HOWE DUALITY OVER FINITE FIELDS III: FULL COMPUTATION AND THE GUREVICH-HOWE CONJECTURES

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Abstract. In this third paper in a series on type I Howe duality for finite fields, we give a complete description of the restriction of the oscillator representation over a finite field to products of dual pairs of symplectic and orthogonal groups in all cases that occur. In particular, this gives an inductive construction of all irreducible complex representations of finite symplectic and orthogonal groups. We also give a proof of the Gurevich-Howe rank and exhaustion conjectures for type I pairs.

Contents

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1. Introduction

The purpose of this paper is to complete the explicit description, started in the previous papers [30, 31], of the decomposition of the oscillator representation of a symplectic group over a finite field into irreducible representations of a product of a dual pair of type I, which consists of a symplectic and an orthogonal group. This problem was suggested by R. Howe in [18]. Since we treat all the applicable cases, this, in particular, gives an inductive construction of all irreducible complex representations of finite symplectic and orthogonal groups. As an application, we prove the Gurevich-Howe rank conjecture [15], Conjecture 0.3.8 on the coincidence of tensor rank and U-rank, and the exhaustion conjecture [15], Conjecture 0.4.12, which follows.

The part of this program completed in [30, 31] was to treat the so-called *stable ranges*, where the rank of one of the groups in the pair is much greater than the other. In [30], the general form of that case of the decomposition was established, in terms of certain correspondences between the sets of irreducible representations of symplectic and orthogonal groups, and, in [31], it was completely described in terms of Lusztig's classification of irreducible representations of finite groups of Lie type.

The strategy of this paper is to break up the remaining cases into two *metastable ranges*. In the metastable ranges, the stable picture breaks down in two ways, both of which are related to the occurence of *generalized Lusztig symbols*, which relax some of the defining conditions of a Lusztig symbol. One type of generalized symbol predicts a 0-dimensional representation - those terms are simply omitted. However, one can also encounter alternating sums of induction terms coupled with generalized Lusztig symbols of the same dimension. These are first shown to be genuine (as opposed to virtual) representations. This is done in Section 4. The other step is to compute them completely, which is done in Section 6.

To describe our results more concretely, we need some notation. Consider a finite field $\mathbb{F}_q$ of characteristic not equal to 2, a symplectic $\mathbb{F}_q$-vector space V, and an $\mathbb{F}_q$-vector space W with a non-degenerate symmetric bilinear form B. In this finite field context, S. Gurevich and R. Howe (see e.g. [15]) proposed the problem of describing explicitly the restriction of the oscillator representation of $Sp(V \otimes W)$ to the subgroup $Sp(V) \times O(W,B)$. The previous two papers in this series [30, 31] described this decomposition explicitly in the *symplectic stable range* (i.e. where $dim(V) \geq 2dim(W)$) and the *orthogonal stable range* (where $dim(V)$ is less than or equal to the dimension of the maximal isotropic subspace of W).

Denote by $\widehat{G}$ the set of isomorphism classes of irreducible complex representations of a finite group G. Then in the symplectic resp. orthogonal stable range, there are correspondences

$$\eta^V_{W,B} : \widehat{O(W,B)} \hookrightarrow \widehat{Sp(V)} \tag{1}$$

$$\zeta^{W,B}_V : \widehat{Sp(V)} \hookrightarrow \widehat{O(W,B)} \tag{2}$$

(explicitly constructed in [30] and described in [31] in terms of G. Lusztig's classification of irreducible representations of finite groups of Lie type, see e.g. [36]) so that the Gurevich-Howe decomposition is a direct sum of terms of the form

$$\rho \otimes \eta^V_{W,B}(\rho) \tag{3}$$

resp.

$$\zeta^{W,B}_V(\pi) \otimes \pi \tag{4}$$

and "degenerate terms," which can be explicitly descried as tensor products analogous to (3), (4) involving parabolic Verma modules induced from smaller choices of V, W.

The purpose of the present paper is to describe the Gurevich-Howe decomposition in the remaining cases, broken up into two *metastable ranges* which, together with the stable ranges, cover all the cases of V, W. The precise definition of the metastable ranges is technical and will be given in Subsection 3.1 below (Definition 3.1.1).

Our main result can now be stated in broad terms as follows:

Theorem 1. *Consider a type I reductive dual pair $(Sp(V), O(W,B))$. Then in the symplectic resp. orthogonal metastable ranges, there are correspondences*

$$\eta^V_{W,B} : \widehat{O(W,B)} \to \widehat{Sp(V)} \cup \{0\} \tag{5}$$

$$\zeta_V^{W,B} : \widehat{Sp(V)} \to \widehat{O(W,B)} \cup \{0\}, \tag{6}$$

explicitly described in terms of Lusztig's classification, such that the restriction of the oscillator representation of $Sp(V \otimes W)$ to $Sp(V) \times O(W,B)$ is a direct sum of terms of the form (3) resp. (4) and explicitly described as tensor products analogous to (3) resp. (4) involving alternating sums of parabolic Verma modules for smaller choices of V, W, each of which adds up to a linear combination of irreducible representations with positive integral coefficients. The alternating sums are explicitly resolved as sums of irreducible representations in terms of Lusztig's classification.

We restate this more precisely after we have introduced the necessary notation, in Theorems 3.5.1 and 3.5.2 below. The description of the alternating sum coefficients appearing with the eta and zeta correspondence is given in Theorem 6.1.1.

One important application of our description of the eta correspondence is that it can be used to answer questions related the character theory of finite symplectic groups (see [15, 16]). In particular, in Section 5, we prove the Gurevich-Howe *rank conjecture*, which predict the equality of a two kinds of "ranks" for certain representations. The first notion of rank defined by Gurevich and Howe is called *U-rank* and is defined as the maximal "rank" of a character in the restriction of a representation to the Siegel unipotent subgroup of a symplectic group. We denote it by rk_U. The second notion of rank is called *tensor rank* and is defined to be the minimal natural number k such that every irreducible summand of the input representation is contained in a tensor product of less than or equal to k oscillator representations. We denote it by $rk_\otimes$. The main result of Section 5 is that Conjecture 0.3.8 of [15] holds:

Theorem 2. *(The Gurevich-Howe Rank Conjecture,* [15]*, Conjecture 0.3.8) For an irreducible representation ρ of a finite symplectic group $Sp_{2N}(\mathbb{F}_q)$ (for q an odd prime power), if the U-rank of ρ is strictly less than N, then it agrees with the tensor rank of ρ:*

$$rk_U(\rho) = rk_\otimes(\rho). \tag{7}$$

We then also have:

Corollary 1. *(The Gurevich-Howe Exhaustion Conjecture,* [15]*, Conjecture 0.4.12) For every choice of $0 < n < N$, every irreducible $Sp_{2N}(\mathbb{F}_q)$-representation of U-rank n is produced in the image of an* *eta correspondence* $\eta_{W,B}^{\mathbb{F}_q^{2N}}$ *for one of the two choices of orthogonal spaces* (W,B) *of dimension* n.

Proof. In [15], Theorem 0.4.13, it is proved that every irreducible representation of $Sp_{2N}(\mathbb{F}_q)$ with tensor rank n is produced in the image of such an eta correspondence. Thus, since Theorem 2 states that, in this range, tensor rank is always equal to U-rank, the exhaustion conjecture follows immediately. □

The correspondences (5), (6) are not formal extensions of the correspondences (1), (2) to the metastable range. If we extended the correspondences (1), (2) formally, we would obtain "generalized Lusztig symbols" which could either contain repeated terms (these translate to the term $\{0\}$ in (5), (6)) or non-monotone terms; those are replaced by different genuine Lusztig symbols. The reason for working in the two metastable ranges is to avoid the appearance of illegal Lusztig symbols with negative terms, which are more complicated to resolve.

Our method for proving Theorem 1 is interpolation of semisimple pre-Tannakian categories [6, 7, 17, 27, 28, 29]. There are other possible methods for approaching this problem. Notably, analogously as the general linear group is embedded into the multiplicative semigroup of all square matrices, the symplectic group is embedded into the *oscillaor semigroup* discovered by Howe [18, 15, 16, 20]. Using this semigroup, one can also obtain results on the Gurevich-Howe decomposition beyond the stable ranges (see e.g. Proposition 8.2.1 of [15]).

Since (5) and (6) cover all the cases of V and W, it is possible to use Theorem 1 for an inductive construction of all irreducible representations of $Sp(V)$, $O(W,B)$ (see Subsection 4.5).

This paper is organized as follows: In Section 2, we discuss some background of the oscillator representations and the relevant cases of Lusztig's classification of irreducible representations of finite groups of Lie type. In Section 3, we describe the metastable ranges, construct an eta or zeta correspondence for every case of type I reductive dual pair, and discuss the alternating sums replacing parabolic induction in the metastable range. This establishes the necessary notation to restate Theorem 1 in concrete terms. In Section 4, we discuss interpolated representation categories and the analogues of the results of [30, 31], prove Theorem 1, and discuss some representation-theoretical implications. In Section 5, we discuss the Gurevich-Howe rank conjecture, proving Theorem 2. In Section 6, we resolve the alternating sums appearing as coefficients in the decomposition of the restricted oscillator representation.

2. Background

We begin by recalling the results of [30, 31] more precisely.

First, we fix notation: For a symplectic group $Sp(\mathbf{V})$, we write $\omega_a[\mathbf{V}]$ to denote the *oscillator representation* arising from the Weil-Shale representation of the Heisenberg group on $\mathbf{V}$ with central (non-trivial, additive) character in $\mathbb{F}_q$ corresponding to $a \in \mathbb{F}_q^\times$ under a fixed identification of $\mathbb{F}_q$ with its Pontrjagin dual. In the case of $a = 1$, we omit the subscript and write $\omega[\mathbf{V}] = \omega_1[\mathbf{V}]$.

Now let us consider a type I reductive dual pair of subgroups of $Sp(\mathbf{V})$, which must be of the form

$$(Sp(V), O(W, B)) \subseteq Sp(\mathbf{V}), \tag{8}$$

for $\mathbb{F}_q$-spaces V and W, say with symplectic and symmetric bilinear forms S and B respectively, so that $\mathbf{V} = V \otimes W$, and we consider its symplectic form to be $S \otimes B$(see, for example, [18]). Tensoring matrices gives an inclusion of the product $Sp(V) \times O(W, B)$ into $Sp(\mathbf{V})$. Note that if there exists a k-dimensional isotropic subspace of W (resp. V), then B (resp. S) can be expressed as a direct sum of k copies of a hyperbolic 2-dimensional symmetric bilinear (resp. symplectic) form with a form $B[-k]$ (resp. $S[-k]$) on a $dim(W) - 2k$-dimensional $W[-k]$ (resp. $dim(V) - 2k$-dimensional space $V[-k]$). If k is a dimension of an isotropic subspace, we write P_k^B (resp. P_k^V) for the maximal parabolic subgroup of $O(W, B)$ (resp. $Sp(V)$) with Levi factor $GL_k(\mathbb{F}_q) \times O(W[-k], B[-k])$ (resp. $GL_k(\mathbb{F}_q) \times Sp(V[-k])$).

In [30], we defined two stable ranges of such type I reductive dual pairs: We say a pair (8) is in the *symplectic stable range* if $dim(W) \leq dim(V)$, and similarly, we say it is in the *orthogonal stable range* if the dimension of V is less than or equal to the dimension of a maximal isotropic subspace of W with respect B. Let us denote by h_W the dimension of a maximal isotropic subspace of W. We proved that, for $(Sp(V), O(W, B))$ in the symplectic stable range, the restriction of $\omega[V \otimes W]$ to a $Sp(V) \times O(W, B)$-representation is

$$\bigoplus_{k=0}^{h_W} \bigoplus_{\rho \in \widehat{O(W[-k], B[-k])}} \eta^V(\rho) \otimes \mathrm{Ind}^{P_k^B}(\rho \otimes \epsilon(\det)) \tag{9}$$

for a system of mutually disjoint injections

$$\eta_{W,B}^{V} : \widehat{O(W,B)} \hookrightarrow \widehat{Sp(V)}$$

called the *eta correspondence* (ommiting the subscript when the source is determined). See also, for this case, the original papers of S. Gurevich and R. Howe finding the eta correspondence [15, 16] and an approach. Similarly, for $(Sp(V), O(W,B))$ in the orthogonal stable range, writing $dim(V) = 2N$, the restriction of $\omega[V \otimes W]$ to $Sp(V) \times O(W,B)$ decomposes as

$$\bigoplus_{k=0}^{N} \bigoplus_{\rho \in \widehat{Sp(V[-k])}} \mathrm{Ind}^{P_k^V}(\rho \otimes \epsilon(\det)) \otimes \zeta^{W,B}(\rho) \tag{10}$$

for a system of mutually disjoint injections

$$\zeta_{V}^{W,B} : \widehat{Sp(V)} \hookrightarrow \widehat{O(W,B)}$$

called the *zeta correspondence* (again, ommiting the subscript when the source is determined).

Therefore, in either stable range, the problem of Howe duality reduces to explicitly computing the eta and zeta correspondences, which we was the main result [31], using Lusztig's classification of irreducible representations: Recall that, broadly, an irreducible representation of a finite group of Lie type is classified by data consisting of a conjugacy class of a semisimple element (s) in the dual group G^D (the "semisimple part"), a unipotent representation of (the dual of) the centralizer of s in G^D, and possible "central sign data" when $Z(G)$ is disconnected; we discuss this in more detail in Section 2 below. For $(Sp(V), O(W,B))$ in the symplectic stable range, our computation of the eta correspondence

$$\eta_{W,B}^{V} : \widehat{O(W,B)} \to \widehat{Sp(V)}$$

can be summarized as transforming the Lusztig data of an irreducible representation of $O(W,B)$ into Lusztig data specifying an irreducible representation of $Sp(V)$ by

- Adding an appropriate number of -1 eigenvalues (and a single 1 eigenvalue, with position depending on the action of $\mathbb{Z}/2 \subseteq O(W,B)$) to the semisimple part if $dim(W)$ is odd, and 1 eigenvalues if $dim(W)$ is even.
- Altering the unipotent part by considering the single changed factor of the centralizer of the new semisimple part (corresponding to -1 eigenvalues if $dim(W)$ is odd and 1 eigenvalues of the

semisimple part if $dim(W)$) and adding a single coordinate to the Lusztig symbol to get the appropriate new rank and defect.

- Central sign data is determined by the quadratic character applied to the semisimple part (as a torus element) multiplied by the discriminant of B when $dim(W)$ is odd, and the central sign data of $SO(W,B)$ when $dim(W)$ is even.

Similarly, for $(Sp(V), O(W,B))$ in the orthogonal stable range, our construction of the zeta correspondence

$$\zeta_V^{W,B} : \widehat{Sp(V)} \to \widehat{O(W,B)}$$

can be summarized by altering the Lusztig data of an irreducible representation of $\widehat{Sp(V)}$ by adding -1 eigenvalues to the semisimple part if $dim(W)$ is odd, 1 eigenvalues if $dim(W)$ is even, altering the affected factor of the unipotent part by adding a single appropriate coordinate, and assigning central sign data determined by the quadratic character of the original semisimple part and $disc(B)$ or the original central sign data.

To be even more precise, we need to also recall Lusztig's classification. Consider a general finite group of Lie type G. Consider a conjugacy class (s) of a semisimple element of the dual group $s \in G^D$. Consider also an irreducible unipotent representation u of the dual of s's centralizer

$$u \in \widehat{(Z_{G^D}(s))^D}_u.$$

This data determines a G-representation we denote by $\rho_{(s),u}$ of dimension

$$dim(\rho_{(s),u}) = \frac{|G|_{q'}}{|(Z_{G^D}(s))^D|_{q'}} \cdot dim(u)$$

(recall that the order of a group of Lie type is equal to the order of its dual). In the case when G has connected center (e.g. $G = SO_{2m+1}(\mathbb{F}_q)$), the $\rho_{(s),u}$ are precisely the irreducible G-representations.

We will also want to consider cases of G with non-trivial center (e.g. $G = Sp_{2N}(\mathbb{F}_q)$ or $O^{\pm}_{2m}(\mathbb{F}_q)$). In these cases, the representation $\rho_{(s),u}$ may split into two non-isomorphic irreducible summands

$$\rho_{(s),u} = \rho_{(s),u,+1} \oplus \rho_{(s),u,-1} \tag{11}$$

of dimensions

$$dim(\rho_{(s),u,+1}) = dim(\rho_{(s),u,-1}) = \frac{dim(\rho_{(s),u})}{2}$$

where the ± 1 corresponds to specifying an action of $Z(G) = \mathbb{Z}/2$ (with $+1$ corresponding to the trivial action and -1 corresponding to the sign). In all other cases $\rho_{(s),u}$ remains irreducible, giving all irreducible representations of G. For example, in the case of $G = Sp_{2N}(\mathbb{F}_q)$, $\rho_{(s),u}$ decomposes as in (11) if and only if s has -1 as an eigenvalue. If $\rho_{(s),u}$ or $\rho_{(s),u,\pm 1}$ is an irreducible G-representation, say $[(s), u]$ or $[(s), u, \pm 1]$ is its corresponding Lusztig classification data.

2.1. **The semisimple data.** The data of a conjugacy class of a semisimple element in a finite group of Lie type is equivalent to the data of its eigenvalues (under the action of the Weyl group). In any symplectic or special orthogonal group, every maximal torus is isomorphic to a product of $SO_2^{\pm}$ factors, possibly on field extensions of $\mathbb{F}_q$

$$T \cong \prod_{i=1}^{k} SO_2^{\pm}(\mathbb{F}_{q^{n_i}}) \tag{12}$$

such that $n_1 + \cdots + n_k$ adds up to the total rank, and in the case of tori in even special orthogonal groups, the sign $\pm$ denoting whether or not the full group is equal to the product of signs appearing in each $SO_2^{\pm}$ factor. Note that each factor is cyclic

$$SO_2^{\pm}(\mathbb{F}_{q^n}) \cong \mu_{q^n \mp 1},$$

giving an identification of T with its Pontrjagin dual. Write $\chi_{(s)}$ for the character of T corresponding to a semisimple $(s) \in T$. Then if, for some G, we consider Lusztig classification data consisting of semisimple part $(s) \in G^D$ and unipotent part u, we have

$$\rho_{(s),u} \subseteq \mathrm{Ind}_G^T(\chi_{(s)}) \tag{13}$$

(noting on the right hand side that, writing (12), each T is self-dual and the maximal tori in G are identified with those in G^D).

