HOWE DUALITY OVER FINITE FIELDS II: EXPLICIT STABLE COMPUTATION

SOPHIE KRIZ

Abstract. In this second paper of a series dedicated to type I Howe duality for finite fields, we explicitly decompose the restriction of an oscillator representation of a finite symplectic group to the product of a symplectic and an orthogonal subgroup which are each other's centralizers in terms of G. Lusztig's classification of irreducible representations of finite groups of Lie type in the two so-called stable ranges.

1. Introduction

This is the second paper of a series dedicated to type I Howe duality for finite fields. We recall that over a finite field, the oscillator representation is a representation of a finite symplectic group, in which a type I reductive dual pair consists of a symplectic and an orthogonal subgroup which are each other's centralizers. In [12], we constructed explicit correspondences between the sets of representations on the symplectic and orthogonal side in the two stable ranges. In this paper, we shall describe these correspondences explicitly in terms of G. Lusztig's classification of representations of finite groups of Lie type (see, for example, [14]).

Consider an oscillator representation $\omega[V \otimes W]$ of a symplecitc group $Sp(V \otimes W)$ restricted to such a reductive dual pair, consisting of a symplectic group $Sp(V)$ and an orthogonal group $O(W, B)$. In the first part of this series [12], we proved the existence of a *symplectic* and an *orthogonal stable range*, where the restriction of the oscillator representation to the product $Sp(V) \times O(W, B)$ decomposes in terms of (twisted) parabolic inductions and a system of injections with mutually disjoint images

$$\eta^V_{W,B} : \widehat{O(W, B)} \hookrightarrow \widehat{Sp(V)} \tag{1}$$

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(where $\widehat{G}$ denotes the set of isomorphism classes of irreducible complex representations of a group G) for $(Sp(V), O(W,B))$ a symplectic stable reductive dual pair (where symplectic stability means that $dim(W) \leq dim(V)/2$), and

$$\zeta_V^{W,B} : \widehat{Sp(V)} \hookrightarrow \widehat{O(W,B)} \tag{2}$$

for $(Sp(V), O(W,B))$ an orthogonal stable reductive dual pair (where $dim(V)$ is less than or equal to the dimension of the maximal isotropic subspace of W). We omit subcripts from the notation of (1), (2) when the source is already established.

The main result of [12] is

Theorem 1. *Let V be a $2N$-dimensional symplectic space and let W be an n-dimensional space with symmetric bilinear form B. Write h_W for the maximal dimension of an isotropic subspace of W.*

(1) If $(Sp(V), O(W,B))$ is a reductive dual pair in $Sp(V \otimes W)$ in the symplectic stable range, then for a system of mutually disjoint injections $\eta^V_{W,B}$ as in (1), the restriction of $\omega[V \otimes W]$ to $Sp(V) \times O(W,B)$ decomposes as

$$\bigoplus_{k=0}^{h_W} \bigoplus_{\rho \in \widehat{O(W[-k], B[-k])}} \eta^V(\rho) \otimes Ind^{P_k^B}(\rho \otimes \epsilon(det)) \tag{3}$$

where $Ind^{P_k^B}$ denotes parabolic induction from the maximal parabolic P_k^B in $O(W,B)$ whose Levi factor is $O(W[-k], B[-k]) \times GL_k(\mathbb{F}_q)$, which we consider $\rho \otimes \epsilon(det)$ a representation of by considering $\epsilon(det)$ as a representation of $GL_k(\mathbb{F}_q)$.

(2) If $(Sp(V), O(W,B))$ is a reductive dual pair in $Sp(V \otimes W)$ in the orthogonal stable range, then for a system of mutually disjoint injections $\zeta_V^{W,B}$ as in (2), the restriction of $\omega[V \otimes W]$ to $Sp(V) \times O(W,B)$ decomposes as

$$\bigoplus_{k=0}^{h_W} \bigoplus_{\rho \in \widehat{Sp(V[-k])}} Ind^{P_k^V}(\rho \otimes \epsilon(det)) \otimes \zeta^{W,B}(\rho) \tag{4}$$

where $Ind^{P_k^V}$ denotes parabolic induction from the maximal parabolic P_k^V in $Sp(V)$ with Levi factor $Sp(V[-k]) \times GL_k(\mathbb{F}_q)$, which we consider $\rho \otimes \epsilon(det)$ a representation of by considering $\epsilon(det)$ as a representation of $GL_k(\mathbb{F}_q)$.

Now to state the main result of this paper, we must briefly recall Lusztig's classification of irreducible representations of finite groups of Lie type G. Most generally, we consider the data of

- The conjugacy class of a semisimple element s in the dual group G^D.
- A unipotent representation u of the dual $(Z_{G^D}(s))^D$ of the centralizer of s in G^D.

We associate to the data $[(s),u]$ a representation of G we denote by $\rho_{(s),u}$ of dimension equal to the dimension of u multiplied by the prime to q part of the quotient order $|G^D/Z_{G^D}(s)|$. Intuitively, $\rho_{(s),u}$ can be thought of as a "faked parabolic induction" of u, with the realization that over a finite field, there are many cases of s (specifically, when it only can live in a torus with a non-split factor SO_2^+) where there is no actual maximal parabolic with Levi factor $Z_{G^D}(s)$. However, $\rho_{(s),u}$ will sit in the the induction from the torus character corresponding to s to G.

In all cases of G, for every irreducible representation $\rho \in \widehat{G}$, there exists a unique choice of data $[(s),u]$ as described above, such that $\rho \subseteq \rho_{(s),u}$. On the one hand, if G has a connected center, for example, in the case of the odd special orthogonal groups $G = SO_{2m+1}(\mathbb{F}_q)$, the representations $\rho_{(s),u}$ are irreducible, and therefore they precisely describe irreducible representations of G. In this case, we call the data $[(s),u]$ the *Lusztig classification data* corresponding to an irreducible representation $\rho = \rho_{(s),u}$.

On the other hand, if G has a disconnected center, $\rho_{(s),u}$ may split further. For example, in the case of symplectic groups $G = Sp_{2N}(\mathbb{F}_q)$ which have center $\mathbb{Z}/2$, a representation $\rho_{(s),u}$ splits if and only if s has -1 eigenvalues. If s has -1 eigenvalues, then $\rho_{(s),u}$ splits into two irreducible non-isomorphic pieces

$$\rho_{(s),u} = \rho_{(s),u,+1} \oplus \rho_{(s),u,-1},$$

both of dimension equal to exactly half of the dimension of $\rho_{(s),u}$. In this case, we call the data $[(s),u,\pm 1]$ the *Lusztig classification data* corresponding to an irreducible representation $\rho = \rho_{(s),u,\pm 1}$ and call the sign ± 1 its *central sign.* If s has no -1 eigenvalues, then as before, we call $[(s),u]$ the *Lusztig classification data* corresponding to the irreducible representation $\rho = \rho_{(s),u}$. A similar but slightly more complicated effect occurs for even orthogonal groups $G = O^{\pm}_{2m}(\mathbb{F}_q)$, with $\rho_{(s),u}$ consisting of one, two, or four distinct irreducible summands, depending on the eigenvalues of s. This effect can also be interpreted according to certain "sign data", and we use similar terminology, calling the collection of data of $(s), u$ and this extra sign data (when needed) the Lusztig classification data associated to an irreducible representation. We discuss this in Section 2.

Using this description, we describe injections

$$\phi^V_{W,B} : \widehat{O(W,B)} \hookrightarrow \widehat{Sp(V)}$$

$$\psi^{W,B}_V : \widehat{Sp(V)} \hookrightarrow \widehat{O(W,B)}$$

roughly defined by altering the semisimple part of the input representation's Lusztig classification data by adding -1 eigenvalues if W is odd dimensional and adding 1 eigenvalues if W is even dimensional, and by changing the unipotent part by adding a coordinate to the symbol of the factor of u corresponding to the altered eigenvalues to achieve the needed new rank and defect (the choice of how to add eigenvalues and where to, if there is one, is determined by the central action of the input irreducible representation). This construction is described in more detail in Section 3 below.

Theorem 2. *Assume the notation of Theorem 1.*

(1) *Suppose we have a reductive dual pair $(Sp(V), O(W,B))$ in $Sp(V \otimes W)$ in the symplectic stable range. Then*
$$\eta^V_{W,B} = \phi^V_{W,B}.$$
(2) *Suppose we have a reductive dual pair $(Sp(V), O(W,B))$ in $Sp(V \otimes W)$ in the orthogonal stable range.Then*
$$\zeta^{W,B}_V = \psi^{W,B}_V.$$

The main tool used to prove Theorems 2 is actually dimension. We state here a key result, which, combined with some combinatorics, will prove that the dimensions of the $Sp(V)$-representations $\eta^V_{W,B}(\rho)$ and $\phi^V_{W,B}(\rho)$ (resp. the $O(W,B)$-representations $\zeta^{W,B}_V(\rho)$ and $\psi^{W,B}_V(\rho)$) always match for $\rho \in \widehat{O(W,B)}$ (resp. $\rho \in \widehat{Sp(V)}$) for $N >> n$ (resp. $n >> N$). We can derive then that they must always match, since for a fixed ρ, the dimensions of $\eta^V_{W,B}(\rho)$ and $\phi^V_{W,B}(\rho)$ both form polynomials of q^N (resp. the dimensions of $\zeta^{W,B}_V(\rho)$ and $\psi^{W,B}_V(\rho)$ both form polynomials of q^n). Since the semisimple part and the sign data of $\eta^V_{W,B}(\rho)$ or $\zeta^{W,B}_V(\rho)$ are already determined by considering the restriction of the oscillator representation to the general linear group, this suffices to prove the representations themselves match.

We define, for ρ an irreducible representation of $Sp_{2N}(\mathbb{F}_q)$, its *N-rank* to be

$$rk_N(\rho) = \lceil \frac{deg_q(dim(\rho))}{N} \rceil.$$

Similarly, for ρ an irreducible representation of $O(W, B)$ with $dim(W) = n$, define its *n-rank* to be

$$rk_n(\rho) = \lceil \frac{deg_q(dim(\rho))}{n} \rceil.$$

Proposition 3. *Assume the notation of Theorem 1.*

(1) *Consider $N >> n$. Then the disjoint union of the images of the eta correspondences*

$$\eta^V_{W,B} : \widehat{O(W, B)} \hookrightarrow \widehat{Sp(V)}$$

for the symplectic space V of dimension $2N$ and the two choices of orthogonal space (W, B) of dimension n is precisely the set of irreducible representations of $Sp(V)$ of N-rank n.

(2) *Consider $n >> N$. Then the disjoint union of the images of the zeta correspondences*

$$\zeta^V_{W,B} : \widehat{O(W, B)} \hookrightarrow \widehat{Sp(V)}$$

for the symplectic space V of dimension $2N$ and the two choices of orthogonal space (W, B) of dimension n is precisely the set of irreducible representations of $Sp(V)$ of N-rank n.

The present paper is organized as follows: In Section 2, we describe Lusztig's classification of irreducible representations. In Section 3, we describe our proposed constructions of the eta and zeta correspondences in more detail. In Section 4, we prove combinatorial identities proving our claimed constructions can be plugged into the decompositions in Theorem 1 and add up to the correct dimension. In Section 5, we use an inductive argument to prove Proposition 3 and conclude Theorem 2. In Section 6, we write down the zeta correspondence in the example where $dim(V) = 2$.

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2. Lusztig's Classification of Irreducible Representations

The purpose of this section is to give more details about Lusztig's classification of irreducible representations in the case of the symplectic and orthogonal groups.

