ \beta_1 < \dots < \beta_b \end{bmatrix} \otimes \begin{bmatrix} \beta'_1 < \dots < \beta'_{b'} \\ \alpha'_1 < \dots < \alpha'_{a'} \end{bmatrix}$$

(with the above corresponding restrictions on α'_i and β'_j), respectively. We can therefore see that by re-labelling the top rows of the symbols as λ and λ' and the bottom rows as μ and μ' , we exactly recover the conditions Pan described. The non-split case of $O_{2m}^-(\mathbb{F}_q)$ proceeds similarly.

Now let us consider $(V, (W, B))$ in the orthogonal stable or metastable range. We use the zeta correspondence and its extension in this case. From the decomposition given in Theorem 3.5.2, we find the unipotent part of the restriction of $\omega[V \otimes W]$ to $\mathrm{Sp}(V) \times O(W, B)$ is the sum over $k = 0, \dots, N$ and over every unipotent irreducible representation ρ of $\mathrm{Sp}(V[-k])$ of summands of the form

$$\mathcal{A}_k(\rho, N'_\rho) \otimes \zeta_{V[-k]}^{W, B}(\rho).$$

Say ρ corresponds to a symbol $\begin{pmatrix} \alpha_1 < \dots < \alpha_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}$, and switch rows so that $a - b$ is 1 mod 4. Then $N'_\rho = N - m + \frac{a+b-1}{2}$. Say $\zeta_V^{W, B}(\rho)$ is constructible. Then, in the split case $O(W, B) = O_{2m}^+(\mathbb{F}_q)$, we get that the underlying $SO_{2m}^\pm(\mathbb{F}_q)$ symbol is

$$\begin{pmatrix} \alpha_1 < \dots < \alpha_a < N'_\rho \\ \mu_1 < \dots < \mu_b \end{pmatrix},$$

with determined central sign data. Recalling the above description of the Pieri rule, the summands surviving in $\mathcal{A}_k(\rho, N'_\rho)$ consist of symbols

$(\alpha'_1 < \dots < \alpha'_{a'})_{\mu_1 < \dots < \mu'_{b'}}$ such that the Young diagram with row lengths $(\alpha'_{a'} - (a' - 1), \alpha'_{a'-1} - (a' - 2), \dots, \alpha'_1)$ can be obtained from adding a row to $(\alpha_a - (a - 1), \alpha_{a-1} - (a - 2), \dots, \alpha_1)$ such that $\alpha_{a'} - (a' - 1) \leq N'_\rho - a$ and such that the Young diagram with row lengths $(\beta'_{b'} - (b' - 1), \beta'_{b'-1} - (b' - 2), \dots, \beta'_1)$ can be obtained from adding a row to $(\beta_b - (b - 1), \beta_{b-1} - (b - 2), \dots, \beta_1)$. Again, the condition on the top row is equivalent to requiring that the Young diagram

$$(N'_\rho - a, \alpha_a - (a - 1), \alpha_{a-1} - (a - 2), \dots, \alpha_1)$$

can be obtained by adding a row to $(\alpha'_{a'} - (a' - 1), \alpha'_{a'-1} - (a' - 2), \dots, \alpha'_1)$.

In the ordered Lusztig symbol notation, these term we have identified are of the form

$$\left[\begin{array}{c} \alpha'_1 < \dots < \alpha'_{a'} \\ \beta'_1 < \dots < \beta'_{b'} \end{array} \right] \otimes \left[\begin{array}{c} \beta_1 < \dots < \beta_b \\ \alpha_1 < \dots < \alpha_a < N'_\rho \end{array} \right]$$

and we can see that by re-labelling the entries of the top row as λ and λ' and the bottom row as μ and μ' , our conditions for α'_i and β'_j are precisely those stated by Pan. The non-split case of $O(W, B) = O_{2m}^-(\mathbb{F}_q)$ proceeds similarly.

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