Given a choice of semisimple part of Lusztig classification data consisting of a conjugacy class $(s) \in G^D$ before proceeding further with describing the rest of the Lusztig classification data, we must write down its centralizer $Z_{G^D}(s)$. To compute this, consider s as an element of some maximal torus T as in (12), minimizing n_i where possible. Coordinates $\pm 1 \neq \lambda \in SO_2^{\pm}(\mathbb{F}_{q^n})$ of multiplicity j give a centralizer factor $U_j^{\pm}(\mathbb{F}_{q^n})$ (where we use the notation $U_j^{+} = GL_j$). Eigenvalues $\pm 1 \in SO_2^{\pm}(\mathbb{F}_q)$ give centralizer factors depending on the specific choice of G^D and G.

If $G^D = Sp_{2r}(\mathbb{F}_q)$ (to describe representations of $G = SO_{2r+1}(\mathbb{F}_q)$), a semisimple $s \in G$ has centralizer

$$Z_{Sp_{2r}(\mathbb{F}_q)}(s) = \prod_{i=1}^{k} U^{\pm}_{j_i}(\mathbb{F}_{q^{n_i}}) \times Sp_{2p}(\mathbb{F}_q) \times Sp_{2\ell}(\mathbb{F}_q) \tag{14}$$

where s has 1 as an eigenvalue of multiplicity $2p$ and -1 as an eigenvalue of multiplicity 2ℓ.

If $G^D = SO_{2r+1}(\mathbb{F}_q)$ (to describe representations of $G = Sp_{2r}(\mathbb{F}_q)$), a semisimple element $s \in G$ has centralizer

$$Z_{SO_{2r+1}(\mathbb{F}_q)}(s) = \prod_{i=1}^{k} U^{\pm}_{j_i}(\mathbb{F}_{q^{n_i}}) \times SO_{2p+1}(\mathbb{F}_q) \times SO^{\pm}_{2\ell}(\mathbb{F}_q) \tag{15}$$

if s has 1 as an eigenvalue of multiplicity $2p+1$ and -1 as an eigenvalue of multiplicity 2ℓ (recall that a single 1 eigenvalue is automatic in this case, since it must be added to embed $T \subset SO_{2r+1}(\mathbb{F}_q)$). The sign of the final factor $SO^{\pm}_{2\ell}(\mathbb{F}_q)$ corresponds to whether we pick $T \subset SO^{+}_{2r}(\mathbb{F}_q)$ or $SO^{-}_{2r}(\mathbb{F}_q)$. We specifically will later consider the semisimple conjugacy classes (σ_r^+) and (σ_r^-) which both have -1 as an eigenvalue of multiplicity $2r$ and a single (automatic) 1 eigenvalue, placed so that $\sigma_r^{\pm}$ is in a torus of the form (12) such that the product of all involved signs of SO_2 is equal to $\pm$. These elements satisfy

$$Z_{SO_{2r+1}(\mathbb{F}_q)}(\sigma_r^{\pm}) = SO^{\pm}_{2r}(\mathbb{F}_q).$$

If $G^D = SO^{\pm}_{2r}(\mathbb{F}_q)$ (to describe representations of $G = G^D = SO^{\pm}_{2r}(\mathbb{F}_q)$), a semisimple $s \in G$ has centralizer

$$Z_{SO^{\pm}_{2r}(\mathbb{F}_q)}(s) = \prod_{i=1}^{k} U^{\pm}_{j_i}(\mathbb{F}_{q^{n_i}}) \times SO^{\pm}_{2p}(\mathbb{F}_q) \times SO^{\pm}_{2\ell}(\mathbb{F}_q) \tag{16}$$

with signs chosen so that their product is the total sign of G, where s has 1 as an eigenvalue of multiplicity $2p$ and -1 as an eigenvalue of multiplicity 2ℓ. For more details, see [4].

2.2. **The unipotent data.** Given a choice of $(s) \in G^D$, the next part of Lusztig classification data is a unipotent representation u of the dual of the centralizer of s $(Z_{G^D}(s))^D$, which by (14), (15), (16), we can write down as a product of unitary groupds and a pair of factors of B, C, D or ${}^2D$-type. We may consider u as a tensor product of irreducible unipotent representations of each factor. It will turn out (recalling the constructions in [31]) that only the tensor factor of u corresponding to one of these final two factors will be "altered" in the description of η or ζ. Therefore, we describe in this subsection the classification of unipotent representations of $Sp_{2r}(\mathbb{F}_q)$, $SO_{2r+1}(\mathbb{F}_q)$, and $SO^{\pm}_{2r}(\mathbb{F}_q)$, using symbols.

The classification of irreducible unipotent representations of $Sp_{2r}(\mathbb{F}_q)$ and $SO_{2r+1}(\mathbb{F}_q)$ are the same, since they are dual groups. A *symbol of B- or C-type* and rank r is defined to be a pair of increasing sequences

$$\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$$

for $\lambda_i, \mu_i \in \mathbb{Z}_{\geq 0}$ such that $(\lambda_1, \mu_1) \neq (0,0)$, $a-b$ is odd (the "defect condition"), and

$$\sum_{i=1}^{a} \lambda_i + \sum_{i=1}^{b} \mu_i = r + \frac{(a+b-1)^2}{4}$$

(the "rank condition"). We take switching rows to give the same symbol. The irreducible unipotents are in bijective correspondence with this combinatorial data (and we denote the representation corresponding to a symbol by the symbol itself).

Similarly, for the case of $SO^{+}_{2r}(\mathbb{F}_q)$ (resp. $SO^{-}_{2r}(\mathbb{F}_q)$), irreducible unipotent representations correspond to symbols of D- (resp. ${}^2D$-type) and rank r, which are defined to consist of pairs of increasing sequences $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ for $\lambda_i, \mu_i \in \mathbb{Z}_{\geq 0}$ such that $(\lambda_1, \mu_1) \neq (0,0)$, the "defect condition" $a-b \equiv 0 \bmod 4$ (resp. $\equiv 2 \bmod 4$, for $SO^{-}_{2r}(\mathbb{F}_q)$) and the "rank condition"

$$\sum_{i=1}^{a} \lambda_i + \sum_{i=1}^{b} \mu_i = r + \frac{(a+b)(a+b-2)}{4}$$

(the same rank condition is used for $SO^{-}_{2r}(\mathbb{F}_q)$). Again, we denote a unipotent representation the same as its corresponding symbol.

Further, the dimension of a unipotent representation $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ of a symplectic or special orthogonal group G can be calculated as the following formula

$$\frac{\prod\limits_{1\leq i<j\leq a} (q^{\lambda_j} - q^{\lambda_i}) \cdot \prod\limits_{1\leq i<j\leq b} (q^{\mu_j} - q^{\mu_i}) \cdot \prod\limits_{1\leq i\leq a, 1\leq j\leq b} (q^{\lambda_i} + q^{\mu_j}) \cdot |G|_{q'}}{2^c \cdot \prod\limits_{1\leq i\leq a} \prod\limits_{j=1}^{\lambda_i} (q^{2j}-1) \cdot \prod\limits_{1\leq i\leq b} \prod\limits_{j=1}^{\mu_i} (q^{2j}-1) \cdot q^{f(a,b)}}$$

where $c = (a+b-1)/2$ if $G = SO_{2r+1}(\mathbb{F}_q)$ or $Sp_{2r}(\mathbb{F}_q)$ and $c = (a+b-2)/2$ if $G = SO^{\pm}_{2r}(\mathbb{F}_q)$, and we write

$$f(a,b) = \sum_{i=1}^{\lfloor (a+b)/2 \rfloor} \binom{a+b-2i}{2}.$$

2.3. **Central sign data.** The above subsections give us all the necessary information about the representation theory of $SO_{2m+1}(\mathbb{F}_q)$, and therefore also $O_{2m+1}(\mathbb{F}_q) = \mathbb{Z}/2 \times SO_{2m+1}(\mathbb{F}_q)$. However, $Sp_{2r}(\mathbb{F}_q)$ and $SO^{\pm}_{2r}(\mathbb{F}_q)$ (and especially $O^{\pm}_{2r}(\mathbb{F}_q) = \mathbb{Z}/2 \ltimes SO^{\pm}_{2r}(\mathbb{F}_q)$, which is the case we ultimately need to consider) have disconnected center, and therefore we also need to consider *central sign data*.

For $Sp_{2r}(\mathbb{F}_q)$, the center is $\mathbb{Z}/2$, and we have a splitting

$$\rho_{(s),u} = \rho_{(s),u,+1} \oplus \rho_{(s),u,-1}$$

if and only if s has any -1 eigenvalues, in which case

$$dim(\rho_{(s),u,\pm 1}) = \frac{dim(\rho_{(s),u})}{2}.$$

Let us now consider $O^{\pm}_{2r}(\mathbb{F}_q)$-representations. Suppose $\rho \in \widehat{O^{\pm}_{2r}(\mathbb{F}_q)}$. Then ρ is classified according to two effects

(1) If ρ is a summand of a toral induction $Ind^{T}_{O^{\pm}_{2r}(\mathbb{F}_q)}(\chi)$ for some character $\chi \neq \chi^{-1}$ in the Pontrjagin dual of T, then the restriction $Res_{SO^{\pm}_{2r}(\mathbb{F}_q)}(\rho)$ splits into two irreducible $SO^{\pm}_{2r}(\mathbb{F}_q)$, with Lusztig classification data for each piece being the same, except semisimple parts are two distinct conjugacy classes of semisimple elements in $SO^{\pm}_{2r}(\mathbb{F}_q)$ which are conjugate in $O^{\pm}_{2r}(\mathbb{F}_q)$.

(2) If ρ is a summand of $Ind^{T}_{O^{\pm}_{2r}(\mathbb{F}_q)}(\chi)$, then
   - If χ has any ± 1 eigenvalues (and no ∓ 1 eigenvalues), then there exists precisely one other irreducible $O^{\pm}_{2r}(\mathbb{F}_q)$-representation ρ' such that
   $$Res_{SO^{\pm}_{2r}(\mathbb{F}_q)}(\rho) = Res_{SO^{\pm}_{2r}(\mathbb{F}_q)}(\rho')$$
   (i.e. this $SO^{\pm}_{2r}(\mathbb{F}_q)$-representation corresponds to a pair of irreducible $O^{\pm}_{2r}(\mathbb{F}_q)$-representations.
   - If χ has both ± 1 eigenvalues, then there exists precisely three other distinct irreducible $O^{\pm}_{2r}(\mathbb{F}_q)$-representations with the same restriction to $SO^{\pm}_{2r}(\mathbb{F}_q)$ as ρ (i.e. this $SO^{\pm}_{2r}(\mathbb{F}_q)$-representation corresponds to four distinct irreducible $O^{\pm}_{2r}(\mathbb{F}_q)$-representations.

(in this case, we must specify signs to describe the action of the center $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$ of $O^{\pm}_{2r}(\mathbb{F}_q)$).

In summary, an irreducible representation of $O^{\pm}_{2r}(\mathbb{F}_q)$ corresponds to the data of

- a semisimple conjugacy class (s) in $O^{\pm}_{2r}(\mathbb{F}_q)$
- a unipotent representation u of $(Z_{SO^{\pm}_{2r}}(s))^D$
- a sign ± 1 if s has 1 eigenvalues and an (indepedently chosen) sign ± 1 if s has -1 eigenvalues.

Call this the Lusztig classification data of an irreducible representation of $O^{\pm}_{2r}$. Denote the corresponding irreducible representation by

$$\rho_{(s),u,(\pm 1, \pm 1)},$$

writing in $(\pm 1, \pm 1)$ the sign arising from $+1$ eigenvalues of s first and the sign arising from -1 eigenvalues of s second, and removing either if s has no such eigenvalues.

3. The general statements

Finally, in this section we precisely state the constructions of the extended eta and zeta correspondences, define the alternating sums of parabolic inductions, and state Theorem 1.

In Subsection 3.1, we precisely define the symplectic and orthogonal metastable ranges. In Subsections 3.2 and 3.3, we describe the constructions of the extended eta and zeta correspondences in terms of Lusztig classification data. The constructions are entirely the same as the constructions of the eta and zeta correspondences in the symplectic and orthogonal stable ranges, when they can be applied. For input represenations where the construction cannot be applied (specifically, where the step of concatenating a new coordinate to a symbol factor in the unipotent part of the Lusztig classification fails to give a legal symbol), we set the extended eta and zeta correspondences to output 0. In Subsection 3.4, we define the alternating sums of Verma modules that play the role of coefficients for the eta and zeta correspondence terms in the decomposition of the restricted oscillator representation in Theorem 1. Finally, having prepared all the necessary notation, we restate Theorem 1 precisely in Subsection 3.5 as Theorems 3.5.1 and 3.5.2.

3.1. **The metastable ranges.** First, for a general unstable choice of symplectic and orthogonal spaces V and (W, B), we must still choose whether it is "closer" to the symplectic or orthogonal stable range.

We separate the pairs $(V, (W, B))$ which do not lie in the symplectic stable range or the orthogonal stable range into "metastable ranges" to indicate whether we intend to approach the decomposition of the restriction of $\omega[V \otimes W]$ by extending the eta or zeta correspondence. We consider the different cases of $O(W, B)$ individually.

Definition 3.1.1. *Consider a choice of symplectic and orthogonal spaces V and (W, B). Write $dim(V) = 2N$.*

- *If W is of odd dimension $dim(W) = 2m + 1$, then we say $(V, (W, B))$ is in the* symplectic metastable range *if*
$$m < N < 2m + 1.$$
Say $(V, (W, B))$ is in the orthogonal metastable range *if*
$$m < 2N \leq 2m$$
- *If W is of even dimension $dim(W) = 2m$ and B is not completely split, then we say $(V, (W, B))$ is in the* symplectic metastable range *if*
$$m \leq N < 2m.$$
Say $(V, (W, B))$ is in the orthogonal metastable range *if*
$$m - 1 < 2N < 2m.$$
- *If W is of even dimension $dim(W) = 2m$ and B is completely split, then we say $(V, (W, B))$ is in the* symplectic metastable range *if*
$$m \leq N < 2m.$$
Say $(V, (W, B))$ is in the orthogonal metastable range *if*
$$m < 2N < 2m.$$

We see that under this definition, every unstable choice of symplectic and orthogonal spaces V and (W, B) is contained in precisely one metastable range. More specifically, the disjoint union of the symplectic stable and metastable ranges consists of all choices of symplectic spaces V and orthogonal spaces (W, B) such that

$$\frac{dim(V)}{2} \geq \lfloor \frac{dim(W)}{2} \rfloor, \tag{17}$$

while the disjoint union of the orthogonal stable and metastable ranges consist of V and (W,B) satisfying the complimentary condition

$$\frac{dim(V)}{2} < \lfloor \frac{dim(W)}{2} \rfloor. \tag{18}$$

Broadly, the conditions (17) and (18) should be thought of as detecting whether it is more computationally viable to decompose the oscillator representation in terms of the eta correspondence (i.e. as a sum of distinct irreducible $Sp(V)$-representations with potentially non-irreducible $O(W,B)$-coefficients), or in terms of the zeta correspondence (i.e. as a sum of distinct irreducible $O(W,B)$-representations with potentially non-irreducible $Sp(V)$-coefficients). Concretely, in our constructions of the eta and zeta correspondence, the conditions (17) and (18) ensure that, in either case, we never attempt to concatenate a negative coordinate to a Lusztig symbol. It is possible to further extend the eta and zeta correspondences to all ranges by interpreting them to output 0 when this occurs (Lusztig's dimension formula for symbols indeed suggests that symbols with negative coordinates are 0-dimensional). However, approaching the decomposition of the oscillator representation from the "wrong side" is in general less computationally efficient, and we do not consider it for the purposes of this paper.

3.2. **The extended eta correspondence.** For a choice of symplectic and orthogonal spaces V, (W,B) which are in the disjoint union of the symplectic stable and metastable ranges, we define a map

$$\eta^V_{W,B} : \widehat{O(W,B)} \to \widehat{Sp(V)} \cup \{0\}$$

selecting the "top" irreducible representation of $Sp(V)$ whose tensor product with the input irreducible representation of $O(W,B)$ is a summand of the restricted oscillator representation (or outputting 0 when no such summand appears). More explicitly, for every $\rho \in \widehat{O(W,B)}$,

$$\eta^V_{W,B}(\rho) \otimes \rho \subset Res_{Sp(V)\times O(W,B)}(\omega[V \otimes W])$$

and there exists some $\pi \in \widehat{Sp(V)}$ with $\pi \otimes \rho$ appearing as a summand of the restricted oscillator representation if and only if $\eta^V_{W,B}(\rho) \neq 0$.

With this forumlation, we may treat $\eta^V_{W,B}$ for $(V,(W,B))$ in the symplectic stable and metastable range at the same time. In the case of symplectic stable choices of V and (W,B), this is precisely a review of the construction of the eta correspondence given in [31]. To define the map $\eta^V_{W,B}$ we treat the different cases of (W,B) separately.

Case 1: Suppose W is of odd dimension $2m+1$, and suppose $(V,(W,B))$ is either in the symplectic stable or metastable range. In this case, writing $dim(V)=2N$, this means $m<N$.

Since $dim(W)$ is odd, we can split

$$O(W,B)=SO_{2m+1}(\mathbb{F}_q)\times\mathbb{Z}/2. \tag{19}$$

For an irreducible representation $\rho\in\widehat{O(W,B)}$, we can then write it as

$$\rho\cong\rho_{(s),u}\otimes(\pm1) \tag{20}$$

for some choice of Lusztig classification data $[(s),u]$ specifying an irreducible $SO_{2m+1}(\mathbb{F}_q)$-representation and (±1) indicating a $\mathbb{Z}/2$-action. More specifically, (s) is a conjugacy class of a semisimple element $s\in Sp_{2m}(\mathbb{F}_q)=(SO_{2m+1}(\mathbb{F}_q))^D$, and u is an irreducible unipotent representation of $(Z_{Sp_{2m}(\mathbb{F}_q)}(s))^D$. Then to define $\eta^V_{W,B}(\rho)$, we must either specify Lusztig classification data for $Sp_{2N}(\mathbb{F}_q)=Sp(V)$, or put it to be 0.