As we described briefly in the introduction, for a finite group of Lie type G, its irreducible representations are classified by certain data we refer to as *G-Lusztig classification data*, consisting of

- (the "semisimple part"): a conjugacy class (s) of a semisimple element s of the dual group G^D
- (the "unipotent part"): a unipotent representation u of the dual of s's centralizer $Z_{G^D}(s)$.
- (possible "sign data"): sign choices describing which piece we take of the representation obtained from (s) and u, if it splits.

More specifically, for every choice of semisimple and unipotent parts (s) and u, there is an associated G-representation we denote by $\rho_{(s),u}$. Its dimension is

$$dim(\rho_{(s),u}) = \frac{|G|_{q'}}{|Z_s(G^D)|_{q'}} dim(u), \tag{5}$$

where $|?|_{q'}$ denotes the prime to q part of the group order (recalling that the a group and its dual have the same order). From the point of view of dimension, the representation $\rho_{(s),u}$ can be thought of as a "faked parabolic induction" of the unipotent representation u where the centralizer of s plays the role of the Levi factor (this does not literally make sense, since there may not be such a parabolic subgroup of G over a finite field). Every irreducible representation is a summand of $\rho_{(s),u}$ for some choice of (s) and u. On the one hand, if G has a connected center e.g., in the case of odd special orthogonal groups, these $\rho_{(s),u}$ are all irreducible and therefore, the G-Lusztig classification data consists only of a semisimple and unipotent part. On the other hand, if $Z(G)$ is not connected e.g., for symplectic groups, there are choices of (s) and u for which $\rho_{(s),u}$ splits further. In these cases, we also need to specify the action of $Z(G)$ on an irreducible representation, to describe which piece of $\rho_{(s),u}$ we are referring to. This is the role of the central sign data in the Lusztig classification data, and denoting it by α, we write

$$\rho_{(s),u,\alpha} \tag{6}$$

for the corresponding irreducible G-representation. Similarly, we write

$$[(s), u, \alpha]$$

for the G-Lusztig classification data corresponding to (6). In the case of even orthogonal groups, α actually consists of two signs rather than a single central sign corresponding to the $\mathbb{Z}/2$ center, which we discuss in Subsection 2.3 below. The purpose of this section is to give more detail about each part of the Lusztig classification data in each case of G we consider in this paper.

In Subsection 2.1, we discuss the maximal tori in symplectic and orthogonal groups and the form of the semisimple elements, up to conjugation. In Subsection 2.2, we discuss the irreducible representations of $O_{2m+1}(\mathbb{F}_q)$. In Subsection 2.3, we discuss the irreducible representations of $O^{\pm}_{2m}(\mathbb{F}_q)$. In Subsection 2.4, we discuss the irreducible representations of $Sp_{2N}(\mathbb{F}_q)$.

2.1. **Tori and the case of rank 1.** In this subsection, let us first take G to be a finite group of Lie type of rank r of the form $Sp_{2r}(\mathbb{F}_q)$, $SO_{2r+1}(\mathbb{F}_q)$, or $SO^{\pm}_{2r}(\mathbb{F}_q)$. The maximal tori in G are all conjugate to a product of $SO^{\pm}_2(\mathbb{F}_{q^n})$ factors

$$SO^{\pm}_2(\mathbb{F}_{q^{n_1}}) \times \cdots \times SO^{\pm}_2(\mathbb{F}_{q^{n_k}}) \tag{7}$$

of maximal rank, so that

$$r = n_1 + \cdots + n_k,$$

and where, in the case of G an even special orthogonal group $SO^{\pm}_{2r}(\mathbb{F}_q)$, the sign in the superscript is equal to the product of the signs appearing in (7). Recall that

$$SO^{+}_2(\mathbb{F}_q) = \{\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{F}_q^{\times}\} \cong \mu_{q-1} \tag{8}$$

$$SO^{-}_2(\mathbb{F}_q) = \{\begin{pmatrix} y & z \\ \varepsilon z & y \end{pmatrix} \mid y, z \in \mathbb{F}_q \; y^2 - \alpha z^2 = 1\} \cong \mu_{q+1}, \tag{9}$$

where in (9), $\varepsilon \in \mathbb{F}_q$ is an element which is not a square, and the isomorphism follows by considering $\mathbb{F}_{q^2} = \mathbb{F}_q[\varepsilon]$, whose norm 1 elements are isomorphic to μ_{q+1}. (In other words, the conjugacy class of a torus element can be identified by its eigenvalues, which lie if μ_{q-1} as $\mathbb{F}_q^{\times}$, or μ_{q+1} as the norm 1 elements of $\mathbb{F}_{q^2}^{\times} = (\mathbb{F}_q[\varepsilon])^{\times}$.)

We note that in the case of G an odd special orthogonal group $SO_{2r+1}(\mathbb{F}_q)$, to embed a torus of the form (7) into G, we need to insert a "forced" diagonal entry 1. The placement of this entry is according to whether the product (7) is a subgroup of $SO^{+}_{2r}(\mathbb{F}_q)$ or $SO^{-}_{2r}(\mathbb{F}_q)$. For $2r$ by $2r$ matrices which can be considered in groups (7) for more than one choice of signs (meaning some factors are equal to the identity matrix I or $-I$), then we must consider whether these two choices of where to insert the final "forced" diagonal entry 1 give different conjugacy classes, or not. They give different conjugacy classes if and only if the $2r$ by $2r$ element of (7) has any -1 eigenvalues. (If there are -1 eigenvalues, the resulting two choices of elements have different centralizers and cannot be conjugate, while if there are no -1 eigenvalues, then the two choices are not distinguishable.)

A semisimple element of G is then an element of a subgroup conjugate to (7), and we may therefore consider it as a product of blocks

$$s \sim A_1 \oplus \cdots \oplus A_t,$$

for $A_i \in SO_2^{\pm}(\mathbb{F}_{q^{n_i}})$ (considered as $2n_i$ by $2n_i$ square matrices). Let us always minimize the field extensions $\mathbb{F}_{q^{n_i}}$ needed to contain the eigenvalues of s e.g., we consider the identity matrix as an element of $(SO_2^{\pm}(\mathbb{F}_q))^r$, rather than an element of $SO_2^{\pm}(\mathbb{F}_{q^r})$. Then the set of eigenvalues of each A_i consists of distinct powers $\lambda_i, \dots, \lambda_i^{n_i}$ and their inverses $\lambda_i^{-1}, \dots, \lambda_i^{-n_i}$, for some $\lambda_i \in \mathbb{F}_q^{n_i}$. In general, the conjugacy class of s is classified by the orbit of the multiset of its eigenvalues consisting of these elements $\lambda_i, \dots, \lambda_i^{n_i}$ for each i, under the action of the Weyl group of G.

Definition 4. *In a symplectic group or an even (special) orthogonal group, we say a semisimple element s is in a* generic conjugacy class *if none of its eigenvalues are ± 1. Otherwise, say s's conjugacy class is* singular of type (p, ℓ) *where 1 is an eigenvalue of multiplicity $2p$ and -1 is an eigenvalue of multiplicity 2ℓ in s.*

Similarly, we say a semisimple element s of an odd special orthogonal group is a generic conjugacy class *if none of its eigenvalues are -1 and exactly one of its eigenvalues is 1. Otherwise, say s's conjugacy class is* singular of type (p, ℓ) *where 1 is an eigenvalue of multiplicity $2p+1$ and -1 is an eigenvalue of multiplicity 2ℓ in s.*

Let us now discuss an example of rank 1. Considering $SO_2^+(\mathbb{F}_q) = \mu_{q-1}$ (resp. $SO_2^-(\mathbb{F}_q) = \mu_{q+1}$) as a torus of $Sp_2(\mathbb{F}_q) = SL_2(\mathbb{F}_q)$ or $SO_3(\mathbb{F}_q)$, for $a \in \mu_{q-1} \smallsetminus \{\pm 1\}$ (resp. $\mu_{q+1} \smallsetminus \{\pm 1\}$), the semisimple element associated to a is conjugate to the semisimple element associated to a^{-1} by the action of the Weyl group. For example, in $SO_3(\mathbb{F}_q)$, an torus element, say $x \in \mu_{q-1} \cong SO_2^+(\mathbb{F}_q)$ is represented by the matrix

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and we have

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Similarly for elements of $\mu_{q+1} \cong SO_2^-(\mathbb{F}_q)$, and in $SL_2(\mathbb{F}_q)$). Therefore, in conjugacy classes of generic semisimple elements in symplectic or special orthogonal groups, its conjugacy class only depends on the sets of eigenvalues $\{x, x^{-1}\}$ in each $SO_2^{\pm}(\mathbb{F}_q)$ block.

We consider the cases when s has ± 1 eigenvalues separately, since they behave differently when taking the centralizer. To give an example of this, again consider $SO_3(\mathbb{F}_q)$, written as the special orthogonal group on $\mathbb{F}_q^3$ with respect to a form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

In this case, s can be singular of type $(1,0)$ or $(0,1)$. The only singular conjugacy class of type $(1,0)$ is the conjugacy class of the identity matrix. However, in $SO_3(\mathbb{F}_q)$, there are two singular conjugacy classes of type $(0,1)$: the conjugacy classes of

$$\sigma_1^+ := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1^- := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which have centralizers $SO_2^+(\mathbb{F}_q)$, $SO_2^-(\mathbb{F}_q)$ in $SO_3(\mathbb{F}_q)$, respectively (since here the centralizer is the special orthogonal group on the two coordinates corresponding to the two -1 entries in $\sigma_1^{\pm}$). Note that in $SL_2(\mathbb{F}_q)$, this effect does not occur, and conjugacy classes of semisimple elements are in fact in all cases determined.

Definition 5. *More generally, for any symmetric bilinear form B on $\mathbb{F}_q^3$, there is a choice of a pair of basis coordinates such that restricting to them gives a split (resp. non-split) symmetric bilinear form on $\mathbb{F}_q^2$. Placing -1's on the diagonal in entries corresponding to these two coordinates gives σ_1^+ (resp. σ_1^-) in general. This ensures that their centralizers are*

$$Z_{SO(\mathbb{F}_q^3,B)}(\sigma_1^{\pm}) = SO_2^{\pm}(\mathbb{F}_q).$$

2.2. **The representation theory of the odd orthogonal groups.** In this subsection, we describe the irreducible representations of the odd orthogonal groups $O_{2m+1}(\mathbb{F}_q)$. First, we can split the center off

$$O_{2m+1}(\mathbb{F}_q) = \mathbb{Z}/2 \times SO_{2m+1}(\mathbb{F}_q),$$

and therefore each irreducible representation can be considered as the tensor product of a sign with its irreducible restriction to the special orthogonal group $SO_{2m+1}(\mathbb{F}_q)$. Now since $SO_{2m+1}(\mathbb{F}_q)$ has no center, the irreducible representations are precisely the representations $\rho_{(s),u}$, corresponding to choices of $SO_{2m+1}(\mathbb{F}_q)$-Lusztig classification data consisting of a conjugacy class (s) of a semisimple element

$$s \in Sp_{2m}(\mathbb{F}_q) = (SO_{2m+1}(\mathbb{F}_q))^D,$$

(recall that the symplectic groups and odd special orthogonal groups are dual) and u an irreducible unipotent representation of the dual of the centralizer of s in $Sp_{2m}(\mathbb{F}_q)$.

To consider (s), we apply the discussion of the previous subsection (putting $r = m$). We may consider s to be an element of the form

$$s = A_1 \oplus A_2 \oplus \cdots \oplus A_k \tag{10}$$

in a torus of the form (7), where each A_i is a $2n_i$ by $2n_i$ matrix in one of the special orthogonal groups $SO_2^{\pm}(\mathbb{F}_{q^{n_i}})$ with eigenvalues

$$\{\lambda_i^j \mid j \in \{-1, -2, \dots, -n_i\} \cup \{1, 2, \dots n_i\}\} \tag{11}$$

for $\lambda_i \in \mathbb{F}_{q^{n_i}}$ (and not contained in a smaller field extension of $\mathbb{F}_q$).

Fix such an (s), and write the semisimple element as in (10). We next need to describe its centralizer in $Sp_{2m}(\mathbb{F}_q)$. Consider the data (11) at each $i = 1, \dots, k$, which may repeat. We need to count the multiplicities of each distinct choice. Suppose there are r different choices of (11) in s corresponding to blocks A which can only be considered in the split groups SO_2^+ (so that the eigenvalues in (11) for these blocks do not contain ± 1). Say they appear with multiplicities $j_1, \dots, j_r$ each, and relabel their corresponding field extension powers as $n'_1, \dots, n'_r$ so that each of these choices of (11) appearing with multiplicity j_i (for $i = 1, \dots, r$) describe the eigenvalues of a block in

$$SO_2^+(\mathbb{F}_{q^{n'_i}}). \tag{12}$$

Similarly, suppose there are t distinct choices of eigenvalue sets (11) appearing in s which are only in not split group SO_2^-, say they appear with multiplicities $k_1, \dots, k_t$, and relabel their corresponding powers of the needed field extensions as $n''_1, \dots, n''_t$, so that they correspond to blocks in

$$SO_2^-(\mathbb{F}_{q^{n''_i}}), \tag{13}$$

for $i = 1, \dots, t$. (The sign of the SO_2 in (12), (13) determines whether the centralizer is U^- or $U^+ = GL$.)

The remaining blocks A have eigenvalues either 1 or -1. Say that there are ℓ blocks with eigenvalue -1 (so -1 is an eigenvalue of total multipliciy 2ℓ) and p blocks with eigenvalue 1 (so 1 is an eigenvalue of total multipliciy $2p$), and therefore

$$m = \sum_{i=1}^{r} j_i \cdot n'_i + \sum_{i=1}^{t} k_i \cdot n''_i + \ell + p.$$

Then the centralizer of s in $Sp_{2m}(\mathbb{F}_q)$ is

$$\prod_{i=1}^{r} U^+_{j_i}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U^-_{k_i}(\mathbb{F}_{q^{n''_i}}) \times Sp_{2\ell}(\mathbb{F}_q) \times Sp_{2p}(\mathbb{F}_q) \tag{14}$$

(We use the notation that $U^+_j(\mathbb{F}_q) = GL_j(\mathbb{F}_q)$.)

Choose a conjugacy class of a semisimple element s in $Sp_{2m}(\mathbb{F}_q)$ with centralizer (14). The remaining data in the Lusztig classification data is a unipotent representation u of the dual of (14), which is

$$\prod_{i=1}^{r} U^+_{j_i}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U^-_{k_i}(\mathbb{F}_{q^{n''_i}}) \times SO_{2\ell+1}(\mathbb{F}_q) \times SO_{2p+1}(\mathbb{F}_q).$$

This consists of a tensor product of unipotent representations of each factor of (14)

$$\bigotimes_{i=1}^{r} u_{U^+_{j_i}} \otimes \bigotimes_{i=1}^{t} u_{U^-_{k_i}} \otimes u^{-1}_{SO_{2\ell+1}} \otimes u^{+1}_{SO_{2p+1}}, \tag{15}$$

where as the notation suggests, $u_{U^+_{j_i}}$ and $u_{U^-_{k_i}}$ are unipotent representations of the A-type factors $U^+_{j_i}(\mathbb{F}_{q^{n'_i}})$ and $U^-_{k_i}(\mathbb{F}_{q^{n''_i}})$ respectively, and $u^{-1}_{SO_{2\ell+1}}$ and $u^{+1}_{SO_{2p+1}}$ are unipotent representations of $SO_{2\ell+1}(\mathbb{F}_q)$ and $Sp_{2p+1}(\mathbb{F}_q)$ (the superscript with the sign for the C-type factors indicates the sign of the eigenvalue ± 1 of the blocks in s which the factor corresponds to).

The irreducible $SO_{2m+1}(\mathbb{F}_q)$ representation $\rho_{(s),u}$ corresponding to (s) and a choice of unipotent representation u as in (15) has dimension equal to the dimension of u multiplied by the prime to q part of the quotient of orders

(16)
$$\frac{|SO_{2m+1}(\mathbb{F}_q)|_{q'}}{|\prod_{i=1}^{r} U^+_{j_i}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U^-_{k_i}(\mathbb{F}_{q^{n''_i}}) \times SO_{2\ell+1}(\mathbb{F}_q) \times SO_{2p+1}(\mathbb{F}_q)|_{q'}}.$$

The unipotent representations can be further described using the theory of *symbols*. We do not discuss the case of the unipotent representations of a finite group of lie type A or ${}^2A$ here. For now, we consider the symbols of type C (which recall are the same as the symbols of type B, since the theory only depends on the Weyl group) of rank r.

Definition 6. Symbols of rank r of type C or B *consist of equivalence classes of two rows of strictly increasing sequences*

$$\begin{pmatrix} \lambda_1 < \lambda_2 < \cdots < \lambda_a \\ \mu_1 < \mu_2 < \cdots < \mu_b \end{pmatrix} \tag{17}$$

under switching rows, for $\lambda_i, \mu_i \in \mathbb{N}_0$ *non-negative integers such that* $(\lambda_1, \mu_1) \neq (0,0)$ *and*

$$\sum_{i=1}^{a} \lambda_i + \sum_{i=1}^{b} \mu_i = r + \frac{(a+b-1)^2}{4},$$

with odd defect, *meaning that* $a-b$ *is odd.*

The dimension of the unipotent representation of $SO_{2\ell+1}(\mathbb{F}_q)$ corresponding to the symbol (17) is the factor

$$\frac{\prod_{1\leq i<j\leq a}(q^{\lambda_j}-q^{\lambda_i}) \cdot \prod_{1\leq i<j\leq b}(q^{\mu_j}-q^{\mu_i}) \cdot \prod_{\substack{1\leq i\leq a \\ 1\leq j\leq b}}(q^{\lambda_i}+q^{\mu_j})}{\prod_{i=1}^{a}\prod_{j=1}^{\lambda_i}(q^{2j}-1) \cdot \prod_{i=1}^{b}\prod_{j=1}^{\mu_i}(q^{2j}-1) \cdot q^{c[a,b]}} \tag{18}$$

multiplied by

$$\frac{|SO_{2\ell+1}(\mathbb{F}_q)|_{q'}}{2^{(a+b-1)/2}},$$

where, for the final power of q in the denominator of (18), we write

$$c[a,b] = \sum_{i=1}^{\lfloor (a+b)/2 \rfloor} \binom{a+b-2i}{2}$$

(We separate the dimension formula into these two factors, since the term (18) will also occur in the symbols of type D.)

2.3. **The representation theory of the even orthogonal groups.** Now consider the special orthogonal group on a $2m$-dimensional $\mathbb{F}_q$-vector space W with respect to a symmetric bilinear form B. Write

$$O(W,B) = O^{\alpha}_{2m}(\mathbb{F}_q),$$

with sign $\alpha = +$ if B is totally split (i.e. there is an m-dimensional isotropic subspace of W), and with sign $\alpha = -$ otherwise. We call this sign α the *total sign* of the symmetric bilinear form B. The even groups of D- and ${}^2D$-type are self-dual. Though we need to be more careful about the role of the center, we still compare with the special orthogonal case (both for easier comparison with the odd case and because the semisimple and unipotent parts of Lusztig classification data are easier to discuss there).

To classify an irreducible representation $\rho \in \widehat{O^{\alpha}_{2m}(\mathbb{F}_q)}$, we look at the of the pair of semisimple and unipotent parts (s) and u of $SO^{\alpha}_{2m}(\mathbb{F}_q)$-Lusztig data such that ρ is a summand of the corresponding $SO^{\alpha}_{2m}(\mathbb{F}_q)$-representation's induction to $O^{\alpha}_{2m}(\mathbb{F}_q)$

$$\rho \subseteq Ind^{SO^{\alpha}_{2m}(\mathbb{F}_q)}_{O^{\alpha}_{2m}(\mathbb{F}_q)}(\rho_{(s),u}). \tag{19}$$

The induction on the right hand side of (19) decomposes into 1, 2, or 4 irreducible representations precisely according to the central action of $SO^{\alpha}_{2m}(\mathbb{F}_q)$ and $O^{\alpha}_{2m}(\mathbb{F}_q)/SO^{\alpha}_{2m} = \mathbb{Z}/2$. Therefore the $O^{\alpha}_{2m}(\mathbb{F}_q)$-Lusztig classification data can be thought of as to consist of semisimple part the conjugacy class of s *as an element of* $O^{\alpha}_{2m}(\mathbb{F}_q)$, unipotent part u, and this central sign data. (Note that the unipotent representations of a group are the same after removing the center, and for simplicity to compare with the odd case, we may consider unipotent parts of Lusztig classification data as irreducible unipotent representations of $SO^{\alpha}_{2m}(\mathbb{F}_q)$.)

Therefore, we begin by describing the special orthogonal group's Lusztig classification data. First, as discussed in Subsection 2.1, the maximal tori in $SO(W,B) = SO^{\alpha}_{2m}(\mathbb{F}_q)$ are, again, all isomorphic to (7) such that the product of the signs appearing in (7) is equal to the total sign α of B. The semisimple elements in $O^{\alpha}_{2m}(\mathbb{F}_q)$ are again then all conjugate to elements of the form

$$s = A_1 \oplus \cdots \oplus A_k$$

where each A_i is a $2n_i \times 2n_i$ matrix determined by its eigenvalues (11) (and -1 to the power of the number of $A_i \in SO_2^-(\mathbb{F}_q)$ is equal to the total sign of B).

To determine the centralizer $Z_{SO^{\pm}_{2m}(\mathbb{F}_q)}(s)$, suppose that there are $j_1, \dots, j_r$ of the A blocks with distinct sets of eigenvalues (11) with corresponding powers of q equal to $n'_1, \dots, n'_r$ corresponding to factors of $SO_2^+(\mathbb{F}_{q^{n'_i}})$, and there are $k_1, \dots, k_t$ blocks with distinct sets of eigenvalues (11) with corresponding powers of q equal to $n''_1, \dots, n''_t$ corresponding to factors of $SO_2^-(\mathbb{F}_{q^{n''_i}})$. The remaininng blocks A have eigenvalues either 1 or -1. Again, say that there are ℓ blocks with eigenvalue -1 and p blocks with eigenvalue 1. (Note that the final multiplicities of the centralizers in the even special orthogonal groups are twice the corresponding multiplicities of the centralizers in the odd special orthogonal groups, since the Weyl group is half the size.) For such a semisimple element s in $SO^{\pm}_{2m}(\mathbb{F}_q)$, its centralizer is then

$$\prod_{i=1}^{r} U^+_{j_i}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U^-_{k_i}(\mathbb{F}_{q^{n''_i}}) \times SO^{\pm}_{2\ell}(\mathbb{F}_q) \times SO^{\pm}_{2p}(\mathbb{F}_q) \tag{20}$$

(where, again, the total product of signs appearing in (20) is the total sign of B).

Fix such a semisimple part (s) whose centralizer in $SO^{\alpha}_{2m}(\mathbb{F}_q)$ is (20). In this case, all the factors of (20) are self-dual, so the unipotent part of the $SO^{\alpha}_{2m}(\mathbb{F}_q)$-Lusztig classification data consists of a unipotent representation u of (20), which can be written as a tensor product

$$\bigotimes_{i=1}^{r} u_{U^+_{j_i}} \otimes \bigotimes_{i=1}^{t} u_{U^-_{k_i}} \otimes u^{-1}_{SO^{\pm}_{2\ell}} \otimes u^{+1}_{SO^{\pm}_{2p}}, \tag{21}$$

with the same notational convention as in the case of the odd special orthogonal groups.