The first part of Lusztig classification data for $Sp_{2N}(\mathbb{F}_q)$ is a conjugacy class of a semisimple element in $(Sp_{2N}(\mathbb{F}_q))^D=SO_{2N+1}(\mathbb{F}_q)$. Let us consider the semisimple part s of the input representation's Lusztig classification data as an element of a maximal torus

$$s\in T=\prod_{i=1}^{k}SO_2^{\pm}(\mathbb{F}_{q^{n_i}})\subseteq(SO_{2m+1}(\mathbb{F}_q))^D=Sp_{2m}(\mathbb{F}_q) \tag{21}$$

(such that $n_1+\cdots+n_k=m$). Identifying each $SO_2^{\pm}(\mathbb{F}_{q^{n_i}})$ with the cyclic group $\mu_{q^{n_i}\mp1}$, define $\epsilon(s)$ to be the product of the quadratic character $\epsilon:\mu_{q^{n_i}\mp1}\to\{\pm1\}$ applied to each coordinate of s in (21). Taking a product with

$$\sigma^{\pm}_{N-m}\in SO_{2(N-m)+1}(\mathbb{F}_q)$$

(recalling that $N>m$ by the range conditions) gives two options of a conjugacy class of a semisimple element in $SO_{2N+1}(\mathbb{F}_q)$. Write

$$\phi^{\pm}(s)=s\oplus\sigma^{\pm}_{N-m}\in SO_{2N+1}(\mathbb{F}_q).$$

We choose the sign of $\pm$ according to the sign of the (±1) factor indicating the $\mathbb{Z}/2$-action in (20). Suppose s has -1 as an eigenvalue of multiplicity 2ℓ, and write its centralizer as

$$Z_{Sp_{2m}(\mathbb{F}_q)}(s)=H\times Sp_{2\ell}(\mathbb{F}_q). \tag{22}$$

Then $\phi^{\pm}(s)$ has -1 as an eigenvalue of multiplicity $2(N-m+\ell)$ and its centralizer is

$$Z_{2O_{2N+1}(\mathbb{F}_q)}(\phi^{\pm}(s))=H^D\times SO^{\pm}_{2(N-m+\ell)}(\mathbb{F}_q).$$

Next, writing (22), and then $Z_{Sp_{2m}(\mathbb{F}_q)}(s)^D = H^D \times SO_{2\ell+1}(\mathbb{F}_q)$, we may consider the unipotent part u of the Lusztig classification data of the input representation as a tensor product

$$u = u_{H^D} \otimes \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$$

where, as the notation suggest, u_{H^D} is an irreducible unipotent representation of H^D, and $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ is a symbol specifying a unipotent representation of $SO_{2\ell+1}(\mathbb{F}_q)$. Since the defect $a-b$ of the symbol is odd, we may therefore permute the rows so that we may assume $a-b$ is 1 mod 4. Let us write

$$N'_\rho := N - m + \frac{a+b-1}{2}. \tag{23}$$

Now if $N'_\rho < \lambda_a$, then

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

describes a symbol of defect 2 mod 4 and rank precisely equal to $N - m + \ell$, and hence taking

$$\phi^-(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

gives a unipotent representation of

$$(Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi^-(s)))^D = H \times SO^-_{2(N-m+\ell)}(\mathbb{F}_q).$$

In this case, say $\phi^-(u)$ is constructible. Say $\phi^-(u)$ is *inconstructible* if $N'_\rho \leq \lambda_a$.

Similarly, if $N'_\rho < \mu_b$, then

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < N'_\rho \end{pmatrix}$$

describes a symbol of defect 0 mod 4 and rank precisely equal to $N - m + \ell$, and hence taking

$$\phi^+(u) = \widetilde{u^{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < N'_\rho \end{pmatrix}$$

gives a unipotent representation of

$$(Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi^+(s)))^D = H \times SO^+_{2(N-m+\ell)}(\mathbb{F}_q).$$

In this case, say $\phi^+(u)$ is constructible. Say $\phi^+(u)$ is *inconstructible* if $N'_\rho \leq \mu_b$.

Definition 3.2.1. *Assume the above notation. We define $\eta^V_{W,B}(\rho)$ to be the irreducible $Sp(V)$-representation with Lusztig classification data $[(\phi^\pm(s)), \phi^\pm(u), \epsilon(s)\cdot disc(B)]$*

$$\eta^V_{W,B}(\rho) := \rho_{(\phi^\pm(s)),\phi^\pm(u),\epsilon(s)\cdot disc(B)}, \tag{24}$$

if $\phi^\pm(u)$ is constructible. We put

$$\eta^V_{W,B}(\rho) := 0$$

if $\phi^\pm(u)$ is inconstrucible.

Case 2: Suppose W is of even dimension $2m$, and write α for the sign so that $O(W,B) = O^\alpha_{2m}(\mathbb{F}_q)$. Suppose also that $(V,(W,B))$ is either in the symplectic stable or metastable range, meaning that if we write $dim(V) = 2N$, we have $m \leq N$.

Let us consider an input irreduicble $O(W,B)$-representation ρ. Let us suppose that its $O(W,B)$-Lusztig classification data consists of a conjugacy class (s) of a semisimple element of $O(W,B)$, an irreducible unipotent representation u of, say, $(Z_{SO(W,B)}(s))^D$ (s is always conjugate to an element of $SO(W,B)$), and possible central sign data depending on which eigenvalues appear in s.

As in the previous case, consider s as an element of a torus

$$s \in T \cong \prod_{i=1}^{k} SO_2^\pm(\mathbb{F}_{q^{n_i}}).$$

We take a direct sum with the identity matrix $I_{2(N-m)+1}$ to obtain a semisimple element

$$\phi(s) = s \oplus I_{2(N-m)+1} \in SO_{2N+1}(\mathbb{F}_q),$$

adding only 1 eigenvalues to s. If s originally does not have any 1 eigenvalues, then its centralizer is altered by taking a product with a special orthogonal group factor

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi(s)) = (Z_{SO^\alpha_{2m}(\mathbb{F}_q)}(s))^D \times SO_{2(N-m)+1}(\mathbb{F}_q).$$

We consider the irreducible unipotent representation

$$\phi(u) := \widetilde{u} \otimes 1$$

of the dual of this group, tensoring $\widetilde{u}$ with the trivial representation of the new factor $(SO_{2(N-m)+1}(\mathbb{F}_q))^D$. On the other hand, if s has 1 as an eigenvalue of multiplicity $2p$ for $p > 0$, then, writing

$$Z_{SO^\alpha_{2m}(\mathbb{F}_q)}(s) = H \times SO^\pm_{2p}(\mathbb{F}_q),$$

we then have

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi(s)) = H^D \times SO_{2(N-m+p)+1}(\mathbb{F}_q)$$

(note that in this case $H = H^D$). Write $(Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(s))^D = H^D \times SO^{\pm}_{2p}(\mathbb{F}_q)$, and

$$u = u_{H^D} \otimes \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$$

for an irreducible unipotent representation u_{H^D} of H^D, and a symbol $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ of $SO^{\pm}_{2p}(\mathbb{F}_q)$. We may switch the symbol's rows so that for the minimal i such that $\lambda_{a-i} \neq \mu_{b-i}$, we have $\lambda_{a-i} < \mu_{b-i}$. Write

$$N'_\rho = N - m + \frac{a+b}{2}.$$

Then

$$\binom{\lambda_1 < \cdots < \lambda_a < N'_\rho}{\mu_1 < \cdots < \mu_b}, \qquad \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b < N'_\rho}$$

define unipotent $(SO_{2(N-m+p)+1}(\mathbb{F}_q))^D = Sp_{2(N-m+p)}(\mathbb{F}_q)$. Therefore, if $\lambda_a < N'_\rho$, $\mu_b < N'_\rho$,

$$\phi^+(u) = \widetilde{u_{H^D}} \otimes \binom{\lambda_1 < \cdots < \lambda_a < N'_\rho}{\mu_1 < \cdots < \mu_b}$$

$$\phi^-(u) = \widetilde{u_{H^D}} \otimes \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b < N'_\rho}$$

respectively define irreducible unipotent representations of the dual group $(Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi(s)))^D = H \times Sp_{2(N-m+p)}(\mathbb{F}_q)$. Say $\phi^+(u)$, $\phi^-(u)$ are *inconstructible* if $\lambda_a \geq N'_\rho$, $\mu_b \geq N'_\rho$, respectively. We choose the sign of $\phi^\pm(u)$ according to the central sign data (± 1) chosen from s having 1 eigenvalues.

Finally, to define an output irreducible $Sp_{2N}(\mathbb{F}_q)$-representation, we need to also choose output central sign data if $\phi(s)$ has -1 eigenvalues. By definition, $\phi(s)$ has the same number of -1 eigenvalues as s. Therefore, in this case, the original s has -1 eigenvalues also, so the $O(W,B)$-Lusztig classification data supplies us with the data of one more central sign ± 1, which we use as the output central sign data.

Definition 3.2.2. *Assume the above notation. We define $\eta^V_{W,B}(\rho)$ to be the irreducible $Sp(V)$-representation with Lusztig classification data $[\phi(s), \phi^\pm(u), \pm 1]$ where the sign in $\phi^\pm(u)$ is the central sign data from s's 1 eigenvalues, the sign in ± 1 is the central sign data from s's -1*

eigenvalues, and signs are ommited when s does not have such eigenvalues

$$\eta^V_{W,B}(\rho) := \rho_{\phi(s),\phi^{\pm}(u),\pm 1}, \tag{25}$$

if $\phi^{\pm}(u)$ is constructible. We put

$$\eta^V_{W,B}(\rho) := 0$$

if $\phi^{\pm}(u)$ is inconstructible.

3.3. **The extended zeta correspondence.** Similarly as in the previous subsection, for a choice of symplectic and orthogonal spaces V, (W,B) which are in the disjoint union of the orthogonal stable and metastable ranges, we define a map

$$\zeta^{W,B}_V : \widehat{Sp(V)} \to \widehat{O(W,B)} \cup \{0\}$$

analogously selecting the "top" irreducible representation of $O(W,B)$ whose tensor product with the input irreducible representation of $Sp(V)$ is a summand of the restricted oscillator representation. This construction proceeds entirely similarly, still considering the cases of $O(W,B)$ separately:

Case 1: Suppose W is of odd dimension $2m+1$, and suppose $(V,(W,B))$ is either in the orthogonal stable or metastable range. In this case, writing $dim(V) = 2N$, this means $m \geq N$.

Now we fix an irreducible representation ρ of $Sp(V) = Sp_{2N}(\mathbb{F}_q)$. Our goal is to construct an irreducible representation of $O(W,B)$, which by applying (19), is equivalent to specifying an irreucible representation of $SO_{2m+1}(\mathbb{F}_q)$ and a sign specifying an action of $\mathbb{Z}/2$. Consider the semisimple conjugacy class part (s) of ρ's Lusztig classification data. We have $s \in (Sp_{2N}(\mathbb{F}_q))^D = SO_{2N+1}(\mathbb{F}_q)$. Say that s has -1 as an eigenvalue of multiplicity 2ℓ, for $0 \leq \ell \leq N$, and let us write

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO^{\pm}_{2\ell}(\mathbb{F}_q).$$

Recall, as a semisimple element of $SO_{2N+1}(\mathbb{F}_q)$, s must have at least one 1 eigenvalue, arising from the embedding of any maximal torus T of the form (12) into $SO_{2N+1}(\mathbb{F}_q)$. Therefore, considering s as an element of the torus, let us write $\widetilde{s} \in T$ (giving a $2N$ by $2N$ matrix), by removing the single "forced" eigenvalue 1 from s. Taking a direct sum with $-I_{2(m-N)}$,

$$\psi(s) := \widetilde{s} \oplus (-I)_{2(m-N)}$$

we obtain a semisimple element of $Sp_{2m}(\mathbb{F}_q) = (SO_{2m+1}(\mathbb{F}_q))^D$, which has -1 as an eigenvalue of multiplicity $2(m-N+\ell)$. Its centralizer is then

$$Z_{Sp_{2m}(\mathbb{F}_q)}(\psi(s)) = H^D \times Sp_{2(m-N+\ell)}(\mathbb{F}_q).$$

For the unipotent part of the Lusztig classification data of ρ, again write $u = u^{H^D} \otimes \binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ for $u^{H^D} \in \widehat{(H^D)}_u$ and a symbol $\binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ specificying an irreducible unipotent representation of $SO^{\pm}_{2\ell}(\mathbb{F}_q)$. We again switch rows of the symbol so that for the minimal i such that $\lambda_{a-i} \neq \mu_{b-i}$ satisfies $\lambda_{a-i} > \mu_{b-i}$. Let us write

$$m'_\rho = m - N + \frac{a+b}{2}.$$

Then if $\lambda_a < m'_\rho$, $\mu_b < m'_\rho$, respectively, the symbols

$$\begin{pmatrix} \lambda_1 < \dots < \lambda_a < m'_\rho \\ \mu_1 < \dots < \mu_b \end{pmatrix}, \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b < m'_\rho \end{pmatrix} \tag{26}$$

have odd defect and rank precisely equal to $m - N + \ell$, and therefore specify irreducible unipotent representations of $SO_{2(m-N+\ell)+1}(\mathbb{F}_q) = (Sp_{2(m-N+\ell)}(\mathbb{F}_q))^D$.

In the case when $\ell = 0$, both of the symbols (26) specify the trivial representation, so let us put

$$\psi(u) = \widetilde{u_{H^D}} \otimes 1.$$

On the other hand, in the case when $\ell > 0$, we are given sign data ± 1 in the Lusztig classification data for ρ, which we can use to select which of the symbols (26) we attempt to use for the unipotent part of the Lusztig classification data of $\zeta^{W,B}_V(\rho)$. Specifically, if $\lambda_a < m'_\rho$, $\mu_b < m'_\rho$, respectively, we put

$$\psi^+(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a < m'_\rho \\ \mu_1 < \dots < \mu_b \end{pmatrix}$$

$$\psi^-(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b < m'_\rho \end{pmatrix}.$$

If $\lambda_a \geq m'_\rho$, resp. $\mu_b \geq m'_\rho$, we say $\psi^+(u)$, resp. $\psi^-(u)$ is inconstructible.

Definition 3.3.1. *Assume the above notation, writing $\rho = \rho_{(s),u}$. In the case where the semisimple part s of ρ's Lusztig classification data has no -1 eigenvalues i.e. $\ell = 0$, we take $\zeta^{W,B}_V(\rho)$ to be the tensor product of the irreducible $SO(W,B)$-representation corresponding to Lusztig* *classification data $[(\psi(s)), \psi(u)]$ with the sign $\epsilon(s) \cdot disc(B)$:*

$$\zeta_V^{W,B}(\rho) = \rho_{(\psi(s)),\psi(u)} \otimes (\epsilon(s) \cdot disc(B)) \tag{27}$$

In the case where s has -1 eigenvalues i.e., $\ell > 0$, writing $\rho = \rho_{(s),u,\pm 1}$, we take $\zeta_V^{W,B}(\rho)$ to be the tensor product of the irreducible $SO(W,B)$-representation corresponding to Lusztig classification data $[(\psi(s)), \psi^{\pm}(u)]$ with the sign $\epsilon(s) \cdot disc(B)$

$$\zeta_V^{W,B}(\rho) = \rho_{(\psi(s)),\psi^{\pm}(u)} \otimes (\epsilon(s) \cdot disc(B)), \tag{28}$$

if the involved $\psi^{\pm}(u)$ is constructible. Put

$$\zeta_V^{W,B}(\rho) = 0$$

if $\psi^{\pm}(u)$ is inconstructible.

Case 2: Suppose W is of even dimension $2m$, and write α for the sign so that $O(W,B) = O_{2m}^{\alpha}(\mathbb{F}_q)$. Suppose also that $(V, (W,B))$ is either in the orthogonal stable or metastable range, meaning that

Consider an irreducible representation ρ of $Sp(V) = Sp_{2N}(\mathbb{F}_q)$. We want to produce $O_{2m}(\mathbb{F}_q)^{\alpha}$-Lusztig classification data, which we recall consists of a semisimple conjugacy class $(s) \in O_{2m}^{\alpha}(\mathbb{F}_q)$, a unipotent part u which can be considered to consist of a unipotent irreducible representaion of the (dual) of the centralizer of s in $SO_{2m}^{\alpha}(\mathbb{F}_q)$, and central sign data. We note that since $Res_{O(W,B)}(\omega[V \otimes W])$ is the permutation representation $\mathbb{C}W$ tensored with the representation $\epsilon(det)$ (corresponding to the sign representation of $O(W,B)/SO(W,B)$), part of the central sign data is already forced. Specifically, as in the case of the symplectic group, we will only need to choose central sign data for the output representation corresponding to -1-eigenvalues.

Write (s) with $s \in SO_{2N+1}(\mathbb{F}_q) = (Sp_{2N}(\mathbb{F}_q))^D$ for the semisimple part of the Lusztig classification data for the input $Sp_{2N}(\mathbb{F}_q)$-representation ρ. Say s has 1 as an eigenvalue of multiplcity $2p+1$, and write

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO_{2p+1}(\mathbb{F}_q).$$

Again, we may remove a single "forced" 1 eigenvalue from s to view it as a $2N$ by $2N$ element of the maximal torus $\widetilde{s} \in T$. Then consider the direct sum with the $2(m-N)$ by $2(m-N)$ identity matrix

$$\psi(s) = \widetilde{s} \oplus I_{2(m-N)},$$

configured to give a $2m$ by $2m$ matrix that can be considered as an element of $SO(W,B) \subseteq O(W,B)$. As in Case 2 of the construction of the eta correspondence, each distinct $SO_{2N+1}(\mathbb{F}_q)$-conjugacy class (s) gives a distinct $O^{\alpha}_{2m}(\mathbb{F}_q)$-conjugacy class $\psi(s)$. We have

$$Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(\psi(s)) = H^D \times SO^{\beta}_{2(N-m+p)}(\mathbb{F}_q), \tag{29}$$

for a single determined choice of sign β (so that its product with the other signs appearing in H agrees with α).

For the unipotent part of the $O^{\alpha}_{2m}(\mathbb{F}_q)$-Lusztig classification data we want to produce, write the unipotent representation u in ρ's Lusztig classification data as $u = u_{H^D} \otimes \binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ for $u_{H^D} \in \widehat{(H^D)}_u$ and a symbol $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ specifying an irreducible unipotent representation of $SO_{2p+1}(\mathbb{F}_q)$. Switch rows so that the defect $a-b$ is 1 mod 4 (which is possible since this symbol has odd defect). Let us write

$$m'_\rho = m - N + \frac{a+b-1}{2}.$$

Then, if $\beta = +$, if $\mu_b < m'_\rho$, putting

$$\psi^+(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < m'_\rho \end{pmatrix}$$

gives a unipotent representation of the group $H \times SO^{+}_{2(N-m+p)}(\mathbb{F}_q) = (Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(\psi(s)))^D$. If $\beta = +$, $\mu_b \geq m'_\rho$, then say $\psi(u)$ is inconstructible. Similarly, if $\beta = -$, if $\lambda_a < m'_\rho$, putting

$$\psi^-(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a < \mu'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

gives a unipotent representation of the group $H \times SO^{-}_{2(N-m+p)}(\mathbb{F}_q) = (Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(\psi(s)))^D$. If $\beta = -$, $\lambda_a \geq m'_\rho$, then say $\psi(u)$ is inconstructible.

Now, as in Case 2 in the symplectic case, (s) and $(\psi(s))$ have the same multiplicity of -1 eigenvalues. Therefore, the undetermined central sign data needed to describe $\zeta^{W,B}_V(\rho)$ if and only if central sign data is given in ρ's original Lusztig classification data. We take it to be the same in this case.

Definition 3.3.2. *Suppose we are given the above notation. We define $\zeta^{W,B}_V(\rho)$ to be the irreducible $O(W,B)$-representation with $O(W,B)$-Lusztig classification data $[\psi(s), \psi(u), \pm 1]$, where the final sign is the central sign of ρ arising if s has -1 eigenvalues and where we omit it if s has no such eigenvalues*

$$\zeta^{W,B}_V(\rho) := \rho_{\psi(s),\psi^{\beta}(u),\pm 1}, \tag{30}$$

for β denoting the sign in (29) (we also neglect to write in the notation the determined central sign data corresponding from the 1 *eigenvalues of s, which is pre-determined). If $\psi(u)$ is inconstructible, we put*

$$\zeta_V^{W,B}(\rho) = 0.$$

Recalling the results of [31], we found that in the symplectic (resp. orthogonal) stable range, every irreducible representation ρ of $O(W, B)$ (resp. $Sp(V)$) appears in the restriction

$$Res_{O(W,B)}(Res_{Sp(V)\times O(W,B)}(\omega[V \otimes W]))$$

(resp. $Res_{Sp(V)}(Res_{Sp(V)\times O(W,B)}(\omega[V \otimes W])))$. In the present paper's notation, for $(V, (W, B))$ in the symplectic stable range, for every $\rho \in \widehat{O(W, B)}$,

$$\eta_{W,B}^V(\rho) \neq 0.$$

Similarly, for $(V, (W, B))$ in the orthogonal stable range, for every $\rho \in \widehat{Sp(V)}$,

$$\zeta_V^{W,B}(\rho) \neq 0.$$

Even in the metastable range, however, the range conditions ensure that we never need to add a negative coordinate to a Lusztig symbol: For $(V, (W, B))$ in the symplectic (stable or) metastable range, for every $\rho \in \widehat{O(W, B)}$, we have

$$N'_\rho > 0.$$

For $(V, (W, B))$ in the orthogonal (stable or) metastable range, for every $\rho \in \widehat{Sp(V)}$, we have

$$m'_\rho > 0.$$

3.4. **Alternating sums of parabolic inductions.** Now we define the alternating sums of parabolic inductions (only needed in the metastable ranges) which form the coefficients of the extended eta or zeta correspondences in the restricted oscillator representation. The description given in this subsection is somewhat technical, but in each case, the principle is to preserve all of the Lusztig classification data, except for the symbol factor of the unipotent part that is altered in the construction of the extended eta or zeta correspondence. We take the sum of the representations obtained by replacing that symbol by those appearing in the alternating sum of parabolic inductions of the symbols obtained by conctenating a final coordinate to one of the rows as in the construction of $\eta^V_{W,B}$ or $\zeta_V^{W,B}$, and removing another coordinate in the same row to recover the original symbol's row lengths (see (33) and (35) below).

Suppose we want to consider the decomposition of the restricted oscillator representation $Res_{Sp(V)\times O(W,B)}(\omega[V\otimes W])$. For this subsection, we now fix our choice of $(V,(W,B))$, specifically fixing a case of the parity of $dim(W)$ and whether we consider the decomposition in terms of an extended eta correspondence (i.e. condition (17) is satisfied) or an extended zeta correspondence (i.e. condition (18) is satisfied).

Now consider a symbol

$$\theta = \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \tag{31}$$

of a group $K[-k]$, where K denotes a possible factor of a centralizer of a semisimple element s in the dual of the domain of whichever correspondence we have chosen to consider ($O(W,B)$ in the extended eta correspondence or $Sp(V)$ for the extended zeta correspondence), which would be altered in the fixed correspondence's construction (corresponding to -1 eigenvalues if $dim(W)$ is fixed to be odd, and 1 eigenvalues if $dim(W)$ is fixed to be even). We use the notation $[-k]$ to refer to the group of the same type and subtracting k from the rank (e.g. for $K = SO(W,B)$, we write $K[-k] = SO(W[-k],B[-k])$). Let us switch rows in (31) so that in the construction of the extended eta or zeta correspondence, if θ appears as the factor of a unipotent part u of the Lusztig classification data for an input representation, ϕ^+ or ψ^+ concatenates a new coordinate to the top row $\lambda_1 < \cdots < \lambda_a$ of (31) (this corresponds to a condition on the row lengths a, b, which varies depending on the case of the extended eta or zeta correspondence we consider). We do this in order to consolidate the notation and treat every case at once.

Fix k and fix some $N' \in \mathbb{N}_0$ such that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_a < N'. \tag{32}$$

Consider the $1 \le c \le a$ such that

$$N' - \lambda_c \le k < N' - \lambda_{c+1}.$$

Then, for $c \le q \le a+1$, let us consider the symbol $\theta^+(N')_q$ with top row given by (32) with the qth coordinate removed, and unmodified bottom row. For $c \le q \le a$ this gives

$$\theta^+(N')_q := \begin{pmatrix} \lambda_1 < \cdots < \widehat{\lambda_q} < \cdots < \lambda_a < N' \\ \mu_1 < \cdots < \mu_b \end{pmatrix}. \tag{33}$$

In the case of $q = a+1$, we the $(a+1)$th coordinate of (32) is N' and thus we have $\theta(N')_q = \theta$. Each of these symbol has the same row lengths as θ, and $\theta(N')_q$ therefore describes a unipotent representation of $K[-(k-(N'-\lambda_q))]$. We may hence consider the alternating sum

$$A_k^+(\theta, N') := \bigoplus_{q=c}^{a+1} (-1)^{a+1-q} \cdot \mathrm{Ind}^{P_{k-(N'-\lambda_q)}}(\theta^+(N')_q) \tag{34}$$

where here P_j denote the maximal parabolics in the full group K with Levi factors $K[-j] \times GL_j(\mathbb{F}_q)$, and we take trivial $GL_j(\mathbb{F}_q)$-action in each induction term.

Similarly, for $N' \in \mathbb{N}_0$ such that $\mu_1 < \cdots < \mu_b < N'$, considering $1 \leq c \leq b$ such that

$$N' - \mu_c \leq k < N' - \mu_{c+1},$$

for $c \leq q \leq b$, we consider the symbol

$$\theta^-(N')_q := \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \widehat{\mu_q} < \cdots < \mu_b < N' \end{pmatrix}. \tag{35}$$

We also put $\theta^-(N')_{b+1} = \theta$. We can then define the alternating sum

$$A_k^-(\theta, N') := \bigoplus_{q=c}^{b+1} (-1)^{b+1-q} \cdot \mathrm{Ind}^{P_{k-(N'-\mu_q)}}(\theta^-(N')_q). \tag{36}$$

We find that (34) and (36), in every case we consider, define genuine representations. In fact, we give a concrete description of which symbols appear in their decompositions in the Appendix.

Again, we approach the case of the extended eta correspondence first.

Definition 3.4.1. *Consider a choice of V and (W, B) in the symplectic (stable or) metastable range. Suppose we are given $0 \leq k \leq h_W$ and $m' > 0$. Consider an irreducible representation $\rho \in \widehat{O(W[-k], B[-k])}$.*

- *Suppose $dim(W) = 2m+1$ is odd. Write the Lusztig classification data of ρ's restriction to $SO(W[-k], B[-k])$ as*

  $$Res_{SO(W[-k],B[-k])}(\rho) = \rho_{(s),u},$$

  and, if s has -1 as an eigenvalue of multiplicity 2ℓ, write $Z_{Sp_{2(m-k)}(\mathbb{F}_q)}(s) = H \times Sp_{2\ell}(\mathbb{F}_q)$, and $u = u_{H^D} \otimes \theta$. As in the construction of the eta correspondence, we can interpret $s \oplus (-I_{2k})$ as a semisimple element of $Sp_{2m}(\mathbb{F}_q)$ with centralizer

$H \times Sp_{2(\ell+k)}(\mathbb{F}_q)$. *Then define* $\mathcal{A}_k(\rho, N')$ *to be the* $O(W,B)$*-representation such that*

$$Res_{SO(W,B)}(\mathcal{A}_k(\rho, N')) = \bigoplus_{\chi \subset A_k^{\pm}(\theta, N')} \rho_{(s \oplus (-I_{2k})), u_{H^D} \otimes \chi},$$

with the sum running over every distinct irreducible unipotent θ *appearing in* $A_k^{\pm}(\theta, N')$ *where the superscript sign is chosen to agree with the sign of* $\phi^{\pm}(u)$ *we take when construction* $\eta_{W,B}^V(\rho)$*, and where we take* $\mathcal{A}_k(\rho, N')$ *itself to have the the same* $\mathbb{Z}/2 = O(W,B)/SO(W,B)$*-action as* ρ.

- *Suppose* $dim(W) = 2m$ *is even. Consider the* $O(W[-k], B[-k])$*-Lusztig classification data of* ρ*, consisting of a semisimple part* (s)*, unipotent part* u*, and possible sign data* α *depending on the eigenvalues of* s*. If* s *has* 1 *as an eigenvalue of multiplicity* $2p$*, write* $Z_{Sp_{2(m-k)}(\mathbb{F}_q)}(s) = H \times Sp_{2p}(\mathbb{F}_q)$*, and* $u = u_{H^D} \otimes \theta$*. As in the construction of the eta correspondence, we can interpret* $s \oplus I_{2k}$ *as a semisimple element of* $Sp_{2m}(\mathbb{F}_q)$ *with centralizer* $H \times Sp_{2(p+k)}(\mathbb{F}_q)$*. Then define* $\mathcal{A}_k(\rho, N')$ *to be the* $O(W,B)$*-representation such that*

$$\mathcal{A}_k(\rho, N') = \bigoplus_{\chi \subset A_k^{\pm}(\theta, N')} \rho_{(s \oplus I_{2k}), u_{H^D} \otimes \chi, \alpha}$$

with the sum running over every distinct irreducible unipotent θ *appearing in* $A_k^{\pm}(\theta, N')$ *where the superscript sign is chosen to agree with the sign of* $\phi^{\pm}(u)$ *we take when construction* $\eta_{W,B}^V(\rho)$.

Similarly, consider the orthogonal metastable range. We note that we still need the separate the cases of the parity of the dimension of the orthogonal space W, though the role of W is somewhat hidden in the notation.

Definition 3.4.2. *Consider a choice of* V *and* (W,B) *in the orthogonal (stable or) metastable range, and write* $dim(V) = 2N$*. Suppose we are given* $0 \leq k \leq N$ *and* $m' > 0$*. Consider an irreducible representation* $\rho \in \widehat{Sp(V[-k])}$.

- *Suppose* $dim(W) = 2m+1$ *is odd. Write the Lusztig classification data of* ρ *as*

$$\rho = \rho_{(s), u, \pm 1},$$

(omitting the central sign data ± 1 *if* s *has no* -1 *eigenvalues). Writing* 2ℓ *for the multiplicity of* -1 *as an eigenvalue of* s*, write*

$Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO^{\pm}_{2\ell}(\mathbb{F}_q)$, and $u = u_{H^D} \otimes \theta$. Then define $\mathcal{A}_k(\rho, m')$ to be the $Sp(V)$-representation

$$\mathcal{A}_k(\rho, m') = \bigoplus_{\chi \subset A^{\pm}_k(\theta, m')} \rho_{(s \oplus (-I)_{2k}), u_{H^D} \otimes \chi, \pm 1}$$

with the sum running over every distinct irreducible unipotent θ appearing in $A^{\pm}_k(\theta, m')$, where the superscript sign is chosen to agree with the sign of $\psi^{\pm}(u)$ we take when construction $\zeta^V_{W,B}(\rho)$, and where the central sign in the Lusztig classification data of each summand is the same as ρ's.

- *Suppose $dim(W) = 2m$ is even. Write the Lusztig classification data of ρ as*

$$\rho = \rho_{(s), u, \pm 1},$$

(omitting the central sign data ± 1 if s has no -1 eigenvalues). Writing $2p$ for the multiplicity of 1 as an eigenvalue of s, write $Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO_{2p+1}(\mathbb{F}_q)$, and $u = u_{H^D} \otimes \theta$. Then define $\mathcal{A}_k(\rho, m')$ to be the $Sp(V)$-representation

$$\mathcal{A}_k(\rho, m') = \bigoplus_{\chi \subset A^{\pm}_k(\theta, m')} \rho_{(s \oplus I_{2k}), u_{H^D} \otimes \chi, \pm 1}$$

with the sum running over every distinct irreducible unipotent θ appearing in $A^{\pm}_k(\theta, m')$, where the superscript sign is chosen to agree with the sign of $\psi^{\pm}(u)$ we take when construction $\zeta^V_{W,B}(\rho)$, where the central sign in the Lusztig classification data of each summand is the same as ρ's.

3.5. **The main statement.** Now that we have established the necessary notation to describe the terms of the decomposition of a restricted oscillator representation claimed in Theorem 1, we may restate it concretely.

The first part of our main result Theorem 1, extending the eta correspondence, can be explicitly restated as:

Theorem 3.5.1. *Suppose $(V, (W, B))$ is in the symplectic metastable range. Then*

$$Res_{Sp(V) \times O(W,B)}(\omega[V \otimes W]) =$$

$$\bigoplus_{k=0}^{h_B} \bigoplus_{\rho \in \widehat{O(W[-k], B[-k])}} \mathcal{A}_k(\rho, N'_\rho) \otimes \eta^V_{W,B}(\rho) \tag{37}$$

where $\mathcal{A}(\rho, N'_\rho)$ is considered as a representation of $O(W,B)$.

Similarly, the second part extending the zeta correspondence can be restated as

Theorem 3.5.2. *Suppose $(V,(W,B))$ is in the orthogonal metastable range. Then*

$$\begin{gathered}Res_{Sp(V)\times O(W,B)}(\omega[V\otimes W]) = \\ \bigoplus_{k=0}^{N} \bigoplus_{\rho\in \widehat{Sp(V[-k])}} \zeta_V^{W,B}(\rho)\otimes \mathcal{A}_k(\rho, m'_\rho)\end{gathered} \tag{38}$$

where $\mathcal{A}(\rho, m'_\rho)$ is considered as a representation of $Sp(V)$.

4. Interpolation and proof of the metastable correspondences

In this section, we describe how the results of [30, 31] for stable range reductive dual pairs can be *interpolated*, which we can use to conclude Theorem 1. We focus on the case of the eta correspondence (interpolating the symplectic stable range result). The case of the zeta correspondence is similar.

In Subsection 4.1, we define interpolated representation categories $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q)$ and $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$, modelling symplectic groups of non-integer rank t. $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q))$ is defined to be tensor generated by a basic object corresponding to the standard permutation representation, and $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ is taken to be generated by basic objects corresponding to the oscillator representations, respectively. While $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q))$ is a more classical construction that can be studied, for example, using oligomorphic groups (see [17]), they do not actually contain objects corresponding to oscillator representations. For our purposes, the "finer" cateory $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ is necessary. These categories form semisimple abelian pre-Tannakian tensor categories for generic complex values of t. However, for certain values, including $t = N \in \mathbb{N}_0$, they are not themselves semisimple, though they are "semisimplifiable," and quotienting out a certain class of "negligible" morphisms does give semisimple pre-Tannakian categories.

In Subsection 4.2, we define subcategories of $\mathscr{C}_B^{int}(t) \subseteq \overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ isolating the ranges of objects appearing in a certain tensor power of oscillator representations (a "partial pseudo-abelian envelope") corresponding to a specific choice of orthogonal space (W, B). For a fixed irreducible representation $\rho \in \widehat{O(W,B)}$, we further consider a subcategory $\mathscr{C}_\rho^{int}(t)$ detecting objects that interpolate representations of $Sp_{2N}(\mathbb{F}_q)$ symplectic stable with $O(W,B)$ whose tensor product with ρ appears in the restriction of $\omega[\mathbb{F}_q^{2N} \otimes W]$. We denote their images under semisimplification by $\widetilde{\mathscr{C}}_B^{int}(t)$, $\widetilde{\mathscr{C}}_\rho^{int}(t)$. In Subsection 4.3, we enumerate the objects of this category in terms of "formal Lusztig symbols" and write down their dimensions. An interpolated decomposition of the restricted oscillator representation holds in these categories (Theorem 4.3.1 below).

In Subsection 4.4, we find that, at $t = N$ with $Sp_{2N}(\mathbb{F}_q)$ in the symplectic metastable range with $O(W,B)$, the relationship between an interpolated category $\widetilde{\mathscr{C}}_\rho^{int}(N)$ and the subcategory of genuine $Sp_{2N}(\mathbb{F}_q)$-representations π such that $\rho \otimes \pi$ appears in the restricted oscillator representation, gives that the decomposition of the restricted oscillator representation as a genuine representation can be derived from the interpolated statement after "cancelling terms." Then we check that simplifying the cancelled terms gives the claimed decomposition as a genuine representation of the symplectic group.

In Subsection 4.5, we describe an application of the statement to give an inductive procedure to explicitly construct representations corresponding to Lusztig classification data.

4.1. **Interpolated representation categories.** First, to begin discussing interpolated representation theory, we briefly recall P. Deligne's cosntruction of the category of representations of a general linear group GL_c for $c \notin \mathbb{Z}$ (see [6, 7, 8]):

This interpolation is based on the fact that in $Rep(GL_N)$, denoting the permutation representation "basic object" of dimension N by X, we have a stable structure of the endomorphism algebras

$$End_{GL_N}(X^{\otimes n}) \cong \mathbb{C}\Sigma_n \tag{39}$$

when $N >> n$. However, in the genuine category $Rep(GL_N)$ of representations of $GL_N(\mathbb{C})$, when the tensor power degree becomes large compared to N, certain Schur functors predicted to occur in $X^{\otimes n}$ have dimension 0, causing (39) to fail. However, consider a formal basic generating object X of dimension $c \in \mathbb{C} \smallsetminus \mathbb{Z}$, we may construct a diagrammatic category where the endomorphism algebra of $X^{\otimes n}$ is always the group algebra $\mathbb{C}\Sigma_n$ (since the polynomial dimensions of the "Schur functors" is never 0 when applied to $dim(X) = c$). Formally adding direct sums and taking a pseudo-abelian envelope gives a category we denote here by $\mathfrak{Rep}(GL_c)$, which is semisimple for $c \in \mathbb{C} \smallsetminus \mathbb{Z}$.

Attempting to apply this construction at $c = N$ gives a non-semisimple category $\mathfrak{Rep}(GL_N)$, which for example, has simple objects of dimension 0. However by applying a procedure of *semisimplification* (see [11]), which is designed to eliminate these simple objects, outputs a semisimple category which is precisely the genuine category of representation $Rep(GL_N)$.