Again, the $SO^{\pm}_{2m}(\mathbb{F}_q)$-representation $\rho_{(s),u}$ corresponding to (s) and a choice of unipotent representation u of (20) has dimension equal to the dimension of u multiplied by the prime to q part of the quotient of orders $|SO^{\alpha}_{2m}(\mathbb{F}_q)|_{q'}/|Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(s)|_{q'}$ which is

$$|SO^{\alpha}_{2m}(\mathbb{F}_q)|_{q'}/|Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(s)|_{q'} = \frac{|SO^{\pm}_{2m}(\mathbb{F}_q)|_{q'}}{|\prod_{i=1}^{r} U^+_{j_i}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U^-_{k_i}(\mathbb{F}_{q^{n''_i}}) \times SO^{\pm}_{2\ell}(\mathbb{F}_q) \times SO^{\pm}_{2p}(\mathbb{F}_q)|_{q'}}. \tag{22}$$

The unipotent representations $u^{-1}_{SO^{\pm}_{2\ell}}$, $u^{+1}_{SO^{\pm}_{2\ell}}$ correspond to the symbols of type D or ${}^2D$ of ranks ℓ and p.

Definition 7. *A symbol of type D and rank r (e.g. corresponding to an irreducible unipotent representation of $SO^{+}_{2r}(\mathbb{F}_q)$) is two rows of strictly increasing sequences (17) (where again, switching rows is counted as giving the same symbol), with λ_i, $\mu_i \in \mathbb{N}_0$ non-negative integers, such that*

$$\sum_{i=1}^{a} \lambda_i + \sum_{i=1}^{b} \mu_i = r + \frac{(a+b)(a+b-2)}{4} \tag{23}$$

and of defect divisible by 4*, i.e. $a-b$ is* 0 *mod* 4*. Similarly, a symbol of type ${}^2D$ and rank r (e.g. corresponding to an irreducible unipotent representation of $SO^{-}_{2r}(\mathbb{F}_q)$) is two rows of strictly increasing sequences (17), with λ_i, $\mu_i \in \mathbb{N}_0$ non-negative integers, such that (23) holds, and it has* defect equal to 2 mod 4*, i.e. $a-b$ is* 2 *mod* 4.

The dimension of the unipotent represent associated to a symbol $\binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ of type D of ${}^2D$ and rank p is again the factor (18), multiplied by

$$\frac{|SO^{\pm}_{2p}(\mathbb{F}_q)|_{q'}}{2^{(a+b-2)/2}} \tag{24}$$

(in the case of $SO^{+}_{2p}(\mathbb{F}_q)$, if the two rows of the symbol are exactly the same, it is called *degenerate*, and splits into two additional equi-dimensional non-isomorphic halves).

This describes the semisimple and unipotent parts of $SO^{\alpha}_{2m}(\mathbb{F}_q)$-Lusztig classification data. As we previously discussed, the induction of the $SO^{\alpha}_{2m}(\mathbb{F}_q)$ representation $\rho_{(s),u}$ corresponding to $(s) \in SO^{\alpha}_{2m}(\mathbb{F}_q)$ and u to $O^{\alpha}_{2m}(\mathbb{F}_q)$ is the $O^{\alpha}_{2m}(\mathbb{F}_q)$-representation (also, by weakness of notation, denoted by $\rho_{(s),u}$) corresponding to the conjugacy class of s as an element of $O^{\alpha}_{2m}(\mathbb{F}_q)$ and u. Therefore, it remains to discuss the "central" sign data (note that here, the word central may be slightly decieving since we potentially have two degrees of freedom: describing the action of the center $\mathbb{Z}/2 = Z(O^{\alpha}_{2m}(\mathbb{F}_q)) = Z(SO^{\alpha}_{2m}(\mathbb{F}_q))$ and another sign describing the action of the determinant $\mathbb{Z}/2 = O^{\alpha}_{2m}(\mathbb{F}_q)/SO^{\alpha}_{2m}(\mathbb{F}_q))$.

In practice, it turns out that $\rho_{(s),u}$ splits in half once for if s has 1 eigenvalues, and once for if s has -1 eigenvalues (menaing that if s has both 1 and -1 eigenvalues, there are a total of four irreducible pieces). To describe an $O^{\alpha}_{2m}(\mathbb{F}_q)$-irreducible representation, the sign data consists of an (independently chosen) sign ± 1-action if s has 1 eigenvalues and a sign ± 1 if s has -1 eigenvalues. We write the corresponding piece of $\rho_{(s),u}$ as

$$\rho_{(s),u,(\pm 1,\pm 1)},$$

listing the sign from 1 eigenvalues first, the -1 eigenvalues second, and removing either when s has no such eigenvalues.

2.4. **The representation theory of the symplectic group.** Finally, in this subsection, we consider the irreducible representations of $Sp_{2N}(\mathbb{F}_q)$. Its dual is $SO_{2N+1}(\mathbb{F}_q)$. Without loss of generality, consider $SO_{2N+1}(\mathbb{F}_q)$ as the special orthogonal group associated to $\mathbb{F}_q^{2N+1}$ with symmetric bilinear form defined by the $2N+1$ by $2N+1$ matrix described by

$$\bigoplus_N \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus (\alpha) \tag{25}$$

considering $\alpha \in \mathbb{F}_q^\times \smallsetminus (\mathbb{F}_q^\times)^2$ as a 1 by 1 matrix. Again, consider a conjugacy class of a semisimple element (s) of $SO_{2N+1}(\mathbb{F}_q)$. Again, as discussed in

$$A_1 \oplus \cdots \oplus A_k \tag{26}$$

where each A_i is a 2 by 2 matrix in $SO_2^{\pm}(\mathbb{F}_{q^{n_i}})$, and 1 is considered as a 1 by 1 matrix, with an additional 1 diagonal entry (to make $2N+1$ by $2N+1$ matrix) either at the very end (if the product of signs of the $SO_2^{\pm}(\mathbb{F}_{q^{n_i}})$ is $+$), or in the second to last entry (if the product of signs is $-$).

First suppose s has no -1 eigenvalues, in which case there is no ambiguity of where the forced 1 diagonal entry is added to (26). In this case, again, say that there are $j_1, \dots, j_r$ blocks A with eigenvalues (11) with corresponding powers of q relabelled as $n'_1, \dots, n'_r$, and there are $k_1, \dots, k_t$ blocks A with eigenvalues (11) with corresponding powers of q relabelled as $n''_1, \dots, n''_t$. The remaining eigenvalues are all 1. Let us write

$$p = N - (\sum_{i=1}^r j_i + \sum_{i=1}^t k_i).$$

Therefore, the centralizer of s in $SO_{2N+1}(\mathbb{F}_q)$ is isomorphic to

$$\prod_{i=1}^r U_{j_i}^+(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^t U_{k_i}^-(\mathbb{F}_{q^{n''_i}}) \times SO_{2p+1}(\mathbb{F}_q). \tag{27}$$

Now suppose that s does have eigenvalues -1. Without loss of generality, then, they are on the diagonal. There must be even number, say 2ℓ. Restricting the form (25) to the coordinates where s has -1's on the diagonal, gives 2ℓ by 2ℓ matrix defining a symmetric bilinear form on $\mathbb{F}_q^{2\ell}$. Say s is of type $+1$ if this form is completely split (i.e. the maximal dimension of an isotropic subspace is ℓ) and say s is of type -1 otherwise (when the maximal dimension of an isotropic subspace is $\ell-1$). Projecting away from these coordinates reduces to the above case with N replaced by $N-\ell$. Let use the same notation for $j_1,\dots j_r$ and $k_1,\dots,k_t$ for the multiplicities of blocks with eigenvalues in (11) not equal to ± 1 and (such that that

$$N=\sum_{i=1}^{r} n_i'\cdot j_i+\sum_{i=1}^{t} n_i''\cdot k_i+\ell+p).$$

Then the centralizer of a semisimple element s of type ± 1 is

$$\prod_{i=1}^{r} U_{j_i}^{+}(\mathbb{F}_{q^{n_i'}})\times\prod_{i=1}^{t} U_{k_i}^{-}(\mathbb{F}_{q^{n_i''}})\times SO_{2\ell}^{\pm}(\mathbb{F}_q)\times SO_{2p+1}(\mathbb{F}_q). \tag{28}$$

(Let us count (27) as the case of (28) with $\ell=0$.)

Fixing a conjugacy class of a semisimple element s in $SO_{2N+1}(\mathbb{F}_q)$ with centralizer (28), as above, the remaining data in the Jordan decomposition is a unipotent irreducible representation u of the dual to (28), which is

$$\prod_{i=1}^{r} U_{j_i}^{+}(\mathbb{F}_{q^{n_i'}})\times\prod_{i=1}^{t} U_{k_i}^{-}(\mathbb{F}_{q^{n_i''}})\times SO_{2\ell}^{\pm}(\mathbb{F}_q)\times Sp_{2p}(\mathbb{F}_q).$$

Again, u consists of a tensor product

$$\bigotimes_{i=1}^{r} u_{U_{j_i}^{+}}\otimes\bigotimes_{i=1}^{t} u_{U_{k_i}^{-}}\otimes u_{SO_{2\ell}^{\pm}}\otimes u_{Sp_{2p}} \tag{29}$$

where $u_{U_{j_i}^{+}}$, $u_{U_{k_i}^{-}}$, $u_{SO_{2\ell}^{\pm}}$, and $u_{Sp_{2p}}$ are unipotent representations of $U_{j_i}^{+}(\mathbb{F}_{q^{n_i'}})$, $U_{k_i}^{-}(\mathbb{F}_{q^{n_i''}})$, $SO_{2\ell}^{\pm}(\mathbb{F}_q)$, and $Sp_{2p}(\mathbb{F}_q)$.

Since $Sp_{2N}(\mathbb{F}_q)$ has a disconnected center $\mathbb{Z}/2$, some of the representations $\rho_{(s),u}$ specified by a conjugacy class of a semisimple element (s) and a unipotent representation has dimension equal to half of the dimension of u multiplied by the prime to q part of the quotient of orders

$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|\prod_{i=1}^r U^+_{j_i}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^t U^-_{k_i}(\mathbb{F}_{q^{n''_i}}) \times SO^\pm_{2\ell}(\mathbb{F}_q) \times Sp_{2p}(\mathbb{F}_q)|_{q'}}. \tag{30}$$

This halving occurs if and only if s has -1 as an eigenvalue, in which case we choose a central sign (± 1) to indicate which half we are want in the Lusztig classification data, according to the action of $Z(Sp_{2N}(\mathbb{F}_q)) = \mathbb{Z}/2$. (Otherwise, the data of (s) and u determines a single irreducible representation of dimension (30).)

The discussion in the previous subsections (Definitions 6 and 7) for symbols applies to the unipotent representation $u_{Sp_{2\ell}}$ (since again, the groups of type C and B have the same symbols), and $u_{SO^+_{2p}}$ and $u_{SO^-_{2p}}$.