One may also attempt to define an interolated category $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q))$ with the standard permutation representation $\mathbb{C}V_t$ of dimension q^{2t} as a basic tensor-generating object, defined so that

$$\begin{gathered}Hom_{\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q)}((\mathbb{C}V_t)^{\otimes m},(\mathbb{C}V_t)^{\otimes n}) = \\ Hom_{Rep(Sp_{2N}(\mathbb{F}_q))}((\mathbb{C}V_N)^{\otimes m},(\mathbb{C}V_N)^{\otimes n}),\end{gathered} \tag{40}$$

for a large enough $N >> m, n$ writing V_N for the q^{2N}-dimensional underlying symplectic space $Sp(V_N) = Sp_{2N}(\mathbb{F}_q)$. The semisimplification of this category at $t = N$ is known to give a semisimple pre-Tannakian category. However, it is not equivalent to the genuine representation category $Rep(Sp_{2N}(\mathbb{F}_q))$. For example, there is no object of dimension q^t corresponding to the oscillator representation.

Another commonly considered interpolated category $\mathfrak{Rep}(GL_t(\mathbb{F}_q))$ is defined similarly to be tensor-generated by the standard representation of dimension q^t written as $\mathbb{C}\mathbb{F}_q^t$, with

$$\begin{gathered}Hom_{\mathfrak{Rep}(GL_t(\mathbb{F}_q)}((\mathbb{C}\mathbb{F}_q^t)^{\otimes m},(\mathbb{C}\mathbb{F}_q^t)^{\otimes n}) = \\ Hom_{Rep(GL_N(\mathbb{F}_q))}((\mathbb{C}\mathbb{F}_q^N)^{\otimes m},(\mathbb{C}\mathbb{F}_q^N)^{\otimes n}),\end{gathered} \tag{41}$$

for a large enough $N >> m, n$. Similarly, we may define an interpolated category $\mathfrak{Rep}(O_t(\mathbb{F}_q))$ with basic object corresponding to the signed $\mathbb{C}W \otimes \epsilon(det)$, writing $\epsilon(det)$ for the sign representation on the center $\mathbb{Z}/2$ corresponding to det. Writing the basic object $\mathbb{C}W_t \otimes \epsilon(det)$ of $\mathfrak{Rep}(O_t(\mathbb{F}_q))$, we set Hom-spaces between its tensor powers to be equal to what they would be in an orthogonal group of high enough rank (which may be chosen to be even or odd), as in (40), (41). Unlike in the symplectic case, modelling the (twisted) permutation representation as the basic generating object in $\overline{\mathfrak{Rep}}(O_t(\mathbb{F}_q))$ will be enough for our purposes, recalling that, in the case of 2-dimensional V with the standard symplectic form, we have

$$Res_{O(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]) = \mathbb{C}W^-$$

denoting the permutation representation $\mathbb{C}W$ tensored with $\epsilon(det)$ i.e. the sign representation of $O(W, B)/SO(W, B)$, (which can be seen from restricting first to $GL(W)$).

Now, both $\mathfrak{Rep}(GL_t(\mathbb{F}_q))$ aand $\overline{\mathfrak{Rep}}(O_t(\mathbb{F}_q))$ are known to, after semisimplification when needed at natural number values of t, give semisimple categories. This can both be proved directly (see, for examples of this method, [6, 7, 27]), or be approached using the theory of oligomorphic groups of A. Snowden and N. Harman (see [17]).

However, we need to define a different interpolated tensor category $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ with the oscillator representations as basic objects (since the structure of endomorphism algebras and homomorphism modules between tensor products of oscillator representation is stable when the dimension of the underlying symplectic space is large enough compared to the tensor product degrees). For more details, see [28]. When attempting to put $t = N \in \mathbb{N}$, as in the case of the interpolated representation categories of the symmetric group, $\overline{\mathfrak{Rep}}(Sp_{2N}(\mathbb{F}_q))$ is not a semisimple category (since, again, it has simple objects of dimension 0).

Consider a category $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))_0$ with objects $\boldsymbol{\omega}_a^{\otimes m} \otimes \boldsymbol{\omega}_b^{\otimes n}$, with Hom-spaces between objects defined to be, as $\mathbb{C}$-vector spaces

$$Hom_{\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))_0}(\boldsymbol{\omega}_a^{\otimes m} \otimes \boldsymbol{\omega}_b^{\otimes n}, \boldsymbol{\omega}_a^{\otimes p} \otimes \boldsymbol{\omega}_b^{\otimes \ell}) :=$$
$$Hom_{Sp_{2N}(\mathbb{F}_q)}(\omega_a^{\otimes m} \otimes \omega_b^{\otimes n}, \omega_a^{\otimes p} \otimes \omega_b^{\otimes \ell})$$

for $N >> m, n, p, l$. Tensor products of morphisms and the actions of the bijections on the coordinates of the different tensor factors are defined in the obvious way from $Rep(Sp_{2N}(\mathbb{F}_q))$ for a large enough N. To construct a category (and involve the constant t), we also need to define a partial trace operation corresponding to matching factors of the generating objects $\boldsymbol{\omega}_a$. In this case, it suffices to define a trace operation for endomorphisms of each $\boldsymbol{\omega}_a$. We do this by considering

$$End_{\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))_0}(\boldsymbol{\omega}_a) = End_{Sp_{2N}(\mathbb{F}_q)}(\omega_a)$$
$$= Hom_{Sp_{2N}(\mathbb{F}_q)}(1, \mathbb{C}V_N) \cong (\mathbb{C}V_N)^{Sp_{2N}},$$

which has a basis consisting of (0) and $\sum_{v \neq 0 \in V_N}(v)$, where we put

$$tr((0)) = q^t, \quad tr(\sum_{v \neq 0 \in V_N}(v)) = 0.$$

Composition can be defined with a combination of tensor product, permutation, and partial trace (for more details, see [28]). This defines a $\mathbb{C}$-linear category $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))_0$.

We then define the category $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ by first formally adding direct sums to $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))_0$, and then applying a pseudo-abelian envelope, adding new objects defined as the images of idempotents in the endomorphism algebras of the objects $\boldsymbol{\omega}_a^{\otimes m} \otimes \boldsymbol{\omega}_b^{\otimes n}$.

In each case, the interpolated category is constructed from the data of a system of *Hom*-spaces between tensor powers of the basic object, with operations of permutation action, tensor product, and partial trace. This data can in fact be captured in the universal algebra structure of a *T-algebra.* To describe a $\mathbb{C}$-linear additive category with strong duality and associative commutative unital tensor product generated by a basic object X, its corresponding T-algebra $\mathcal{T}$ consists of vector spaces for every pair of finite sets S, T

$$\mathcal{T}_{S,T} = Hom(X^{\otimes S}, X^{\otimes T}),$$

with appropriate functoriality over the category of finite sets, linear partial trace operations corresponding to partial bijections between S and T and tensor product operations corresponding the set disjoint union, with appropriate axioms. Here, we denote the T-algebra corresponding to $\mathfrak{Rep}(GL_t(\mathbb{F}_q)$ with basic object corresponding to the standard representation $X = \mathbb{C}\mathbb{F}_q^t$ by $\mathcal{T}[GL_t(\mathbb{F}_q)]$ and the T-algebra corresponding to $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q)$ with basic object $X = \boldsymbol{\omega}_a \otimes \boldsymbol{\omega}_b$ by $\overline{\mathcal{T}}[Sp_{2t}(\mathbb{F}_q)]$. For more details, see [28].

Let us denote the *semisimplification* of a $\mathbb{C}$-linear additive category with strong duality and associative commutative unital tensor product $\mathscr{C}$ by $\mathscr{S}(\mathscr{C})$. Recall that semisimplification refers to a construction quotienting out *negligible morphisms*, such as simple idempotents with trace 0, [11].

We recall a result of [28].

Proposition 4.1.1. *In every case of t, the semisimplification of the category $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ is a semisimple (and, in particular, abelian) pre-Tannakian category. For values of $t \in \mathbb{C}$ such that $q^t \neq \pm q^n$ for $n \in \mathbb{N}_0$, the category $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ itself is semisimple.*

Proof. First, there is an inclusion of T-algebras

$$\overline{\mathcal{T}}[Sp_{2t}(\mathbb{F}_q)] \hookrightarrow \mathcal{T}[GL_t(\mathbb{F}_q)],$$

since for every N, the restriction of an oscilaltor representation ω_a to $GL_N(\mathbb{F}_q) \subseteq Sp_{2N}(\mathbb{F}_q)$ is isomorphic to

$$Res_{GL_N(\mathbb{F}_q)}(\omega_a) \cong (\mathbb{C}V_N) \otimes \epsilon(\det).$$

In particular, then for finite sets S, T, restriction gives an inclusion from each Hom-space

$$Hom_{\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))}(\boldsymbol{\omega}_a^{\otimes S_1} \otimes \boldsymbol{\omega}_b^{\otimes S_2}, \boldsymbol{\omega}_a^{\otimes T_1} \otimes \boldsymbol{\omega}_b^{\otimes T_2}) =$$
$$Hom_{Sp_{2N}(\mathbb{F}_q)}(\omega_a^{\otimes S_1} \otimes \omega_b^{\otimes S_2}, \omega_a^{\otimes T_1} \otimes \omega_b^{\otimes S_2})$$

making up $\overline{\mathcal{T}}[Sp_{2t}(\mathbb{F}_q)]_{S,T}$ for $S_1 \amalg S_2 = S, T_1 \amalg T_2 = T$, into the $GL_N(\mathbb{F}_q)$-equivariant Hom-space on the restrictions

$$Hom_{GL_N(\mathbb{F}_q)}((\mathbb{C}V_N \otimes \epsilon(\det))^{S_1 \amalg S_2}, (\mathbb{C}V_N \otimes \epsilon(\det))^{T_1 \amalg T_2}),$$

which is isomorphic to $\mathcal{T}[GL_t(\mathbb{F}_q)]_{S,T}$. Partial trace, tensor product, and functoriality (and therefore composition) are all compatible.

We then apply

Lemma 4.1.2. *The semisimplification of a $\mathbb{C}$-linear additive category with strong duality and an associative, commutative, unital tensor product generated by a basic object X is semisimple if and only if for every endomorphism $f \in End(X^{\otimes n})$, if the trace of f is non-zero, then for every n, there exists an $m > n$ such that*

$$tr(f^{\circ m}) \neq 0.$$

Proof of Lemma 4.1.2. To prove sufficiency, consider an endomorphism f of some tensor power $X^{\otimes n}$ is non-negligible, i.e. there exists some morphism $g \in End(X^{\otimes n})$ such that the trace of $f \circ g$ is non-zero. The trace condition then gives that for every n, there exists a $m > n$ such that

$$tr((f \circ g)^{\circ m}) \neq 0,$$

and hence, f is not an element of the Jacobian ideal of the endomorphism algebra $End(X^{\otimes n})$, and in particular, is not nilpotent. Therefore, the semisimplification of the category is semisimple.

Necessity follows from the general result that in a semisimple $\mathbb{C}$-algebra (e.g. the endomorphism algebra of $X^{\otimes n}$ in the semisimplification), if some general trace operation (i.e. a linear combination of trace on each factor, consider the endomorphism algebra as a product of matrix algebras) is non-zero on an element f, then for every n, there exists an $m > n$ such that the trace operation is non-zero on $f^{\circ m}$. (This follows, for example, by considering the Vanermonde determinant.) □

In particular, if the semisimplification of a category defined by a T-algebra $\mathcal{T}$ is semisimple, then the semisimplifications of any category defined by a sub-T-algebra $\mathcal{T}' \subseteq \mathcal{T}$ is semisimple as well. Therefore, since the semisimplification of $\mathfrak{Rep}(GL_t(\mathbb{F}_q))$ is semisimple, so is the semisimplification of $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$.

The second claim follows, for example, by examining the polynomial order of $Sp_{2N}(\mathbb{F}_q)$, and replacing N by t, to conclude that every indecomposable object is non-vanishing in the semisimplification. $\square$

For our purposes here, we will want to consider the case of $t = N$, making the first part of this statement more relevant. The second part of this statement for generic values of t can be used to conclude an interpolated version of our decomposition statement, for example, describing an interpolated eta correspondence

$$\eta^t_{W,B} : \widehat{O(W,B)} \hookrightarrow \overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q)),$$

sending the irreducible representations of $O(W,B)$ to simple objects of $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$.

4.2. **Partial pseudo-abelian envelopes and "isotypical" subcategories.** In this subsection, we give an argument using interpolation for our full statement of Howe duality. From here on, we will focus on the case of interpolating the eta correspondence with target in odd orthogonal group representations, for simplicity. In this part of the argument, every other case is entirely similar. Fix a choice of orthogonal space and bilinear form (W,B).

Recall that in this case, for a choice of V lying with (W,B) in the symplectic stable range, the eta correspondence is an injective map $\eta^V_{W,B}$ sending the irreducible representations of $O(W,B)$ to irreduicble representations of $Sp(V)$. The methods of [30, 31] ultimately only rely on information carried by the endomorphism algebras of (tensor powers of) oscillator representations, whose structure is stable under enlarging the dimension of V, in the stable symplectic range, and therefore, by definition, they pass to the interpolated categories.

Write $dim(W) = n$, and suppose B is, as a symmetric bilinear form, equivalent to a diagonal matrix with entries $a_1, \dots, a_n \in \mathbb{F}_q^\times$. We may then consider the object

$$\boldsymbol{\omega}^{\otimes B} = \boldsymbol{\omega}_{a_1} \otimes \cdots \otimes \boldsymbol{\omega}_{a_n}$$

in $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$. Let us consider the "partial pseudo-abelian envelope" $\mathscr{C}^{int}_B(t)$ defined as the subcategory of $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ consisting of images of idempotents of $End_{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))(\boldsymbol{\omega}^{\otimes B})$. We do not consider a tensor product on $\mathscr{C}^{int}_B(t)$, only working with its structure as an additive $\mathbb{C}$-linear category.

Further, considering

$$End_{\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))}(\boldsymbol{\omega}^{\otimes B}) \cong End_{Sp_{2N}(\mathbb{F}_q)}(\omega_{a_1}[V_N] \otimes \cdots \otimes \omega_{a_n}[V_N]) = End_{Sp_{2N}(\mathbb{F}_q)}(\omega[V_N \otimes W]),$$

for a large enough rank N, where on the right hand side, we consider the restriction of $\omega[V_N \otimes W]$ along the inclusion

$$Sp_{2N}(\mathbb{F}_q) = Sp(V_N) \subseteq Sp(V_N) \times O(W, B) \subseteq Sp(V_N \otimes W).$$

Therefore, there is a built in action of $O(W, B)$ on the endomorphism algebras $End_{\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))}(\boldsymbol{\omega}^{\otimes B})$.

In particular, for a fixed irreducible representation ρ of $O(W, B)$, we may consider the full *ρ-isotypical* subcategory $\mathscr{C}_\rho^{int}(t) \subseteq \mathscr{C}_B(t)$, with objects consisting of images of idempotents ι in $End_{\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))}(\boldsymbol{\omega}^{\otimes B})$ where, considering the $O(W, B)$-fixed point algebra

$$\begin{array}{c} End_{O(W,B)\times\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))}(\boldsymbol{\omega}^{\otimes B}) = \\ (End_{\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))}(\boldsymbol{\omega}^{\otimes B}))^{O(W,B)} \end{array},$$

there is a simple idempotent of the form $\kappa \otimes \iota$, for κ an idempotent in $Rep(O(W, B))$ with image isomorphic to ρ. We may also describe these objects as the images of ι lying in the ρ-isotypical part of $End_{\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))}(\boldsymbol{\omega}^{\otimes B})$ as an $O(W, B)$-representation. We cannot take a semisimplification of $\mathscr{C}_\rho^{int}(t)$, since we have given up its tensor structure. However, we may consider the images of $\mathscr{C}_B^{int}(t)$, $\mathscr{C}_\rho^{int}(t)$ under the quotient semisimplification functor

$$\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q)) \to \mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))).$$

Writing $\widetilde{\mathscr{C}}_B^{int}(t)$, $\widetilde{\mathscr{C}}_\rho^{int}(t)$ for these images, they form semisimple abelian subcategories of $\mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q)))$. Note that this is only non-trivial for a choice of $t = N$ a natural number.

On the other hand, we may also consider the full subcategories $\mathscr{C}_\rho^{gen}(N)$ of $Rep(Sp_{2N}(\mathbb{F}_q))$ consisting of direct sums of all genuine irreducible representations $\pi \in Rep(Sp_{2N}(\mathbb{F}_q))$ such that

$$\pi \otimes \rho \subseteq Res_{O(W,B)\times Sp(V)}(\omega[V \otimes W]).$$

4.3. **Interpolating correspondences.** The purpose of this subsection is to describe how the objects of the subcategories $\mathscr{C}_B^{int}(t)$ of the interpolation $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ constructed in tensor powers of oscillator representations of odd degree such that the total product of central characters has a certain quadratic character α (i.e. corresponding to odd orthogonal spaces and symmetric bilinear forms of a certain prescribed discriminant α), can be written down according to an "formally interpolated Lusztig classification." We discuss this in detail for the case of $dim(W)$ odd. All other cases are similar.

More specifically, an object of $\mathscr{C}_B^{int}(t)$ is of the form

$$\eta^t_{W,B}(\rho),$$

for W of dimension $2m+1$ for $m \in \mathbb{N}$, with a form B of discriminant $disc(B) = \alpha$, and an irreducible representation ρ of $O(W,B)$. Say that as a representation of $O(W,B) = SO_{2m+1}(\mathbb{F}_q) \times \mathbb{Z}/2$, ρ is of the form $\rho_{(s),u} \otimes (\pm 1)$ where $\rho_{(s),u}$ is an irreducible representation of $SO_{2m+1}(\mathbb{F}_q)$ corresponding to Lusztig classification data with semisimple part $(s) \in Sp_{2m}(\mathbb{F}_q) = (SO_{2m+1}(\mathbb{F}_q))^D$ and unipotent part $u \in \widehat{(Z_{Sp_{2m}(\mathbb{F}_q)}(s))^D}_u$. More concretely, further say that s has -1 as an eigenvalue of multiplicity 2ℓ, writing $Z_{Sp_{2m}(\mathbb{F}_q)}(s) = H \times Sp_{2\ell}(\mathbb{F}_q)$, and

$$u = u_{H^D} \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}$$

for $u_{H^D} \in \widehat{H^D}_u$ and $\binom{\lambda_a < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ (switching rows so that $a-b$ is 1 mod 4) specifying a unipotent representation of $SO_{2\ell+1}(\mathbb{F}_q) = (Sp_{2\ell}(\mathbb{F}_q))^D$. Then, for the sign $+$, we say that $\eta^t_{W,B}(\rho_{(s),u} \otimes (+1))$ corresponds to "interpolated Lusztig classification data"

$$[\phi^+(s), \widetilde{u_{H^D}} \otimes \begin{pmatrix} & \lambda_1 < \dots < \lambda_a & \\ \mu_1 < \dots < \mu_b & & t'_\rho \end{pmatrix}], \tag{42}$$

writing $t'_\rho = t - m + \frac{a+b-1}{2}$. This is exactly the Lusztig classification data of a stable range eta correspondence $\eta^t_{W,B}(\rho)$ for $dim(V) = 2N$, with N replaced by t (we omit the final $<$ sign in the second row of the symbol notation, since at an interpolated value of t, writing $\mu_b < t'_\rho$ may be false or incomparable). Again, $Sp_{2t}(\mathbb{F}_q)$ is not a genuine group, and writing $\phi^+(s)$ indicates an element with finitely many eigenvalues not equal to -1 (which would contribute genuine factors in its "centralizer") and has -1 as an eigenvalue of "multiplicity $2(t-m+\ell)$." Interpolating the stable formula one would obtain for $Sp_{2N}(\mathbb{F}_q)$, replacing N by t, its dimension is

$$dim(\eta^t_{W,B}(\rho_{(s),u} \otimes (+1))) =$$

$$\frac{dim(\rho) \cdot \prod_{i=t'+1}^{t} (q^{2i}-1) \cdot \prod_{i=1}^{a} (q^{t'_\rho} + q^{\lambda_i}) \cdot \prod_{i=1}^{b} (q^{t'_\rho} - q^{\mu_i})}{2 \cdot q^{(a+b-1)(a+b+1)/4} \cdot |SO_{2m+1}(\mathbb{F}_q)|_{q'}}. \tag{43}$$

Similarly, at the sign -1, we say that $\eta^t_{W,B}(\rho_{(s),u}\otimes(-1))$ corresponds to "interpolated Lusztig classification data"

$$[\phi^-(s),\widetilde{u_{H^D}}\otimes\begin{pmatrix}\lambda_1<\cdots<\lambda_a & t'_\rho\\ \mu_1<\cdots<\mu_b & \end{pmatrix}],\tag{44}$$

writing $t'_\rho=t-m+(a+b-1)/2$. Interpolating the stable formula one would obtain for $Sp_{2N}(\mathbb{F}_q)$, replacing N by t, its dimension is

$$dim(\eta^t_{W,B}(\rho_{(s),u}\otimes(-1)))=$$
$$\frac{dim(\rho)\cdot\prod\limits_{i=t'+1}^{t}(q^{2i}-1)\cdot\prod\limits_{i=1}^{a}(q^{t'_\rho}-q^{\lambda_i})\cdot\prod\limits_{i=1}^{b}(q^{t'_\rho}+q^{\mu_i})}{2\cdot q^{(a+b-1)(a+b+1)/4}\cdot|SO_{2m+1}(\mathbb{F}_q)|_{q'}}.\tag{45}$$

Now we may consider the "restriction" functor

$$\mathbf{Res}:\mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2nt}(\mathbb{F}_q)))\to\mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q)))\boxtimes Rep(O(W,B)).$$

The interpolated Howe duality statement then is

Theorem 4.3.1. *In the semisimplification $\mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q)))$, the original decomposition of $\mathbf{Res}(\boldsymbol{\omega}_1)$ as*

$$\bigoplus_{k=0}^{m_W}\bigoplus_{\rho\in\widehat{O(W[-k],B[-k])}}\eta^t_{W,B}(\rho)\boxtimes Ind^{P_k}(\rho\otimes\epsilon(det))\tag{46}$$

holds, as objects of $\mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q)))\boxtimes Rep(O(W,B))$.

In summary, the objects of $\mathscr{C}^{int}_B(t)$ are precisely direct sums of all

$$\eta^t_{W[-k],B[-k]}(\rho)\tag{47}$$

for irreducible representations $\rho\in\widehat{O(W[-k],B[-k])}$. For a fixed irreducible representation ρ of $O(W,B)$, the objects of $\mathscr{C}^{int}_\rho(t)$ consist of direct sums of objects $\eta^t_{W[-k],B[-k]}(\rho')$ corresponding to irreducible representations $\rho'\in\widehat{O(W[-k],B[-k])}$ such that ρ is a summand of the parabolic induction

$$\rho\subseteq\mathrm{Ind}^{P_k}(\rho'\otimes\epsilon(\mathrm{det}))$$

writing $P_k\subseteq O(W,B)$ for the maximal parabolic subgroup with Levi factor $O(W[-k],B[-k])\times GL_k(\mathbb{F}_q)$, considering $\epsilon(\mathrm{det})$ as a representation of the factor $GL_k(\mathbb{F}_q)$.

At $t=N$ corresponding to a reductive dual pair $(Sp_{2N}(\mathbb{F}_q),O(W,B))$ in the symplectic metastable range, the semisimplification images $\widetilde{\mathscr{C}}_B(N)$,

$\widetilde{\mathscr{C}}_\rho(N)$ are semisimple categories with objects consisting of direct sums of simple objects corresponding to all formal interpolated Lusztig classification, eliminating 0-dimensional objects, which occur precisely when $N'_\rho = \lambda_i$ or μ_i for some i in (42) or (44). Note that the remaining formal interpolated Lusztig classification objects, say where $\lambda_i < N'_\rho < \lambda_{i+1}$ in (42) or $\mu_i < N'_\rho < \mu_{i+1}$ in (44), have dimension equal to a genuine irreducible $Sp_{2N}(\mathbb{F}_q)$-representation where N'_ρ is inserted in the appropriate place, multiplied by $(-1)^{a-i}$ or $(-1)^{b-i}$. Call this the *true permutation* of the formal interpolated Lusztig data as $t = N$.

4.4. **The proof of the metastable eta correspondence.** Now, to approach a choice of $(V,(W,B))$ in the symplectic metastable range, we attempt to apply Theorem 4.3.1 to $t = N$, giving a decomposition in the semisimplification $\mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q)))$ in terms of objects of $\widetilde{\mathscr{C}}_B(N)$. We must relate this category to genuine $Sp_{2N}(\mathbb{F}_q)$-representations. In fact, we claim that replacing the formal Lusztig classification data by its true permutation, with the corresponding sign, gives a genuine decomposition of the restricted oscillator representation in the Grothendieck group $K(Rep(Sp_{2N}(\mathbb{F}_q)))$. Simplifying will precisely give the claimed decomposition in Theorem 3.5.1.

To see this, for each $\rho \in \widehat{O(W,B)}$, consider functors

$$\Phi : \mathscr{C}^{gen}_\rho(N) \to \widetilde{\mathscr{C}}^{int}_\rho(N)$$

defined as follows: Consider a simple object π of the source, such that

$$\pi \otimes \rho \subseteq Res_{Sp(V)\times O(W,B)}(\omega[V \otimes W]).$$

We may consider an idempotent ι_π in the $Sp(V)$-equviariant endomorphism algebra of $Res_{Sp(V)}(\omega[V\otimes W]) \cong \omega^{\otimes B}$ whose image is one of these copies of π. By duality, since each oscillator representation has $\omega_a \otimes (\omega_a)^\vee \cong \mathbb{C}V)$, we may conisder

$$\iota_\pi \in End_{Sp(V)}(\omega^{\otimes B}) \cong Hom_{Sp(V)}(1,(\mathbb{C}V)^{\otimes B}) = (\mathbb{C}(V\otimes W))^{Sp(V)}$$

as a linear combination of $Sp(V)$-orbits on $V \otimes W = V^{\oplus n}$ (recall that an $Sp(V)$-orbit consits of a set of n-tuples of vectors $(v_1,\dots,v_n)$ satisfying some linear (in)dependence conditions, and equations for the values of the symplectic form on pairs of them). Theses orbits can also be considered as orbits of any $Sp_{2M}(\mathbb{F}_q)$ acting on $(\mathbb{F}_q^{2M})^{\oplus n}$ for any higher

M, and therefore ι_π corresponds to an interpolated endomorphism

$$End_{\overline{\mathfrak{Rep}}(Sp_{2N}(\mathbb{F}_q))}(\boldsymbol{\omega}^{\otimes B}) \cong ((\mathbb{C}(\mathbb{F}_q^{2M} \otimes W))^{Sp_{2M}(\mathbb{F}_q)}$$

for $M >> n$ (by the definition of $\overline{\mathfrak{Rep}}(Sp_{2N}(\mathbb{F}_q))$). Since partial trace operations (and therefore compositions) are computed the same in $Rep(Sp_{2N}(\mathbb{F}_q))$ and $\overline{\mathfrak{Rep}}(Sp_{2N}(\mathbb{F}_q))$ (the difference between them arising instead from certain morphisms in the interpolated category not existing in the genuine category), this new endomorphism is still an idempotent, with image equal to an object in $\widetilde{\mathscr{C}}_\rho^{int}(N)$ of the same dimension as π. We put $\Phi(\pi)$ to be this object.

On the other hand, define

$$\Psi : K(\widetilde{\mathscr{C}}_\rho^{int}(N)) \to K(\mathscr{C}_\rho^{gen}(N))$$

by sending a simple object in $\widetilde{\mathscr{C}}_\rho^{int}(N)$ which must be of the form (42) or (44) at $t = N$ for some choice of s, u, $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ to $(-1)^{b-i}$, where i is the index so that $\mu_i < t'_\rho < \mu_{i+1}$ or $\lambda_i < t'_\rho < \lambda_{i+1}$, times the genuine irreducible representation of $Sp(V)$ whose Lusztig calssification data is the same as (42) or (44) with the formal interpolated symbol replaced by

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_i < t'_\rho < \mu_{i+1} < \cdots < \mu_b \end{pmatrix}$$

or

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_i < t'_\rho < \lambda_{i+1} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix},$$

respectively. In other words, we them precisely to their signed true permutations.

Proposition 4.4.1. *The composition of*

$$K(\mathscr{C}_\rho^{gen}(N)) \xrightarrow{K(\Phi)} K(\widetilde{\mathscr{C}}_\rho^{int}(N)) \xrightarrow{\Psi} K(\mathscr{C}_\rho^{gen}(N))$$

is $Id_{K(\mathscr{C}_\rho^{gen}(N))}$.

Proof. Note that this holds immediately when formal Lusztig classification data is actually genuine Lusztig classification data defining a $Sp_{2N}(\mathbb{F}_q)$-representation, since both $K(\Phi)$ and Ψ act as the identity on these objects, considered in either categories. The general statement follows since dimensions are preserved by Ψ and Φ, and it is not possible for $\Psi \circ K(\Phi)$ when applied to an irreducible representation of $\mathscr{C}_{\rho}^{gen}(N)$ to output a linear combination of multiple different irreducible representations in $Rep(Sp_{2N}(\mathbb{F}_q)$ with integer coefficients, both by dimension and the fact that it would violate the decomposition (46). $\square$

Therefore, the decomposition of the restricted oscillator representation as a genuine representation of $Sp_{2N}(\mathbb{F}_q)$ can be obtained from (46) by applying Ψ, and cancelling terms as in $K(\mathscr{C}_B(N))$. It remains to reconcile this cancellation with the claimed answer. Consider a term arising from $\rho \in O(\widehat{W[-j], B}[-j])$. Write $\rho = \rho_{(s),u} \otimes (\pm 1)$, for $\rho_{(s),u}$ an irreducible unipotent representation of the special orthogonal group $SO(W[-j], B[-j]) = SO_{2(m-j)+1}(\mathbb{F}_q)$. Suppose $s \in Sp_{2(m-j)}(\mathbb{F}_q) = (SO_{2(m-j)+1}(\mathbb{F}_q))^D$ has -1 as an eigenvalue of multiplicity 2ℓ, and write $Z_{Sp_{2(m-j)}(\mathbb{F}_q)}(s) = H \times Sp_{2\ell}(\mathbb{F}_q)$. Then $\psi^{\pm}(s)$ remains as defined in the stable range, so that $Z_{SO_{2N+1}(\mathbb{F}_q)}(\psi^{\pm}(s)) = H^D \times SO^{\pm}_{2(N-m+\ell)}(\mathbb{F}_q)$. If $\psi^{\pm}(u)$ is constructible, then the interpolated eta correspondence outputs a genuine $Sp_{2N}(\mathbb{F}_q)$-representation, which is our proposed metastable construction of $\eta^{V}_{W,B}$. In the case corresponding to when $\psi(u)$ is "inconstructible," however, the interpolated eta correspondence outputs a representation involving an illegal symbol

$$\psi(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a & N'_\rho \\ \mu_1 < \cdots < \mu_b & \end{pmatrix}. \tag{48}$$

This term

$$(\rho_{\psi^{\pm}(s),\psi(u),\pm 1} \otimes (\epsilon(s)disc(B))) \otimes \mathrm{Ind}^{P_i}(\rho_{(s),u})$$

must be eliminated when we semisimplify.

To see where this term cancels, suppose that

$$\lambda_1 < \cdots < \lambda_c < N'_\rho < \lambda_{c+1} < \cdots < \lambda_a,$$

then, for $c+1 \leq k \leq a$, we have that the multiplicity $2p$ of -1 as an eigenvalue of s must be greater than or equal to $2(\lambda_k - N'_\rho)$. first let us write

$$j^{(k)} = j - \lambda_k + N'_\rho.$$

Note that by the rank conditions of Lusztig symbols, the multiplicity $2p$ of -1 as an eigenvlaue must be greater than or equal to $2(\lambda_k - N'_\rho)$. Then write $s^{(k)}$ for the semisimple element of

$$Sp_{2(m-j+\lambda_k - N'_\rho)}(\mathbb{F}_q) = Sp_{2(m-j^{(k)})}(\mathbb{F}_q) = (SO_{2(m-j^{(k)})+1}(\mathbb{F}_q))^D$$

obtained by removing $2(\lambda_k - N'_\rho)$ eigenvalues -1. Write

$$u^{(k)} = u_{H^D}\otimes$$

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_c < N'_\rho < \lambda_{c+1} < \cdots < \lambda_{k-1} < \lambda_{k+1} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

Consider the representation corresponding to the Lusztig data

$$\rho_{(s^{(k)}),u^{(k)}}$$

giving an irreducible representation of $SO_{2(m-j^{(k)})+1}(\mathbb{F}_q)$. Then $\psi^\pm(s) = \psi^\pm(s^{(k)})$, and we have

$$N'_{\rho_{(s^{(k)}),u^{(k)}}} = \lambda_k.$$

Therefore,

$$\psi(u^{(k)}) = \widetilde{u_{H^D}}\otimes$$

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_c < N'_\rho < \lambda_{c+1} < \cdots < \lambda_{k-1} < \lambda_{k+1} < \cdots < \lambda_a \quad \lambda_k \\ \mu_1 < \cdots < \mu_b \end{pmatrix}.$$

Note that the interpolated symbol part has dimension $(-1)^{a-k}$, multiplied by the dimension of the genuine symbol

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_c < N'_\rho < \lambda_{c+1} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}. \tag{49}$$

Therefore, at $j^{(k)}$, this contributes a term

$$(\rho_{\psi^\pm(s),\psi(u^{(k)}),\pm 1} \otimes (\epsilon(s)disc(B))) \otimes \mathrm{Ind}^{P_{j^{(k)}}}(\rho_{(s^{(k)}),u^{(k)}}), \tag{50}$$

which has terms that can cancel recursively (e.g. $k = 1$ completely cancels (48), though it introduces new summands that are illegal if $a > 1$), since switching coordinates in interpolated Lusztig symbols gives a change in the sign of dimension. At $k = a$, we have the legal symbol (49).

Putting together the terms (48) and (50) for $c+1 \leq k \leq a$, this gives the final genuine term, which is exactly $\eta^V_{W,B}(\rho_{(s^{(a)}),u^{(a)}})$, tensored with a coefficient of the $O(W,B)$-representation given by the alternating sum

$$\bigoplus_{k=c}^{a} (-1)^{a-k}\mathrm{Ind}^{j_{(k)}}(\rho_{(s^{(k)}),u^{(k)}})$$

(writing $j^{(0)} = j$, $s^{(0)} = s$, $u^{(0)} = u$). By definition, this $O(W,B)$-representation is precisely the proposed alternating sum

$$\mathcal{A}_{j_{(a)}}(\rho_{(s^{(a)}),u^{(a)}}, N'_\rho).$$

Therefore, in the final decomposition of $Res_{Sp(V)\times O(W,B)}(\boldsymbol{\omega}[V\otimes W])$ in the genuine representation categories $Rep(Sp(V))$, $Res(O(W,B))$, we obtain that the illegal constructions should be taken to be 0, in exchange for replacing $\mathrm{Ind}^{P_{j(a)}}$ in the corresponding final legal level by $\mathcal{A}_{j(a)}$, exactly as claimed.

We also note that Theorems 3.5.1, 3.5.2 can be checked elementarily, using the global dimension formula calculated in [30, 31] and observations about how the dimensions of endomorphism algebras of tensor powers of oscillator representations degenerate in the degrees corresponding to metastable reductive dual pairs.

4.5. **An "inductive construction".** As used above, the semisimplification of $\overline{\mathfrak{Rep}}(Sp_{2N}(\mathbb{F}_q))$ is $Rep(Sp_{2N}(\mathbb{F}_q))$, with interpolated Lusztig classification data taken to be true Lusztig classification data defining obejcts of the genuine representation category $Rep(Sp_{2N}(\mathbb{F}_q))$ when it involves legal symbols, and semisimplified to be the 0 object when it involve illegal symbols.

In particular, note that the stable and metastable eta correspondences provides a concrete construction of every irreducible representation of $Sp_{2N}(\mathbb{F}_q)$. Specifically, every irreducible representation $\rho\in Sp_{2N}(\mathbb{F}_q)$ is contained , and using our main result, we can pinpoint the minimal $0\leq n\leq 2N$ and degree n tensor product

$$\omega[V]_{a_1}\otimes\omega[V]_{a_2}\otimes\cdots\otimes\omega[V]_{a_n} \tag{51}$$

of oscillator representations corresponding to non-trivial additive characters of $\mathbb{F}_q$ corresponding to $a_1,\dots,a_n\in\mathbb{F}_q^\times$, since (51) can be calculated as the restriction of $Res_{Sp(V)\times O(W,B)}(\omega[V\otimes W])$ down to $Sp(V)$-representation, where W is a n-dimensional $\mathbb{F}_q$-representation and B is can be described by the n by n matrix with diagonal entries $a_1,\dots,a_b$

$$B=\begin{pmatrix} a_1 & 0 & \dots & 0\\ 0 & a_1 & & 0\\ \vdots & & & \vdots\\ 0 & 0 & \dots & a_n\end{pmatrix}.$$

(Note that outside the stable range, while n is unique, in certain cases, both distinct choices of B may contain ρ).