Example: The oscillator representations

$$\omega_a = \omega_a^+ \oplus \omega_a^- \quad \omega_b = \omega_b^+ \oplus \omega_b^- \tag{31}$$

of $Sp_{2N}(\mathbb{F}_q)$ can be recovered using this classification by considering the conjugacy classes of elements $s \in SO_{2N+1}(\mathbb{F}_q)$ of type ± 1, which are singular of type $(0, N)$ (i.e. -1 is an eigenvalue of multiplicity $2N$ in s):

Definition 8. *Consider B a general symmetric bilinear form on $\mathbb{F}_q^{2N+1}$. Without loss of generality,*

$$B = \bigoplus_N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus Z. \tag{32}$$

Consider the sum of the final Nth hyperbolic and the (1-dimensional) Z term as a symmetric bilinear form on $\mathbb{F}_q^3$:

$$SO_3(\mathbb{F}_q) = SO(\mathbb{F}_q^3, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus Z). \tag{33}$$

Consider $\sigma_1^\pm$ as an element of (33) (see Definition 5 above). Put

$$\sigma_N^\pm = (-I_{2(N-1)}) \oplus \sigma_1^\pm, \tag{34}$$

considering $-I_{2(N-1)}$ as a $2(N-1)$ by $2(N-1)$ matrix in

$$SO^+_{2(N-1)}(\mathbb{F}_q) = SO(\mathbb{F}_q^{2(N-1)}, \bigoplus_{N-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}),$$

corresponding to the first $N-1$ hyperbolic summands of B in (32).

The centralizer of $\sigma_N^\pm$ in $SO_{2N+1}(\mathbb{F}_q)$ is

$$Z_{\sigma_N^\pm}(SO_{2N+1}(\mathbb{F}_q)) = SO^\pm_{2N}(\mathbb{F}_q), \tag{35}$$

which is self dual. The factor (30) is then

$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO^{\pm}_{SN+1}(\mathbb{F}_q)|_{q'}} = \frac{\prod_{i=1}^{N}(q^{2i}-1)}{(q^N \mp 1)\prod_{i=1}^{N-1}(q^{2i}-1)} = q^N \pm 1 \tag{36}$$

In both cases, take u to be the trivial representation 1.

The representations $\rho_{(\sigma_N^{\pm}),1}$ obtained from the Jordan decomposition of $(\sigma_N^{\pm})$ and $u = 1$ are then of dimension (36). They then decompose into equi-dimensional halves, with

$$\rho_{(\sigma_N^+),1} = \omega_a^+ \oplus \omega_b^+,$$

$$\rho_{(\sigma_N^-),1} = \omega_a^- \oplus \omega_b^-,$$

recovering the pieces of the oscillator representations (31). In our notation, we have

$$\omega_a^{\pm} = \rho_{(\sigma_N^{\pm}),1,\epsilon(a)},$$

where ϵ denotes the quadratic character

$$\epsilon : \mathbb{F}_q^{\times} \to \{\pm 1\}. \tag{37}$$

3. The claimed construction

The purpose of this section is to describe in more detail the claimed construction that we are proposinng gives the eta and zeta correspondences. We define the proposed constructions

$$\phi^V_{W,B} : \widehat{O(W,B)} \hookrightarrow \widehat{Sp(V)}$$

in the symplectic stable range, and

$$\psi^{W,B}_V : \widehat{Sp(V)} \hookrightarrow \widehat{O(W,B)}$$

in the orthogonal stable range.

First, we establish a piece of notation that will be important to the definitions of $\phi^V_{W,B}$ and $\psi^{W,B}_V$. Recalling Lusztig's description of irreducible unipotent representations (see Definitions 6 and 7 above, and [13]), we see the set of irreducible unipotent representations for G and G^D are identified. For an irreducible unipotent representation u of G, we denote by $\widetilde{u}$ the corresponding irreducible unipotent representation of G^D.

In Subsection 3.1, we treat the case of $(Sp(V), O(W,B))$ in the symplectic symplectic stable range where the dimension of W is odd. In Subsection 3.2, we treat the case of the symplectic stable range where the dimension of W is even. In Subsection 3.3, we treat the case of a reductive dual pair $(Sp(V), O(W,B))$ in the orthogonal stable range where the dimension of W is odd. In Subsection 3.4, we treat the case of the orthogonal stable range where the dimension of W is even.

3.1. **The odd symplectic stable case.** Consider a choice of type I reductive dual pair $(Sp(V), O(W, B))$ in the symplectic stable range for odd-dimensional W. Write $dim(V) = 2N$, $dim(W) = 2m + 1$. The range condition requires that $N \geq 2m + 1$. In this case, the center splits off of the orthogonal group, and we may consider $O(W, B) = \mathbb{Z}/2 \times SO_{2m+1}(\mathbb{F}_q)$. Our goal is to define a construction whose input is an irreducible representation of $(\pm 1) \otimes \rho_{(s),u}$ of $O(W, B)$ (considering ± 1 as a representation of $\mathbb{Z}/2$ and $\rho_{(s),u} \in \widehat{SO_{2m+1}(\mathbb{F}_q)}$ for some Lusztig classification data $[(s), u]$ for $SO_{2m+1}(\mathbb{F}_q)$) and whose output is a unique irreducible $Sp(V) = Sp_{2N}(\mathbb{F}_q)$ representation. In other words, we must associate to $(\pm 1) \otimes \rho_{(s),u}$ a choice of new $Sp_{2N}(\mathbb{F}_q)$-Lusztig classification data:

$$[(\phi^{\pm}(s)), \phi^{\pm}(u), disc(B) \cdot \varepsilon(s)]. \tag{38}$$

Broadly, we construct $\phi^{\pm}(s)$ by adding -1 eigenvalues to s, and we alter the symbol in the affected factor of the unipotent part by adding a single coordinate to one of the rows (to get the needed new rank and defect).

To be more specific, we recall that in the input data, as Lusztig classification data for $SO_{2m+1}(\mathbb{F}_q)$, $[(s), u]$ consists of a conjugacy class (s) of a semisimple element s in the dual group $Sp_{2m}(\mathbb{F}_q) = (SO_{2m+1}(\mathbb{F}_q))^D$ and an irreducible unipotent representation u of the dual of s' centralizer

$$(Z_{Sp_{2m}(\mathbb{F}_q)}(s))^D.$$

Now, to specify (38), we begin with describing the semisimple part $(\phi^{\pm}(s))$. As discussed in the previous section, s is an element of a maximal torus of the form (7), and it is deermined by the data of the orbit of its eigenvalues. Recalling the notation (34), we then define $\phi^{\pm}(s)$ to be the semisimple element

$$\begin{aligned} \phi^{\pm}(s) := s \oplus \sigma^{\pm}_{N-m} \in \prod_{i=1}^{k} SO_2^{\pm}(\mathbb{F}_{q^{n_i}}) \times SO_{2(N-m)+1}(\mathbb{F}_q) \\ \subseteq SO_{2N+1}(\mathbb{F}_q) = (Sp_{2N}(\mathbb{F}_q))^D. \end{aligned}$$

On the level of eigenvalues, $\phi^{\pm}(s)$ are obtained precisely by adding $2(N-m)$ eigenvalues -1 and a single 1 eigenvalue to s's original eigenvalues, in a position where projecting away from the coordinate of the 1 eigenvalue gives a subspace of W where B is completely split if the sign is + and non-split if the sign if $-$. Suppose s has eigenvalue -1 of multiplicity 2ℓ, and 1 of multiplicity $2p$, and its centralizer is of the form (14). For simplicity, let us separate the factors corresponding to the eigenvalues of s not equal to -1

$$H = \prod_{i=1}^{r} U^{+}_{j_i}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U^{-}_{k_i}(\mathbb{F}_{q^{n''_i}}) \times Sp_{2p}(\mathbb{F}_q),$$

so that

$$Z_{Sp_{2m}(\mathbb{F}_q)}(s) = H \times Sp_{2\ell}(\mathbb{F}_q). \tag{39}$$

Then we find that the centralizer of our new elements $\phi^{\pm}(s)$ in $SO_{2N+1}(\mathbb{F}_q)$ is precisely

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi^{\pm}(s)) = H^D \times SO^{\pm}_{2(N-m+\ell)}(\mathbb{F}_q).$$

Now let us describe the unipotent part $\phi^{\pm}(u)$ of (38). Let us factor u as in (15), and let us consider ths symbol

$$u^{-1}_{SO_{2\ell+1}} = \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$$

and write u_H for the unipotent H-representation consisting of a product of the other factors $\bigotimes_{i=1}^{r} u_{U^{+}_{j_i}} \otimes \bigotimes_{i=1}^{t} u_{U^{-}_{k_i}} \otimes u^{-1}_{SO_{2\ell+1}} \otimes u^{+1}_{SO_{2p+1}}$, so that we can write

$$u = u_H \otimes \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}.$$

The defect $a-b$ of the symbol is odd, so we may switch rows to assume without loss of generality that $a-b$ is 1 mod 4. Write

$$N'_{\rho(s),u} = N - m + \frac{a+b-1}{2}$$

(note that by the symplectic stable range condition, we automativally have $N'_{\rho(s),u} \geq N-m \geq m+1$). Then we may concatenate $N'_{\rho(s),u}$ onto the end of either row of the symbol $\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$, obtaining new symbols

$$\phi^{+}(\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}) = \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b < N'_{\rho(s),u}},$$

describing a unipotent representation of $SO^{+}_{2(N-m+\ell)}(\mathbb{F}_q)$ and

$$\phi^{-}(\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}) = \binom{\lambda_1 < \dots < \lambda_a < N'_{\rho(s),u}}{\mu_1 < \dots < \mu_b},$$

describing a unipotent representation of $SO^-_{2(N-m+\ell)}(\mathbb{F}_q)$. We then put

$$\phi^{\pm}(u) := \widetilde{u_H} \otimes \phi^{\pm}(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}),$$

giving a unipotent representation of the group $H^D \times SO^{\pm}_{2(N-m+\ell)}(\mathbb{F}_q) = Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi^{\pm}(s))$.

Finally, we need a central sign to complete the Lusztig classification data (38) since by definition $\phi^{\pm}(s)$ has -1 eigenvalues. Consider again s as an element of the torus (7). Further, consider each factor $SO^{\pm}_2(\mathbb{F}_{q^{r_i}}) \cong \mu_{q^{r_i} \mp 1}$. Then define $\varepsilon(s)$ to be the product of applying the quadratic character on each $\mathbb{Z}/(q^{r_i} \mp 1)$ to each coordinate, giving a total sign. Multiplying with the discriminant $disc(B)$ gives the central sign of (38).

Definition 9. *Given the above notation we define $\phi^V_{W,B}(\rho)$ to be the irreducible $Sp(V)$-representation with the new Lusztig classification data we constructed:*

$$\phi^V_{W,B}((\pm 1) \otimes \rho_{(s),u}) = \rho_{(\phi^{\pm}(s)),\phi^{\pm}(u),disc(B)\cdot\varepsilon(s)}.$$

3.2. **The even symplectic stable case.** Now suppose the reductive dual pair $(Sp(V), O(W,B))$ is in the symplectic stable case and W is even dimensional. Write $dim(V) = 2N$ and $dim(W) = 2m$ Write α for the sign so that $O(W,B) = O^{\alpha}_{2m}(\mathbb{F}_q)$. In both cases, the orthogonal stable range condition requires that $N \geq 2m$.

Fix an input irreduicble $O(W,B)$-representation ρ. As described in Subsection 2.3, we take its $O(W,B)$-Lusztig classification data to consist of a conjugacy class in $O(W,B)$ (s) of a semisimple element $s \in SO(W,B)$, an irreducible unipotent representation u of the dual of s's centralizer $(Z_{SO(W,B)}(s))^D$, and possible central sign data depending on which eigenvalues appear in s. Broadly, we obtain new $Sp_{2N}(\mathbb{F}_q)$-Lusztig classification data

$$[(\phi(s)), \phi^{\pm}(u), \pm 1]$$

by adding 1 eigenvalues to s, then altering the affect factor of the unipotent part by adding a single coordinate to one row of the symbol (according to the original $+1$ part of the sign data of ρ if there is a choice), and keeping the original -1 part of the sign data if it occurs.