Further, by our description, at such a minimal n, ρ appears as a summand of the top part of $Res_{Sp(V)\times O(W,B)}(\omega[V\otimes W]))$, and we can pinpoint the irreducible $O(W,B)$-representation π whose tensor product with ρ appears in $Res_{Sp(V)\times O(W,B)}(\omega[V\otimes W]))$, i.e. such that

$$\rho = \eta^V_{W,B}(\pi). \tag{52}$$

Given this information, we can explicitly construct an endomorphism of (51) whose image is ρ: In [30, 31], in our proof of the decomposition (51) in the stable range, we used an identification of the endomorphism algebra of (51) with the $Sp(V)$-fixed points on a free $\mathbb{C}$-algebra with basis equal to a direct sum of n copies of V

$$End_{Sp(V)}(\bigotimes_{i=1}^n \omega[V]_{a_i}) \cong (\mathbb{C}(\bigoplus_n V))^{Sp(V)}.$$

In this description, we produced endomorphisms corresponding to reflections in $O(W,B)$ (generating a full group algebra $\mathbb{C}O(W,B)$), and we proved that the subalgebra $\mathbb{C}O(W,B)$ is the endomorphism algebra of the top part of $Res_{Sp(V)\times O(W,B)}(\omega[V\otimes W]))$. Therefore, knowing $\pi\in\widehat{O(W,B)}$ satisfying (52), we may consider the idempotent

$$\frac{dim(\pi)}{|O(W,B)|}\sum_{g\in O(W,B)}\chi_\pi(g)^{-1}\cdot g\in \mathbb{C}O(W,B) \tag{53}$$

in $End_{Sp(V)}(\bigotimes_{i=1}^n \omega[V]_{a_i})$, where

$$\chi_\pi : O(W,B)\to\mathbb{C}^\times$$

denotes the character corresponding to π. The image of (53) in (51) recovers ρ. Outisde of the stable range, the constructions of the reflection elements are readily interpolated. In the metastable range, if $\rho\otimes\pi$ is in the top part of the restricted oscillator representation, then the corresponding idempotent (53) survives and is unaltered by semisimplification.

This process also works for $Sp(V)$ and $O(W,B)$ switched, to write down $\zeta^{W,B}_V$. Therefore, we can produce any Lusztig symbol in a choice of Lusztig classification data inductively, adding one coordinate at a time.

5. The Gurevich-Howe rank conjecture

Finally, as advertised in the introduction, the purpose of this section is to apply our calculation of the eta correspondence to prove Theorem 2, verifying the rank conjecture of S. Gurevich and R. Howe, which predicts the equality of U-rank and tensor-rank for every representation of a symplectic group $Sp_{2N}(\mathbb{F}_q)$ not attaining top possible U-rank N.

First, in Subsection 5.1, we recall the definitions of U-rank and tensor rank in more detail and recall the results of Gurevich and Howe in [15, 16] which reduce Theorem 2 to a statement that all irreducible representations with tensor rank larger than N have U-rank equal to N (see Proposition 5.1.1). Next, in Subsection 5.2, we use our explicit description of the extended eta correspondence to obtain an induction relation (see Proposition 5.2.1) between tensor ranks from an analogue of the Pieri rule, which reduces the claimed statement to lower tensor rank, again.

5.1. ***U*-rank and the eta correspondence.** First we recall the definitions of U-rank and tensor rank given in [15, 16]. We begin with U-rank. Consider a symplectic group $Sp_{2N}(\mathbb{F}_q)$. The Siegel unipotent subgroup is defined as

$$U_N = \{\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \mid A \in M_{N\times N}(\mathbb{F}_q) \text{ symmetric}\} \subseteq Sp_{2N}(\mathbb{F}_q).$$

Note that, as a group, U_N is isomorphic to the abelian group of symmetric $N \times N$ matrices, with respect to addition. In particular, we may fix an identification of U_N with its Pontrjagin dual U_N^* in the standard way, i.e. by fixing a non-trivial additive character $\chi_0 : \mathbb{F}_q \to \mathbb{C}^\times$ and identifying each element of U_N corresponding to a symmetric matrix A with the character

$$\begin{gathered} U_N \to \mathbb{C}^\times \\ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \mapsto \chi_0(tr(AB)). \end{gathered}$$

Gurevich and Howe [15, 16] then define the *U-rank* of an $Sp_{2N}(\mathbb{F}_q)$-representation ρ as the maximal rank of a character appearing in its restriction to U_N:

$$rk_U(\rho) := \max\{rk(\chi) \mid \chi \in U_N^* \text{ and } \chi \subseteq Res_{U_N}(\rho)\}, \tag{54}$$

where for a character $\chi \in U_N^*$, its rank is defined to be the matrix rank of the symmetric $N \times N$-matrix specifying its corresponding element of U_N.

On the other hand, the tensor rank of a $Sp_{2N}(\mathbb{F}_q)$-representation is defined according to the oscillator representations. Recall that each oscilaltor representation of a symplectic group $Sp(V)$ decomposes into two irreducible summands

$$\omega_a[V] = \omega_a^+[V] \oplus \omega_a^-[V]$$

(from the perspective of the eta correspondence, $\omega_a^{\pm}[V]$ is obtained by applying the eta correspondence to the representations (± 1) of $\mathbb{Z}/2 = O_1(\mathbb{F}_q)$. These pieces of the oscillator representation are the smallest non-trivial irreducible representations of $Sp(V)$, and they each have U-rank 1. The *tensor rank* of a representation ρ is then defined as the minimal degree n such that every irreducible component of ρ appears in a tensor product of less than or equal to n oscillator representations:

$$rk_{\otimes}(\rho) := min\{n \mid \text{for } \pi \in \widehat{Sp(V)},\ \pi \subseteq \rho,\ \text{there exists}$$
$$m \leq n,\ a_1, \dots, a_m \in \mathbb{F}_q^{\times} \text{ with } \pi \subseteq \omega_{a_1}[V] \otimes \dots \otimes \omega_{a_m}[V]\}$$

Now recalling again that a degree n tensor product

$$\omega_{a_1}[V] \otimes \dots \otimes \omega_{a_n}[V]$$

can be considered as the restirction of the oscillator representation of a larger symplectic group $Sp(V \otimes W)$ along the inclusion

$$Sp(V) \hookrightarrow Sp(V) \times O(W, B) \hookrightarrow Sp(V \otimes W),$$

where (W, B) denotes an n-dimensional $\mathbb{F}_q$-space with a non-degenerate symplectic form B corresponding to the diagonal matrix with entries $a_1, \dots, a_n$, we see that understanding the restricted oscillator representation $Res_{Sp(V)\times O(W,B)}(\omega[V \otimes W])$ is key to. In particular, the results of this paper for pairs $(Sp(V), O(W, B))$ in the symplectic stable or metastable range explicitly classify the irreducible representations of each tensor rank $0 \leq rk_{\otimes} \leq 2N$.

In [15, 16], Gurevich and Howe also found a connection between the restricted oscillator representations $Res_{Sp(V)\times O(W,B)}(\omega[V \otimes W])$ and U-rank, which was one of their original motiviation for defining the eta correspondence:

The original statement describing the eta correspondence given in, say, Theorem 4.3.3 of [16], is that for choices of V and (W, B) in the symplectic stable range (which we recall means $dim(W) \leq dim(V)/2$, there is a system of injections

$$\eta^V_{W,B} : \widehat{O(W, B)} \hookrightarrow \widehat{Sp(V)}$$

(we omit the subscript when the source is determined) such that for every irreducible representations $\rho \in \widehat{O(W, B)}$, the tensor product $\rho \otimes \eta^V_{W,B}(\rho)$ is a summand of $Res_{Sp(V)\times O(W,B)}(\omega[V \otimes W])$, and

$$rk_U(\eta^V_{W,B}(\rho)) = dim(W). \tag{55}$$

Further, every other $\pi \in \widehat{Sp(V)}$ such that $\rho \otimes \pi$ appears in the restricted oscillator representation has strictly lower U-rank. (Note that though Theorem 4.3.3 of [16] does not include the case of $dim(W) = dim(V)/2$, the result still applies to this case as described in Remark 4.3.6.)

First, we note that the results of [15, 16] immediately imply the agreement of tensor- and U-rank in cases covered by the symplectic stable range:

Corollary 1. *For irreducible $Sp_{2N}(\mathbb{F}_q)$-representations ρ of tensor rank $\leq N$, the notions of rank coincide:*

$$rk_{\otimes}(\rho) = rk_U(\rho).$$

Therefore, it only remains to prove the following

Proposition 5.1.1. *Consider an irreducible representation ρ of a symplectic group $Sp_{2N}(\mathbb{F}_q)$ obtained first in the restriction of an oscillator representation to an unstable reductive dual pair, meaning*

$$N < rk_{\otimes}(\rho) \leq 2N.$$

Then ρ attains top U-rank

$$rk_U(\rho) = N.$$

5.2. **An induction rule and concluding Theorem 2.** As described in the previous subsection, to prove Proposition 5.1.1, we need more explicit information about the symplectic group representations of each tensor rank, which can be obtained from Theorem 3.5.1. In particular, in the decomposition (37), we may further restrict to $Sp(V)$-represenations by treating the coefficient $O(W, B)$-representations as multiplicity spaces, obtaining a classification of the irreducible $Sp(V)$-represntations of tensor rank $rk_{\otimes} = r$ for each $0 \leq r \leq 2N$ as precisely those constructed in the image of an eta correspondence

$$\eta^V_{W,B} : \widehat{O(W, B)} \to \widehat{Sp(V)} \cup \{0\},$$

for one of the two non-equivalent choices of (W, B) with dimension r.

The key step we use to conclude Propostion 5.1.1 and Theorem 2 is the following result, which gives a relationship between the different rank layers of the eta correspondence, according to parabolic induction:

Proposition 5.2.1. *Fix a $\mathbb{F}_q$-vector space W with symmetric bilinear form B. Consider symplectic spaces V, U of dimension $2N \leq 2M$ respectively, such that both reductive dual pairs $(Sp(V), O(W, B))$ and*

$(Sp(U), O(W,B))$ are in the symplectic stable or metastable ranges. Then we may consider the eta correspondences

$$\eta^V : \widehat{O(W,B)} \to \widehat{Sp(V)} \cup \{0\}$$

$$\eta^U : \widehat{O(W,B)} \to \widehat{Sp(U)} \cup \{0\}.$$

For an irreducible representation $\pi \in \widehat{O(W,B)}$ such that $\eta^V(W) \neq 0$, we have

$$\eta^U(\pi) \subseteq Ind^{P^U_{M-N}}(\eta^V(\pi)^{\pm}), \tag{56}$$

where the $\pm$ denotes whether we consider a sign character on the factor $GL_{M-N}(\mathbb{F}_q)$ of the Levi subgroup before inflating to P^U_{M-N} and applying the induction. The sign is $+$ when W is even dimensional and is $-$ when W is odd dimensional.

To prove this, we now recall briefly the analogue of the Pieri rule for symbols. We give a more concrete statement in Subsection **??** below with a more detailed explanation on how it can be derived from the results of [35]. For the purposes of proving Theorem 2, we only need one case of it, so we do not give the full general statement in this subsection.

Consider a unipotent representation of $Sp_{2N}(\mathbb{F}_q)$ corresponding to a symbol

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}.$$

Write P_1 for the maximal parabolic subgroup of $Sp_{2(N+1)}(\mathbb{F}_q)$ with Levi factor $Sp_{2N}(\mathbb{F}_q) \times GL_1(\mathbb{F}_q)$, and consider $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ as a its representation by letting the $GL_1(\mathbb{F}_q)$ factor of the Levi subgroup act trivially and inflating on the unipotent radical trivially. Then its parabolic induction to a $Sp_{2(N+1)}(\mathbb{F}_q)$-representation

$$Ind^{P_1}_{Sp_{2(N+1)}(\mathbb{F}_q)}(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix})$$

is a direct sum of unipotent representations corresponding to symbols

$$\begin{gathered} \begin{pmatrix} \lambda_1 < \cdots < \lambda_{i-1} < \lambda_i + 1 < \lambda_{i+1} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}, \\ \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_{i-1} < \mu_i + 1 < \mu_{j+1} < \cdots < \mu_a \end{pmatrix} \end{gathered} \tag{57}$$

when possible, i.e. for $1 \le i \le a$ or $1 \le j \le b$ where $\lambda_i + 1 < \lambda_{i+1}$ or $\mu_j + 1 < \mu_{j+1}$, respectively, and the unipotent representations

$$\begin{pmatrix} 1 < \lambda_1 + 1 < \lambda_2 + 1 < \cdots < \lambda_a + 1 \\ 0 < \mu_1 + 1 < \cdots < \mu_b + 1 \end{pmatrix} \tag{58}$$
$$\begin{pmatrix} 0 < \lambda_1 + 1 < \cdots < \lambda_a + 1 \\ 1 < \mu_1 + 1 < \mu_2 + 1 < \cdots < \mu_b + 1 \end{pmatrix}$$

when possible, i.e. when $\lambda_1 > 0$ or $\mu_1 > 0$, respectively. This is a full description of the "one step" Pieri rule.

More generally, for the "r step" Pieri rules, describing the parabolic induction from a maximal parabolic P_r with Levi subgroup $Sp_{2N}(\mathbb{F}_q) \times GL_r(\mathbb{F}_q)$ to $Sp_{2(N+r)}(\mathbb{F}_q)$ (still taking $GL_r(\mathbb{F}_q)$ and the unipotent radical to act trivially on the input representation), instead of adding a single "box" to the underlying Young diagrams corresponding to a symbol $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$, we must add a "row of r boxes." More specifically, one must undo the procedures described in Proposition 3.2 and Subsection 4.6 of [35], then apply the classical Pieri rule adding a "row of r boxes" as a Weyl group representation, before re-applying the procedures of [35] to recover the original defect and the new rank $N + r$. This rule can be derived directly from the definition of the symbols (see [35], Subsection 4.8).

In particular, summands that always appear in $Ind^{P_r}(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b})$ are symbols obtained by adding r to the final coordinate in a row:

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_{a-1} < \lambda_a + r \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \tag{59}$$
$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_{b-1} < \mu_b + r \end{pmatrix}.$$

To apply such a parabolic induction Ind^{P_r} to a general representation ρ of $Sp_{2N}(\mathbb{F}_q)$, the resulting $Sp_{2(N+r)}(\mathbb{F}_q)$ representation consists of summands which add 1's to the semisimple part of ρ's Lusztig classification data and have unipotent part consisting of the input unipotent part with the factor corresponding to the centralizer of 1 eigenvalues replaced by the possible pieces of its r step parabolic induction. We also consider the "signed parabolic induction" $Ind^{P_r}(\rho^-)$, by which we denote the $Sp_{2(N+r)}(\mathbb{F}_q)$-representation obtained by tensoring ρ with the sign character of the $GL_r(\mathbb{F}_q)$ factor of the Levi subgroup of P_r before inflating and inducing. The procedure on Lusztig classification data giving the signed parabolic induction is completely similar to the unsigned case, except that -1's are added to the semisimple part of the data corresponding to the input representation (instead of 1's) and the symbol corresponding to this factor of its centralizer is altered, instead.

In particular, by combining the symbol Pieri rule with our description of the eta correspondence given in Definitions 3.2.1 and 3.2.2 we are able to conclude Proposition 5.2.1:

Proof of Proposition 5.2.1. The choice of sign in (56) precisely specifies whether the induction operation will add 1's or -1's to the Lusztig classification data of the input representation. Since it is chosen according to the partiy of $dim(W)$, the semisimple part of the Lusztig classification data of $\eta^U(\pi)$ agrees with that of the irreducible summands of $Ind^{P^U_{M-n}}(\eta^V(\pi)^{\pm})$, reducing the claim to the fact that in the "altered factor" of the unipotent parts of $\eta^U(\rho)$ and $\eta^V(\rho)$,

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < M'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \subseteq Ind^{P_{M-N}}(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}),$$

which follows from the symbol Pieri rule. □

Considering the effect of parabolic induction on U-rank then allows us to reduce Proposition 5.1.1 to Corollary 1, and conclude Theorem 2:

Proof of Proposition 5.1.1 and Theorem 2. Consider a representation ρ of $Sp(V)$ of tensor rank

$$N < rk_\otimes(\rho) \leq 2N.$$

Then by Theorem 3.5.1, there exsits a choice of (W, B) with $dim(W) = rk_\otimes(\rho) > N$, and an irreducible representation $\pi \in \widehat{O(W, B)}$ such that

$$\eta^V_{W,B}(\pi) = \rho.$$

Let us denote by V' the symplectic space of dimension $dim(V') = 2 \cdot dim(W)$, i.e. the maximal dimensional symplectic space such that $(Sp(V'), O(W, B))$ is a reducive dual pair in the symplectic stable range. Consider the eta correpsondence

$$\eta^{V'} : \widehat{O(W, B)} \hookrightarrow \widehat{Sp(V')},$$

and its image of π. Let us write

$$\rho' = \eta^{V'}(\rho) \in \widehat{Sp(V')}.$$

Applying Corollary 1, we know that as a representation of $Sp(V') = Sp_{2\cdot dim(W)}(\mathbb{F}_q)$, its U-rank is

$$rk_U(\rho') = rk_\otimes(\rho') = dim(W).$$

Now applying Proposition 5.2.1 gives that ρ' appears as a summand of a (possibly signed) parabolic induction of ρ from a parabolic subgroup with Levi factor

$$Sp(V') \times GL_{dim(W)-N}(\mathbb{F}_q),$$

which is an operation that can only increase U-rank by at most the difference $dim(W) - N$. In other words, the U-rank of ρ is at least

$$rk_U(\rho) \geq rk_U(\rho') - (dim(W) - N) =$$
$$rk_\otimes(\rho') - (dim(W) - N) = N,$$

and therefore we must have equality $rk_U(\rho) = N$, obtaining (7). $\square$

6. Resolving the alternating sums

In Theorem 3.5.1 and 3.5.2, we decompose the restricted oscillator representation $Res_{Sp(V)\times O(W,B)}(\omega[V \otimes W])$ in terms of the eta correspondence (in the case of the symplectic stable or metastable range, corresponding to the condition (17)) or the zeta correspondence (in the orthogonal metastable or stable ranges, corresponding to the complementary condition (18)). Further, we described the eta and zeta correspondences in terms of Lusztig classification data, allowing us to directly compute the $Sp(V)$-representation and $O(W, B)$-representation summands occuring in the restriction of $\omega[V \otimes W]$.

However, in (37) and (38), the eta and zeta correspondence terms appear with coefficients $\mathcal{A}_k(\rho, N'_\rho)$ (giving an $O(W, B)$-representation, for ρ an irreducible $O(W[-k], B[-k])$-representation) or $\rangle(\rho, m'_\rho)$ (giving a $Sp(V)$-representations, for ρ an irreducible $Sp(V[-k])$, respectively. Our description of these terms for the purpose of proving the Theorem was as certain alternating sums of parabolic inductions. The purpose of this section is to simplify these sums $\mathcal{A}_k(\rho, N'_\rho)$ (resp. $\mathcal{A}_i(\rho, m'_\rho)$) and describe their irreducible $O(W, B)$- (resp. $Sp(V)$-) representation summands in a way that can be used for concrete computations.

6.1. **The main statement.** Our main result is

Theorem 6.1.1. *Fix a reductive dual pair* $(Sp(V), O(W,B))$ *in the symplectic stable or metastable range. For an irreducible representation* ρ *of* $O(W[-k], B[-k])$*, consider the factor of the unipotent part of its Lusztig classification data writable as a symbol*

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}, \tag{60}$$

such that it is replaced by a symbol

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

in the construction of $\eta^V_{W[-k],B[-k]}(\rho)$ *(switch rows if necessary). Then the* $O(W,B)$*-representation* $\mathcal{A}_k(\rho, N'_\rho)$ *consists of the irreducible summands appearing in the parabolic induction* $Ind^{P_k}(\rho \otimes \epsilon(det))$ *such that, when performing the Pieri rule (see Proposition 6.2.1) on the row* $\lambda_1 < \cdots < \lambda_a$ *of (60), the highest coordinate* $\lambda'_{a'}$ *of the corresponding row of the new symbol satisfies*

$$\lambda'_{a'} < N'_\rho + (a' - a).$$

There is a similar description in the case of $(Sp(V), O(W,B))$ *in the orthogonal stable or metastable range, of the* $Sp(V)$*-representations* $\mathcal{A}_i(\rho, m'_\rho)$ *for* $Sp(V[-i])$*-representations* ρ.

To prove this, we use the Pieri rule for Lusztig symbols, which we state in Proposition 6.2.1 below. Recall that the combinatorial data of a symbol (classifying the irreducible unipotent representations) is equivalent to the data of its defect (which gives the information of the underlying cuspidal representation) and a pair of Young diagrams corresponding to the irreducible representation of the remaining Weyl group specifiying which piece of the induced cuspidal representation the symbol corresponds to (this data comes from undoing the procedures given in Proposition 3.2 and Subsection 4.6 or 4.7 of [35]; we give more details below).

First, as we discussed in Subsection 3.4, in each case, in every summand of the alternating sums of parabolic inductions all factor through precisely to the symbol altered by the eta or zeta correspondence (the symbol of the factor of the unipotent part corresponding to $(-1)^{dim(W)}$ eigenvalues), with the other Lusztig classification data being preserved.

Therefore, the problem of simplifying $\mathcal{A}_k(\rho, N'_\rho)$ (resp. $\mathcal{A}_k(\rho, m'_\rho)$) reduces to simplifying an alternating sum of parabolic inductions of symbols. As in Definitions 3.4.1 and 3.4.2, write $\theta = \binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ for the symbol corresponding to the factor of the unipotent part of ρ's Lusztig classification data which is altered in the extended eta correspondence (resp. the extended zeta correspondence). Arrange the rows so that the eta correspondence alters the top row. Write

$$k^{(i)} = k - N'_\rho + \lambda_i, \tag{61}$$

so that we have

$$k^{(a)} \geq k^{(a-1)} \geq \cdots \geq k^{(1)}.$$

Pick the minimal i_0 such that $k^{(i_0)} \geq 0$ We then need to find the symbols χ appearing in the alternating sum $A_k^{\pm}(\theta, N'_\rho)$ (with superscript sign agreeing with the sign of $\phi^{\pm}(u)$ or $\psi^{\pm}(u)$ appearing in the the construction of $\eta^V_{W,B}(\rho)$ or $\zeta^{W,B}_V(\rho)$), which can be written out as

$$\bigoplus_{i=i_0}^{a+1} (-1)^{a+1-i} \cdot Ind^{P_{k^{(i)}}}(\begin{pmatrix} \lambda_1 < \cdots < \widehat{\lambda_i} < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}) \tag{62}$$

(for the case of the extended zeta correspondence, replace N'_ρ by m'_ρ). The irreducible components of $\mathcal{A}_k(\rho, N'_\rho)$ will then consist of of $O(W, B)$-representations obtained with Lusztig classification data obtained by adding $(-I)_{2k}$ to ρ's original semisimple part, and taking the unipotent part obtained by tensoring a symbol in (62) (and choosing the same central sign data as ρ).

In other words, Theorem 6.1.1 can be more precisely stated as

Theorem 6.1.2. *Assume the above notation. The alternating sum (62) simplifies as the sum of all symbols*

$$\begin{pmatrix} \lambda'_1 < \cdots < \lambda'_{a'} \\ \mu_1 < \cdots < \mu'_{b'} \end{pmatrix} \subseteq Ind^{P_k}(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix})$$

such that

$$\lambda_{a'} < N'_\rho + (a' - a).$$

This statement can now be proved directly by considering the induction rule on symbols, which can be done by translating the symbols into Young diagram data.

Consider a symbol of type B, C, D or ${}^2D$-type

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}. \tag{63}$$

We want to consider the "k-step induction"

$$Ind^{P_k}(\begin{pmatrix}\lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b\end{pmatrix}), \tag{64}$$

where P_k denotes the standard maximal parabolic $P_k^{W,B}$ if we take (63) to correspond to a unipotent representation of $O(W[-k], B[-k])$ for some orthogonal space (W, B), or P_k^V if we take (63) to correspond to a unipotent representation of $Sp(V[-k])$ for some symplectic space V. To consider a symbol (63) as a representation of P_k in either of these cases, let the factor $GL_k(\mathbb{F}_q)$ of the Levi subgroup of P_k act trivially, and inflate by letting. Then (63) denotes the induction of the resulting representation to $O(W, B)$ or $Sp(V)$. The decomposition of (64) is according to a *Pieri rule*, which we concretely state in Proposition 6.2.1 below.

6.2. **The Pieri rule.** First, let us briefly recall the role of symbols as representations. In (4.6.2), (4.7.1) of [35], Lusztig described how the combinatorial data of a pair of increasing sequences (63) corresponds to an irreducible representation of a certain Hecke algebra $\mathscr{H}_n(q, q^d)$ defined according to certain relations (see Subection 4.1 of [35]) which are equivalent to the classical Iwahori relations and recover the group algebra of the Weyl group (see [5], §68A). In Subsection 4.8 of [35], Lusztig also describes how induction is preserved by these correspondences. For the Weyl groups of the groups we consider here, the irreducible representations in each case are classified by pairs of Young diagrams whose total numbers of boxes add up to the rank. Therefore, the induction (64) can be computed by applying the Pieri rule to these Young diagrams i.e., by considering all choices of $k_1 + k_2 = k$, and adding a row of length k_1 to the top row's corresponding Young diagram and a row of length k_2 to the bottom row's corresponding Young diagram.

To find the Weyl group representation corresponding to a symbol (63), without loss of generality, switch rows so that $a \geq b$, and denote the defect by $d = a - b$.

First suppose d is strictly positive. The symbol notation then indicates that the unipotent representation $\binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ is constructed in an induction of the cuspidal unipotent representation corresponding to the symbol

$$\begin{pmatrix}0 < 1 < 2 < \dots < d-1 \\ \emptyset\end{pmatrix}$$

(the minimal rank symbol of defect d). The first step of Lusztig's procedure is to "remove" this cuspidal representation from the symbol (i.e. by undoing the bijection j of Proposition 3.2 of [35]), to obtain a defect one symbol

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ 0 < 1 < \cdots < d-2 < \mu_1 + (d-1) < \cdots < \mu_b + (d-1) \end{pmatrix}, \tag{65}$$

(using the convention of [35] describing how to reduce a symbol if the coordinate of its first two rows is 0).

The next step is to undo the procedure described in Subsection 4.6 of [35] to obtain Young diagrams. In the case of (65), we obtain a Young diagram

$$(\lambda_1 \leq \lambda_2 - 1 \leq \cdots \leq \lambda_a - (a-1)) \tag{66}$$

where the ith row has length $\lambda_{a-i+1} - (a-i)$ corresponding to the top row, and a Young diagram

$$(\mu_1 \leq \mu_2 - 1 \leq \cdots \leq \mu_b - (b-1)) \tag{67}$$

where the ith row has length $\mu_{b-i+1} - (b-i)$ corresponding to the bottom row (not writing the rows with length 0 corresponding to the coordinates $0 < 1 < \cdots < d-2$ concatenated onto the bottom row in (65)).

In the case of defect $d = 0$, we undo the procedure in Subsection 4.7 of [35] to obtain this same answer, of a Weyl group representation corresponding to Young diagrams (66), (67).

We will denote the Young diagrams (66), (67), by α, β, denoting the ith row lengths by

$$\begin{aligned} \alpha_i &:= \lambda_{a-i+1} - (a-i) \\ \beta_i &:= \mu_{b-i+1} - (b-i). \end{aligned} \tag{68}$$

(We use the convention that the first row of the Young diagram is the longest.)

We use the terminology that, for a Young diagram $(\gamma_n \leq \cdots \leq \gamma_1)$ and for a natural number k, we say Young diagrams of the form

$$(k_{n+1} \leq \gamma_n + k_n \leq \gamma_n + k_{n-1} \leq \cdots \leq \gamma_1 + k_1)$$

where k_i are natural numbers satisfying $k_1 + \cdots + k_{n+1} = k$ and, for every $i = 1, \dots, n$, we have $k_{i+1} \leq \gamma_i - \gamma_{i+1}$ (putting $\gamma_{n+1} = 0$ are the Young diagrams *obtained by adding a row of length k to γ*.

The Pieri rule for symbols can then be stated as follows:

Proposition 6.2.1. *Fix an orthogonal space (W, B) (resp. a symplectic space V) and consider a symbol $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ defining a unipotent representation of $O(W[-k], B[-k])$ (resp. $Sp(V[-k])$). Recall that* *we denote by P_k the standard maximal parabolic with Levi subgroup $O(W[-k], B[-k]) \times GL_k(\mathbb{F}_q)$ (resp. $Sp(V[-k]) \times GL_k(\mathbb{F}_q)$), and consider the symbol as a P_k-representation by letting $GL_k(\mathbb{F}_q)$ act trivially and inflating by letting the unipotent part of the parabolic act trivially. Then its parabolic induction to an $O(W, B)$-representation decomposes as a sum of symbols*

$$Ind^{P_k}(\begin{pmatrix}\lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b\end{pmatrix}) = \bigoplus_{\mathcal{S}_k[\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}]} \begin{pmatrix}\lambda'_1 < \cdots < \lambda'_{a'} \\ \mu'_1 < \cdots < \mu'_{b'}\end{pmatrix}$$

where the sum runs over the set of symbols $\mathcal{S}_k[\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}]$ consisting of $\binom{\lambda'_1 < \cdots < \lambda'_{a'}}{\mu'_1 < \cdots < \mu'_{b'}}$ where, for some $k_1 + k_2 = k$, the Young diagram

$$(\lambda'_1 \leq \lambda'_2 - 1 \leq \cdots \leq \lambda'_{a'} - (a' - 1))$$

is obtained by adding a row of length k_1 to

$$(\lambda_1 \leq \lambda_2 - 1 \leq \cdots \leq \lambda_a - (a - 1)),$$

and the Young diagram

$$(\mu'_1 \leq \mu'_2 - 1 \leq \cdots \leq \mu'_{b'} - (b' - 1))$$

is obtained by adding a row of length k_2 to

$$(\mu_1 \leq \mu_2 - 1 \leq \cdots \leq \mu_b - (b - 1)).$$

(The awkwardness of this statement is in order to accomodate all cases of input symbols, including those where one of the rows of the original symbol begins with a 0 coordinate, and to properly address the case when a' and b' are larger than a and b.)

6.3. **The proof of the main statement.** We then conclude Theorem 6.1.2 by considering it in terms of Young diagrams, and applying this Pieri rule.

Proof of Theorem 6.1.2. Let us use the above notation (68) for an alterable unipotent part of a representation ρ for which we want to find the coefficient $\mathcal{A}_k(\rho, N'_\rho)$ appearing with $\eta^V_{W,B}(\rho)$. Since the resulting Young diagrams (66), (67) ultimately do no depend on which row was on top or longer, we may say without loss of generality that the top row is the one where we add the coordinate N'_ρ when constructing $\eta^V_{W,B}(\rho)$.

Applying this to the context of Theorem 6.1.2, the symbol

$$\begin{pmatrix}\lambda_1 < \cdots < \lambda_{i-1} < \lambda_{i+1} < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b\end{pmatrix} \tag{69}$$

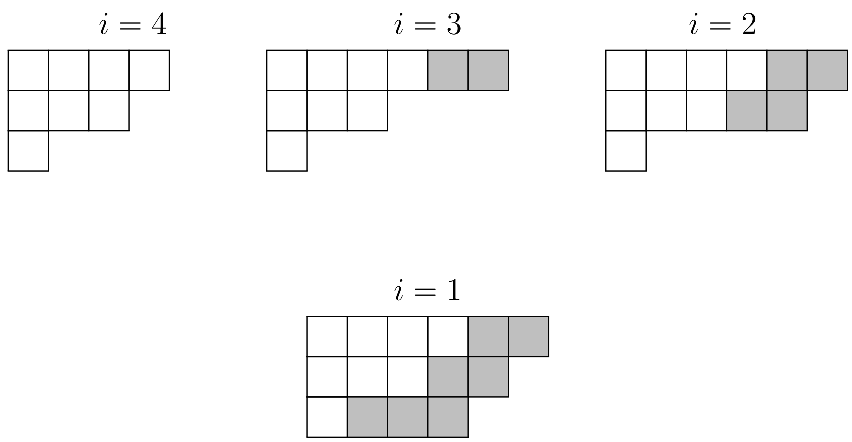

FIGURE 1. These are the top row Young diagrams corresponding to the symbols (69), in the case of $\alpha = (1 \leq 3 \leq 4)$, $N'_\rho - a = 6$. The boxes highlighted gray show the skew semistandard tableau which, after removal from each Young diagram recovers the original α. At each i, just enough boxes are added to a row of the $(i+1)$th Young diagram to not be obtainable from the $(i+2)$th.

appearing in the ith term of the alternating sum (62) corresponds to the same cuspidal representation $\binom{0<1<\cdots<d-1}{\emptyset}$ as the original symbol (since it has the same row lengths) and the Young diagrams

$$\begin{matrix} (\alpha_a \leq \cdots \leq \alpha_{a-i+2} \leq \alpha_{a-i} + 1 \leq \cdots \leq \alpha_1 + 1 \leq N'_\rho - a) \\ (\beta_b \leq \cdots \leq \beta_1). \end{matrix} \tag{70}$$

(See Figure 1 for an example.)

The claimed reduction of the alternating sum consists of symbols corresponding to the same underlying cuspidal representation and a pair of Young diagrams (α', β') obtained from applying the k-step Pieri rule to (α, β) such that the first row length of α' is strictly bounded

$$\alpha'_1 < N'_\rho - a. \tag{71}$$

First we note that the summands of the initial parabolic induction satisfying this condition must survive in the alternating sum, since the condition (71) guarantees that such an (α', β') cannot appear in the parabolic induction of any of the other terms for $i \leq a$, since the first row is shorter than the first row of any top Young diagram in (70).

It remains to see that every other term in the alternating sum vanishes. First, for summands of the initial parabolic induction term $Ind^{P_k}(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})$ which fail the condition (71) are cancelled in the next term at $i=a$, since its corresponding top Young diagram

$$(\alpha_a \leq \cdots \leq \alpha_2 \leq N'_\rho - a).$$

Similarly, we may proceed inductively to see that every pair of Young diagrams appearing from the ith term of the alternating sum which was not used to cancel the $(i+1)$th term is cancelled at the $(i-1)$th term. This follows since, at each step i, the condition on a pair of Young diagrams to have already appeared in the $(i+1)$th term (and therefore be cancelled in the previous step) is that the length of the $(a-i+2)$th row does not exceed what can be attained from the previous step without adding boxes directly above each other, i.e.

$$\alpha'_{a-i+2} \leq \alpha_{a-i+1}. \tag{72}$$

On the other hand, the condition on α' to appear in the $(i-1)$th term is complementary

$$\alpha'_{a-i+2} \geq \alpha_{a-i+1} + 1,$$

since the condition is that the $(a-i+2)$th row is at least as long as the corresponding row of the top Young diagram in the $(i-1)$th term, so all the cases are covered.

It remains therefore to check that every term from the final term (corresponding to the lowest i) in the alternating sum cancels. Recall that the sum (62) has final term at $i=i_0$, which is the smallest. First, suppose $i_0 > 1$. Then by definition

$$k^{(i_0-1)} = k - N'_\rho + \lambda_{i_0-1} < 0, \tag{73}$$

and thus, when considering the final induction

$$\mathrm{Ind}^{k^{(i_0)}}(\begin{pmatrix} \lambda_1 < \cdots < \widehat{\lambda_{i_0}} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}),$$

on the Young diagram level, every top Young diagram appearing satisfies the condition (72), since it we would need to add at least $\lambda_{i_0} - \lambda_{i_0-1}$ boxes, which is strictly more than $k^{(i_0)}$ by (73). Therefore, all terms are used up in the previous step, giving the claim. In the case when $i_0 = 1$, we again know that (72) must also follow, by the metastable dimension condition, since the final number of boxes to be added is

$$k^{(1)} = k - N'_\rho + \lambda_1 < \lambda_1$$

(since $k \leq m$ and $N'_\rho > m$ because we assumed that V and (W, B) lie in the symplectic stable or metastable range).

$\square$

Example: We emphasize the point that this alternating sum, even in extreme cases, need not by irreducible. The largest symbol (giving the unipotent irreducible representation of maximal dimension) for $SO_{2m+1}(\mathbb{F}_q)$ is

$$\begin{pmatrix} 0 < 1 < 2 < \cdots < m \\ 1 < 2 < \cdots < m \end{pmatrix},$$

which gives the Steinberg representation St_m, of dimension q^{m^2}. In this case, its corresponding pair of Young diagrams consists of a column of m boxes (corresponding to the top row) and the empty Young diagram (corresponding to the bottom row In this case, the Pieri rule adding a single box gives

$$\mathrm{Ind}^{P_1}(St_m) = \begin{pmatrix} 0 < 1 < 2 < \cdots < m-1 < m+1 \\ 1 < 2 < \cdots < m \end{pmatrix} \oplus$$

$$\begin{pmatrix} 0 < 1 < 2 < \cdots < m \\ 1 < 2 < \cdots < m-1 < m+1 \end{pmatrix} \oplus St_{m+1}$$

Then, for example, the alternating sum of symbols $A_1^{\pm}(St_m, m+1)$ with sign chosen to alter the top row outputs

$$\mathrm{Ind}^{P_1}(St_m) - \begin{pmatrix} 0 < 1 < 2 < \cdots < m-1 < m+1 \\ 1 < 2 < \cdots < m \end{pmatrix} =$$

$$\begin{pmatrix} 0 < 1 < 2 < \cdots < m \\ 1 < 2 < \cdots < m-1 < m+1 \end{pmatrix} \oplus St_{m+1}$$

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