Consider, again, s as an element of a torus (7). One may take a direct sum with the identity matrix $I_{2(N-m)+1}$ to obtain a semisimple element

$$\phi(s) = s \oplus I_{2(N-m)+1} \in SO_{2N+1}(\mathbb{F}_q) = (Sp_{2N}(\mathbb{F}_q))^D.$$

(Note that each different class (s) considered as a conjugacy class in $O(W,B)$ corresponds to a different $\phi(s)$, whereas if we only considered (s) as a conjugacy class in $SO(W,B)$, in cases with eigenvalues not equal to ± 1, there would be another $SO(W,B)$-conjugacy class (s') with $(\psi(s)) = (\psi(s'))$ in $SO_{2N+1}(\mathbb{F}_q)$.)

If s does not have any 1 eigenvalues, then we have

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi(s)) = (Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(s))^D \times SO_{2(N-m)+1}(\mathbb{F}_q),$$

and we consider the unipotent representation

$$\phi(u) := \widetilde{u} \otimes 1$$

of its dual, tensoring $\widetilde{u}$ with the trivial representation of the new factor $(SO_{2(N-m)+1}(\mathbb{F}_q))^D$.

If s has 1 as an eigenvalue of multiplicity $2p$ for $p > 0$, then suppose its centralizer is of the form (20). We separate out the factors arising from the eigenvalues not equal to 1 by writing

$$H = \prod_{i=1}^{r} U^{+}_{j_i}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U^{-}_{k_i}(\mathbb{F}_{q^{n''_i}}) \times SO^{\pm}_{2\ell}(\mathbb{F}_q)$$

so that we have

$$Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(s) = H \times SO^{\pm}_{2p}(\mathbb{F}_q),$$

and

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(\psi(s)) = H^D \times SO_{2(N-m+p)+1}(\mathbb{F}_q)$$

(note that in this case $H = H^D$). Factoring u as in (21), let us consider the symbol of the factor corresponding to the 1 eigenvalue

$$u^{+1}_{SO^{\pm}_{2p}} = \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

and write u^{H^D} for the representation of $H = H^D$ consisting of the other factors $\bigotimes_{i=1}^{r} u_{U^{+}_{j_i}} \otimes \bigotimes_{i=1}^{t} u_{U^{-}_{k_i}} \otimes u^{-1}_{SO^{\pm}_{2\ell}}$. Switch rows so that for the minimal i such that $\lambda_{a-i} \neq \mu_{b-i}$, we have $\lambda_{a-i} < \mu_{b-i}$. Write

$$N'_{\rho} = N - m + \frac{a+b}{2}.$$

Then

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_{\rho} \\ \mu_1 < \cdots < \mu_b \end{pmatrix}, \qquad \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < N'_{\rho} \end{pmatrix}$$

define unipotent $(SO_{2(N-m+p)+1}(\mathbb{F}_q))^D = Sp_{2(N-m+p)}(\mathbb{F}_q)$. Therefore, if $\lambda_a < N'_\rho$, $\mu_b < N'_\rho$,

$$\phi^+(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

$$\phi^-(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < N'_\rho \end{pmatrix}$$

respectively define irreducible unipotent representations of the dual group $(Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi(s)))^D = H \times Sp_{2(N-m+p)}(\mathbb{F}_q)$. We choose the sign of $\phi^\pm(u)$ according to the central sign data (± 1) chosen from s having 1 eigenvalues.

Finally, to define an output irreducible $Sp_{2N}(\mathbb{F}_q)$-representation, we need to also choose output central sign data if $\phi(s)$ has -1 eigenvalues. By definition, $\phi(s)$ has the same number of -1 eigenvalues as s. Therefore, in this case, the original s has -1 eigenvalues also, so the $O(W,B)$-Lusztig classification data supplies us with the data of one more central sign ± 1, which we use as the output central sign data.

Definition 10. *Given the above notation, we define $\eta^V_{W,B}(\rho)$ to be the irreducible $Sp(V)$-representation with Lusztig classification data $[\phi(s), \phi^\pm(u), \pm 1]$ where the sign in $\phi^\pm(u)$ is the central sign data from s's 1 eigenvalues, the sign in ± 1 is the central sign data from s's -1 eigenvalues, and signs are ommited when s does not have such eigenvalues*

$$\phi^V_{W,B}(\rho) := \rho_{(\phi(s)),\phi^\pm(u),\pm 1}. \tag{40}$$

3.3. **The odd orthogonal stable case.** Suppose $(Sp(V), O(W,B))$ is in the orthogonal stable case and W is odd dimensional. Write $dim(V) = 2N$ an $dim(W) = 2m+1$. In this case, writing $dim(V) = 2N$, this means $m \geq N$.

Consider an irreducible representation ρ of $Sp(V) = Sp_{2N}(\mathbb{F}_q)$. Again, we consider $O(W,B) = \mathbb{Z}/2 \times SO_{2m+1}(\mathbb{F}_q)$, and therefore, our goal is to specify an irreducible representation of $SO_{2m+1}(\mathbb{F}_q)$ and a sign specifying an action of $\mathbb{Z}/2$. Writing (s) and u for the semisimple and unipotent parts of ρ's Lusztig classification data (and considering the central sign data ± 1 if it occurs), we will produce $SO_{2m+1}(\mathbb{F}_q)$-Lusztig classification data

$$[(\psi(s)), \psi^\pm(u)]$$

(where, if there is no central sign data, there ill only be a single choice of unipotent part $\psi(u)$), and tensor it with the $\mathbb{Z}/2$-representation corresponding to the sign

$$(\epsilon(s) \cdot disc(B)).$$

Broadly, in this case, we construct $\psi(s)$ by adding -1 eigenvalues to s and adding a single new coordinate to the symbol corresponding to the affected factor of the unipotent part (to achieve the new needed rank and defect), according the central sign data of ρ if it occurs.

To be more specific, consider the semisimple conjugacy class part (s) of ρ's Lusztig classification data. We have $s \in (Sp_{2N}(\mathbb{F}_q))^D = SO_{2N+1}(\mathbb{F}_q)$. Recall again that we can consider s as an element of a torus of the form (7), by removing the single "forced" eigenvalue 1 from s. Write $\widetilde{s}$ for the $2N$ by $2N$ matrix obtained in this way. Taking a direct sum with $-I_{2(m-N)}$,

$$\psi(s) := \widetilde{s} \oplus (-I)_{2(m-N)}$$

specifies a semisimple element of $Sp_{2m}(\mathbb{F}_q) = (SO_{2m+1}(\mathbb{F}_q))^D$, which has -1 as an eigenvalue of multiplicity $2(m - N + \ell)$.

Say that s has -1 as an eigenvalue of multiplicity 2ℓ with centralizer of the form (28). Again, let us separate the factors corresponding to eigenvalues not equal to -1 and write

$$H = \prod_{i=1}^{r} U_{j_i}^{+}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U_{k_i}^{-}(\mathbb{F}_{q^{n''_i}}) \times SO_{2p+1}(\mathbb{F}_q)$$

so that

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO_{2\ell}^{\pm}(\mathbb{F}_q)$$

and

$$Z_{Sp_{2m}(\mathbb{F}_q)}(\psi(s)) = H^D \times Sp_{2(m-N+\ell)}(\mathbb{F}_q).$$

For the unipotent part of the Lusztig classification data of ρ, write its factorization as in (29). Specially consider the symbol corresponding to the factor

$$u_{SO_{2\ell}^{\pm}} = \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

and write u^{H^D} for the unipotent H^D-representation corresponding to the rest of the factors $\bigotimes_{i=1}^{r} u_{U_{j_i}^{+}} \otimes \bigotimes_{i=1}^{t} u_{U_{k_i}^{-}} \otimes u_{Sp_{2p}}$. Again, switch rows so that for the minimal i such that $\lambda_{a-i} \neq \mu_{b-i}$ has $\lambda_{a-i} > \mu_{b-i}$. Let us write

$$m'_\rho = m - N + \frac{a+b}{2}.$$

By the orthogonal stable range condition, we must have $\lambda_a < m'_\rho$ and $\mu_b < m'_\rho$. Concatenating m'_ρ to the end of one of the rows, we obtain symbols

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < m'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}, \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < m'_\rho \end{pmatrix} \tag{41}$$

which have odd defect and rank precisely equal to $m-N+\ell$, and therefore specify irreducible unipotent representations of $SO_{2(m-N+\ell)+1}(\mathbb{F}_q) = (Sp_{2(m-N+\ell)}(\mathbb{F}_q))^D$.

In the case when $\ell = 0$, both (41) specify the same representation - the trivial representation. In this case put

$$\psi(u) = \widetilde{u_{H^D}} \otimes 1.$$

In the case when $\ell > 0$, we have additional central sign data ± 1 in the Lusztig classification data for ρ, which we use to select which symbol (41) should appear as the new factor of the unipotent part of the Lusztig classification data of $\zeta_V^{W,B}(\rho)$. Specifically, we put

$$\psi^+(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a < m'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

$$\psi^-(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < m'_\rho \end{pmatrix}.$$

Definition 11. *Suppose we are given the above notation. If $\ell = 0$, writing $\rho = \rho_{(s),u}$, we take $\psi_V^{W,B}(\rho)$ to be the tensor product of the irreducible $SO(W,B)$-representation corresponding to Lusztig classification data $(\psi(s)), \psi(u)$ with the sign $\epsilon(s) \cdot disc(B)$:*

$$\psi_V^{W,B}(\rho) = \rho_{(\psi(s)),\psi(u)} \otimes (\epsilon(s) \cdot disc(B)) \tag{42}$$

If $\ell > 0$, writing $\rho = \rho_{(s),u,\pm 1}$, we take $\phi_V^{W,B}(\rho)$ to be the tensor product of the irreducible $SO(W,B)$-representation corresponding to Lusztig classification data $(\psi(s)), \psi^\pm(u)$ with the sign $\epsilon(s) \cdot disc(B)$

$$\psi_V^{W,B}(\rho) = \rho_{(\psi(s)),\psi^\pm(u)} \otimes (\epsilon(s) \cdot disc(B)). \tag{43}$$

3.4. **The even orthogonal stable case.** Suppose W is of even dimension $2m$, and write α for the sign so that $O(W,B) = O^\alpha_{2m}(\mathbb{F}_q)$. Suppose that $(Sp(V), O(W,B))$ is in the orthogonal stable range.

Consider an irreducible representation ρ of $Sp(V) = Sp_{2N}(\mathbb{F}_q)$. We want to produce $O_{2m}(\mathbb{F}_q)^\alpha$-Lusztig classification data, which we recall consists of a semisimple conjugacy class $(s) \in O^\alpha_{2m}(\mathbb{F}_q)$, a unipotent part u which can be considered to consist of a unipotent irreducible representaion of the (dual) of the centralizer of s in $SO^\alpha_{2m}(\mathbb{F}_q)$, and central sign data. We note that since $Res_{O(W,B)}(\omega[V \otimes W])$ is the permutation representation $\mathbb{C}W$ tensored with the representation $\epsilon(det)$ (corresponding to the sign representation of $O(W,B)/SO(W,B)$), part of the central sign data is already forced. Specifically, as in the case of the symplectic group, we will only need to choose central sign data for the output representation corresponding to -1-eigenvalues. Broadly, we will construct the new semisimple and unipotent parts of the $O_{2m}^{\pm}(\mathbb{F}_q)$-Lusztig classification data

$$(\psi(s)), \psi(u)$$

by adding 1 eigenvalues to s and altering the symbol of the affect factor of the unipotent part by adding a single new coordinate to one of the rows to achieve the new needed rank and defect.

To be more specific, write (s) with $s \in SO_{2N+1}(\mathbb{F}_q) = (Sp_{2N}(\mathbb{F}_q))^D$ for the semisimple part of the Lusztig classification data for the input $Sp_{2N}(\mathbb{F}_q)$-representation ρ. Say s has 1 as an eigenvalue of multiplcity $2p+1$. Again, we may remove a single "forced" 1 eigenvalue from s to view it as a $2N$ by $2N$ element of a maximal torus (7). Then consider the direct sum with the $2(m-N)$ by $2(m-N)$ identity matrix

$$\psi(s) = \widetilde{s} \oplus I_{2(m-N)},$$

configured to give a $2m$ by $2m$ matrix that can be considered as an element of $SO(W,B) \subseteq O(W,B)$. As in Subsection 3.2, each distinct $SO_{2N+1}(\mathbb{F}_q)$-conjugacy class (s) gives a distinct $O_{2m}^{\alpha}(\mathbb{F}_q)$-conjugacy class $\psi(s)$. Writing the centralizer of s as (28), we again separate out the factors corresponding to the eigenvalues not equal to 1, writing

$$H = \prod_{i=1}^{r} U_{j_i}^{+}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U_{k_i}^{-}(\mathbb{F}_{q^{n''_i}}) \times SO_{2\ell}^{\pm}(\mathbb{F}_q)$$

so that

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO_{2p+1}(\mathbb{F}_q)$$

and

$$Z_{SO_{2m}^{\alpha}(\mathbb{F}_q)}(\psi(s)) = H^D \times SO_{2(N-m+p)}^{\beta}(\mathbb{F}_q),$$

for a single determined choice of sign β (so that its product with the other signs appearing in H agrees with α).

To construct the unipotent part of the $O_{2m}^{\alpha}(\mathbb{F}_q)$-Lusztig classification data $\psi(u)$, specially consider the symbol

$$u_{SO_{2p+1}} = \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

and write u^{H^D} for the product of the remaining factors $\bigotimes_{i=1}^{r} u_{U^+_{j_i}} \otimes \bigotimes_{i=1}^{t} u_{U^-_{k_i}} \otimes u_{SO^{\pm}_{2\ell}}$. Switch the symbol rows so that the defect $a-b$ is 1 mod 4 (which is possible since this symbol has odd defect). Let us write

$$m'_\rho = m - N + \frac{a+b-1}{2}.$$

Then, if $\beta = +$, if $\mu_b < m'_\rho$, putting

$$\psi(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < m'_\rho \end{pmatrix}$$

gives a unipotent representation of the group $H \times SO^+_{2(N-m+p)}(\mathbb{F}_q) = (Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(\psi(s)))^D$. Similarly, if $\beta = -$, if $\lambda_a < m'_\rho$, putting

$$\psi(u) = \widetilde{u_{H^D}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a < \mu'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

gives a unipotent representation of the group $H \times SO^-_{2(N-m+p)}(\mathbb{F}_q) = (Z_{SO^{\alpha}_{2m}(\mathbb{F}_q)}(\psi(s)))^D$.

Now, as in Subsection 3.2, (s) and $(\psi(s))$ have the same multiplicity of -1 eigenvalues. Therefore, the undetermined central sign data needed to describe $\zeta_V^{W,B}(\rho)$ if and only if central sign data is given in ρ's original Lusztig classification data. We take it to be the same in this case.

Definition 12. *Suppose we are given the above notation. We define $\zeta_V^{W,B}(\rho)$ to be the irreducible $O(W,B)$-representation with $O(W,B)$-Lusztig classification data $[\psi(s), \psi(u), \pm 1]$, where the final sign is the central sign of ρ arising if s has -1 eigenvalues and where we omit it if s has no such eigenvalues*

$$\psi_V^{W,B}(\rho) := \rho_{\psi(s),\psi(u),\pm 1}, \tag{44}$$

(we also neglect to write in the notation the determined central sign data corresponding from the 1 eigenvalues of s, which is pre-determined).

4. A combinatorial identity

Now recalling [12], a key step in decomposition the restriction of an oscillator representation $\omega[V \otimes W]$ to $Sp(V) \times O(W,B)$ is to separate off its "top part," which specifically singles out summands arising from the eta or zeta correspondence with source corresponding to the appropriate full-rank orthogonal or symplectic group, respectively. In the symplectic stable range, we write

$$\omega[V \otimes W]^{\text{top}} := \bigoplus_{\rho \in O(W,B)} \eta^V_{W,B}(\rho) \otimes \rho,$$

and call it the *top part* of $\omega[V \otimes W]$. Similarly, in the orthogonal stable range, we write

$$\omega[V \otimes W]^{\text{top}} := \bigoplus_{\rho \in O(W,B)} \rho \otimes \zeta^{W,B}_V(\rho),$$

and call it the *top part* of $\omega[V \otimes W]$.

From here, the proof of Theorem 2 separates into two key steps: A combinatorial verification that the dimension of the direct sum of matches the dimension of the top part of $\omega[V \otimes W]$, and an inductive argument showing that the claimed correspondence in Theorem 2 is the only possible one. The first step is the goal of this section.

Theorem 13. *If $(Sp(V), O(W,B))$ is in the symplectic stable range, the dimension of the top part of the restriction of $\omega[V \otimes W]$ matches the sum of products of the dimensions of irreducible representations of $O(W,B)$ and their $\psi^V_{W,B}$ correspondences:*

$$dim(\omega[V \otimes W]^{top}) = \sum_{\rho \in \widehat{O(W,B)}} dim(\rho) \cdot dim(\phi^V_{W,B}(\rho)). \tag{45}$$

Similarly, if $(Sp(V), O(W,B))$ is in the orthogonal stable range, the dimension of the top part of the restriction of $\omega[V \otimes W]$ matches the sum of products of the dimensions of irreducible representations of $Sp(V)$ and their $\phi^{W,B}_V$ correspondences:

$$dim(\omega[V \otimes W]^{top}) = \sum_{\rho \in \widehat{Sp(V)}} dim(\rho) \cdot dim(\psi^{W,B}_V(\rho)). \tag{46}$$

4.1. **The dimension of the top part of the oscillator representation.** First, we need a more explicit formula for the left hand side of (45):

Proposition 14. *Consider symplectic and orthogonal spaces V and (W,B) whose dimensions are in the symplectic stable range. Writing $dim(V) = 2N$ and $dim(W) = 2m+1$, the dimension of the top part of the restriction of $\omega[V \otimes W]$ is*

$$\sum_{i=0}^{m} (-1)^{m-i} \cdot q^{\binom{m-i}{2}} \cdot \binom{m}{i}_q \cdot \prod_{k=i+1}^{m} (q^k+1) \cdot q^{(2i+1)N} \tag{47}$$

Proof. Write, for $j < i$

$$C_{i,j} := -\binom{i}{j}_q \cdot \prod_{k=j+1}^{i}(q^k+1) = \tag{48}$$
$$\frac{(q^{2i}-1)(q^{2(i-1)}-1)\dots(q^{2(j+1)}-1)}{(q^{i-j}-1)(q^{i-j-1}-1)\dots(q-1)}$$

Let X_m denote the dimension of the top part $\omega[V \otimes W]^{\text{ top}}$, where W is a $2m+1$-dimensional $\mathbb{F}_q$-space. Taking the dimension of (3) then gives the recursive equation

$$X_m = q^{(2m+1)N} + \sum_{i=0}^{m-1} C_{m,i} \cdot X_i. \tag{49}$$

Our goal to prove (47) is to re-express the right-hand side of (49) in terms of a sum of $q^{(2i+1)N}$ for $0 \leq i \leq m$ some lower coefficient. Now, iteratively applying (49), we find that

$$X_m = \sum_{i=0}^{m}\left(\sum_{i=\ell_1<\dots<\ell_j=m}\prod_{k=1}^{j-1} C_{\ell_{k+1},\ell_k}\right) q^{(2i+1)N}.$$

It suffices to prove

$$\sum_{i=\ell_1<\dots<\ell_j=m}\prod_{k=1}^{j-1} C_{\ell_{k+1},\ell_k} = C_{m,i}\cdot q^{\binom{i}{2}}. \tag{50}$$

Using (48), each term $\prod_{k=1}^{j-1} C_{\ell_{k+1},\ell_k}$ where $i = \ell_1 < \dots < \ell_j = m$, factors as

$$\frac{(q^{2m}-1)(q^{2(m-1)}-1)\dots(q^{2(m-i+1)}-1)}{\prod_{k=1}^{j-1}\prod_{r=1}^{\ell_{k+1}-\ell_k}(q^r-1)},$$

which can be simplified as

$$C_{m,i}\cdot\frac{(q^{m-i}-1)(q^{m-i-1}-1)\dots(q-1)}{\prod_{k=1}^{j-1}\prod_{r=1}^{\ell_{k+1}-\ell_k}(q^r-1)},$$

reducing the claim to

$$q^{\binom{m-i}{2}} = \sum_{i=\ell_1<\dots<\ell_j=m}\frac{(q^{m-i}-1)(q^{m-i-1}-1)\dots(q-1)}{\prod_{k=1}^{j-1}\prod_{r=1}^{\ell_{k+1}-\ell_k}(q^r-1)}. \tag{51}$$

The right-hand side of (51) can also be written as

$$\sum_{0=\ell'_1<\cdots<\ell'_j=m-i} \binom{\ell'_j}{\ell'_{j-1}}_q \binom{\ell'_{j-1}}{\ell'_{j-2}}_q \cdots \binom{\ell'_2}{\ell'_1}_q,$$

by substituting $\ell'_j = \ell_j - i$, so (51) follows from a q-version of the multinomial theorem. □

We re-write (47) again as follows, to separate it into terms which correspond to levels of singularity of semisimple elements (more specifically, the multiplicity of eigenvalue -1) in the classification of irreducible representations of $SO_{2m+1}(\mathbb{F}_q)$:

Proposition 15. *The top dimension of* $\omega[V\otimes W]^{top}$ *is*

$$\sum_{\ell=0}^{m}(-1)^{\ell}\, q^{N+(m-\ell)(m-\ell-1)+\ell^2} \binom{m}{\ell}_{q^2} \prod_{j=0}^{m-\ell-1} (q^{2(N-i)}-1). \tag{52}$$

Proof. Substituting $i=m-\ell$, (47) can be re-written as

$$\sum_{\ell=0}^{m}(-1)^{\ell} q^{\binom{\ell}{2}} \binom{m}{\ell}_{q^2} (\prod_{j=1}^{\ell}(q^j+1))q^{(2(m-\ell)+1)+N}. \tag{53}$$

Now in (52), using

$$(m-\ell-1)(m-\ell) = \sum_{j=0}^{m-\ell-1} 2k,$$

we have

$$q^{(m-\ell-1)(m-\ell)} \prod_{j=0}^{m-\ell-1} (q^{2(N-i)}-1) = \prod_{j=0}^{m-\ell-1} (q^{2N}-q^{2i}).$$

Hence, (52) reduces to

$$\sum_{\ell=0}^{m}(-1)^{\ell} q^{N+\ell^2} \binom{m}{\ell}_{q^2} \prod_{i=0}^{m-\ell-1} (q^{2N}-q^{2i}).$$

Finally, at each ℓ,

$$\prod_{i=0}^{m-\ell-1}(q^{2N}-q^{2i})=\sum_{j=0}^{m-\ell}q^{2Nj}\cdot\sum_{1\leq i_1<\cdots<i_{m-\ell-j}\leq m-\ell-1}q^{2(i_1+\dots i_{m-\ell-j})}=$$

$$\sum_{j=0}^{m-\ell}q^{2Nj}\cdot\binom{m-\ell}{j}_{q^2}.$$

Therefore, the coefficient of $q^{(2(m-\ell)+1)N}$ in (53) for each ℓ is

$$\sum_{k=0}^{\ell}(-1)^k q^{k^2}\binom{m}{m-k}_{q^2}\binom{m-k}{m-\ell}_{q^2}$$

(identifying $\binom{m}{k}_{q^2}$ with $\binom{m}{m-k}_{q^2}$). Hence, the claim reduces to verifying that

$$\sum_{k=0}^{\ell}(-1)^k q^{k^2}\binom{m}{m-k}_{q^2}\binom{m-k}{m-\ell}_{q^2}= \tag{54}$$

$$q^{\binom{\ell}{2}}\binom{m}{m-\ell}_{q^2}\prod_{j=1}^{\ell}(q^j+1)$$

Further, we have

$$\binom{m}{m-k}_{q^2}\binom{m-k}{m-\ell}_{q^2}=\binom{m}{m-\ell}_{q^2}\binom{\ell}{\ell-k}_{q^2}=\binom{m}{m-\ell}_{q^2}\binom{\ell}{k}_{q^2},$$

reducing (54) again to a q-multinomial theorem. $\square$

The purpose of re-writing the dimension of the top part of $\omega[V\otimes W]$ as (52) is because, for each ℓ, the prime to q part of the ℓth term of (52) is

$$\binom{m}{\ell}_{q^2}\prod_{i=0}^{m-\ell-1}(q^{2(N-i)}-1)= \tag{55}$$

$$\frac{|Sp_{2m}(\mathbb{F}_q)|_{q'}\cdot|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|Sp_{2(m-\ell)}(\mathbb{F}_q)|_{q'}\cdot|Sp_{2\ell}(\mathbb{F}_q)|_{q'}\cdot|Sp_{2(N-m-\ell)}(\mathbb{F}_q)|_{q'}}.$$

We use Proposition 15 to conclude (45) by approximating the right hand recursively by considering terms $dim(\pi)dim(\phi_{W,B}(\pi))$ separately for $\pi \in \widehat{O(W,B)}$ arising from a conjugacy class of a semisimple element of the dual group $Sp_{2m}(\mathbb{F}_q)$, which is singular of type $(m-\ell,\ell)$ (i.e. has -1 as an eigenvalue with multiplicity 2ℓ), using the elementary fact that the sum of the squares of the dimensions of all irreducible representations of a group G recover its group order. This gives that the "level ℓ" approximation of the right hand side of (45) (which counts correctly the terms from π arising from conjugacy classes of semisimple elements $Sp_{2m}(\mathbb{F}_q)$ with eigenvalue -1 of multiplicity less than or equal to 2ℓ, and miss-counts the terms from π arising from conjugacy classes with eigenvalue -1 of multiplicity more than 2ℓ) is the sum of the first ℓ terms of (52).

More formally:

Proof of Theorem 13. Suppose W is an $\mathbb{F}_q$-vector space of dimension $2m+1$ with symmetric bilinear form B. First, consider irreducible representations $\pi \in \widehat{O(W,B)}$ whose restrictions $Res_{SO(W,B)}(\pi)$ to $SO(W,B)$ correspond to a conjugacy class of a semisimple element $s \in Sp_{2m}(\mathbb{F}_q)$ where -1 is not an eigenvalue. Call such irreducible representations of $SO(W,B)$ the *"level 0" representations of* $SO(W,B)$. For each such $\pi' \in \widehat{SO(W,B)}$, say with Lusztig classification given by the Jordan decomposition $((s),u)$, the centralizer of s first must be of the form

$$Z_s(Sp_{2m}(\mathbb{F}_q)) = \prod_{i=1}^{r} U^+_{j_i}(\mathbb{F}_q) \times \prod_{i=1}^{t} U^-_{k_i}(\mathbb{F}_q) \times Sp_{2p}(\mathbb{F}_q)$$

for $\sum_{i=1}^r j_i + \sum_{i=1}^t k_i + p = m$ (where s has 1 as an eigenvalue with multiplicity $2p$), with the unipotent representation u then consisting of the data of unipotent representations of $U^+_{j_i}(\mathbb{F}_q)$, $U^-_{k_i}(\mathbb{F}_q)$, and a symbol of rank p and type C. Then,

$$\begin{gathered} Z_{s\oplus\sigma^{\pm}_{N-m}}(SO_{2N+1}(\mathbb{F}_q)) = \\ \prod_{i=1}^{r} U^+_{j_i}(\mathbb{F}_q) \times \prod_{i=1}^{t} U^-_{k_i}(\mathbb{F}_q) \times SO_{2p+1}(\mathbb{F}_q) \times SO^{\pm}_{2(N-m)}(\mathbb{F}_q) \end{gathered},$$

wich has order

$$|Z_{s\oplus\sigma^{\pm}_{N-m}}(SO_{2N+1}(\mathbb{F}_q))| = |Z_s(Sp_{2m}(\mathbb{F}_q))| \cdot |SO^{\pm disc(B)}_{2(N-m)}(\mathbb{F}_q)|.$$

For both choices, the dimension of $\phi_{W,B}(u)$ is equal to the dimension of u. Hence, for every $\pi' \in \widehat{SO(W,B)}$, the sum of dimensions $dim(\phi_{W,B}(\pi\otimes 1)) + dim(\phi_{W,B}(\pi \otimes -1))$ is equal to the dimension of π, multiplied by

$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{2|SO^+_{2(N-m)}(\mathbb{F}_q)\times Sp_{2m}(\mathbb{F}_q)|_{q'}}+\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{2|SO^-_{2(N-m)}(\mathbb{F}_q)\times Sp_{2m}(\mathbb{F}_q)|_{q'}}=$$

$$\frac{1}{|Sp_{2m}(\mathbb{F}_q)|_{q'}}q^{N-m}\prod_{i=0}^{m-1}(q^{2(N-i)}-1)$$

Hence, since the dimensions of the $O(W,B)$ representations $\pi\otimes 1$ and $\pi\otimes -1$ are equal to $dim(\pi)$, the sum of the two terms

(56)

$$\dim(\pi'\otimes 1)\,\dim(\phi_{W,B}(\pi'\otimes 1))+\dim(\pi'\otimes -1)\,\dim(\phi_{W,B}(\pi'\otimes -1))=$$

$$\frac{\dim(\pi')^2}{|Sp_{2m}(\mathbb{F}_q)|_{q'}}q^{N-m}\prod_{i=0}^{m-1}(q^{2(N-i)}-1)$$

If all representations $\pi'\in\widehat{SO(W,B)}$ satisfied (56), then the right hand side of (45) would equal

$$\frac{|SO(W,B)|}{|Sp_{2m}(\mathbb{F}_q)|_{q'}}q^{N-m}\prod_{i=0}^{m-1}(q^{2(N-i)}-1)=$$

(57)

$$q^{N-m(m-1)}\prod_{i=0}^{m-1}(q^{2(N-i)}-1)$$

(recalling that $|SO(W,B)|=|Sp_{2m}(\mathbb{F}_q)|$, with q-part equal to q^{m^2}).

We call (57) the *level 0 approximation* of (45). Note that it is precisely equal to the 0th term of (45). The remainder of the argument consists of considering the ranges of irreducible representations arising from semisimple elements one ℓ at a time (from $\ell=1$ to $\ell=m$). We must compute that adding the ℓth term of (52) cancels the "level ℓ-error," arising from miscounting the terms (45) for π' arising from (s) with exactly 2ℓ in the level $(\ell-1)$-approximation of (45) (though it may create more error at higher levels), so that we can take the sum

$$\sum_{i=0}^{\ell}(-1)^i q^{N+(m-i)(m-i-1)+i^2}\binom{m}{\ell}_{q^2}\prod_{j=0}^{m-i-1}(q^{2(N-j)}-1)$$

to be the *level ℓ approximation* of (45).

We may therefore prove Theorem 13 inductively by verifying that for every ℓ, the level ℓ-approximation is equal to the sum of the 0th to

ℓth terms of (52), up to an error of terms with N-degree less than or equal to $2(m-\ell)+1$.

Lemma 16. *Fix a symbol* $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ *of rank* ℓ*, type* C*, and write* $c=(a+b-1)/2$*. Choosing the sign of* $SO^{\pm}_{2N}(\mathbb{F}_q)$ *in the denominator according to matching the defect of the written symbols with the appropriate groups (depending on* $a-b$ *mod 4), then the sum*

$$\begin{aligned}&\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO^{\pm}_{2N}(\mathbb{F}_q)|_{q'}}dim(\begin{pmatrix}\lambda_1<\cdots<\lambda_a\\ \mu_1<\cdots<\mu_b<N-\ell+c\end{pmatrix})+\\ &\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO^{\mp}_{2N}(\mathbb{F}_q)|_{q'}}dim(\begin{pmatrix}\lambda_1<\cdots<\lambda_a<N-\ell+c\\ \mu_1<\cdots<\mu_b\end{pmatrix})\end{aligned} \tag{58}$$

is the product

$$(\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO_{2\ell+1}(\mathbb{F}_q)\times SO^{+}_{2(N-\ell)}(\mathbb{F}_q)|_{q'}}+\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO_{2\ell+1}(\mathbb{F}_q)\times SO^{-}_{2(N-\ell)}(\mathbb{F}_q)}).$$
$$dim(\begin{pmatrix}\lambda_1<\cdots<\lambda_a\\ \mu_1<\cdots<\mu_b\end{pmatrix}), \tag{59}$$

up to an error term equal to (59), multiplied again by $dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})$ *and the factor*

$$q^{N-(m-\ell)(m-\ell-1)}\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|Sp_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}\frac{|SO_{2m+1}(\mathbb{F}_q)|_{q'}}{|SO_{2\ell+1}(\mathbb{F}_q)\times SO_{2(m-\ell)+1}(\mathbb{F}_q)|_{q'}|}$$

Proof. Suppose, without loss of generality, $a-b$ is 1 mod 4. Recalling how to compute the dimensions of symbols, we have that

$$\frac{dim(\begin{pmatrix}\lambda_1<\cdots<\lambda_a\\ \mu_1<\cdots<\mu_b<N-\ell+c\end{pmatrix})}{q^{\binom{\frac{a+b-1}{2}}{}+\binom{\frac{a+b-3}{2}}{}+\binom{\frac{a+b-5}{2}}{}+\cdots}\cdot|SO^{+}_{2N}(\mathbb{F}_q)|_{q'}}=$$

$$\frac{dim(\begin{pmatrix}\lambda_1<\cdots<\lambda_a\\ \mu_1<\cdots<\mu_b\end{pmatrix})\prod_{i=1}^{a}(q^{N-\ell+c}+q^{\lambda_i})\prod_{i=1}^{b}(q^{N-\ell+c}-q^{\mu_i})}{q^{\binom{\frac{a+b-2}{2}}{}+\binom{\frac{a+b-4}{2}}{}+\binom{\frac{a+b-6}{2}}{}+\cdots}\cdot|SO_{2\ell+1}(\mathbb{F}_q)|_{q'}\prod_{i=1}^{N-\ell+c}(q^{2i}-1)}$$

and, similarly,

$$\frac{dim(\begin{pmatrix}\lambda_1 < \dots < \lambda_a < N-\ell+c\\ \mu_1 < \dots < \mu_b\end{pmatrix})}{q^{\binom{a+b-1}{2}+\binom{a+b-3}{2}+\binom{a+b-5}{2}+\dots}\cdot |SO^-_{2N}(\mathbb{F}_q)|_{q'}} =$$

$$\frac{dim(\begin{pmatrix}\lambda_1 < \dots < \lambda_a\\ \mu_1 < \dots < \mu_b\end{pmatrix})\prod_{i=1}^{a}(q^{N-\ell+c}-q^{\lambda_i})\prod_{i=1}^{b}(q^{N-\ell+c}+q^{\mu_i})}{q^{\binom{a+b-2}{2}+\binom{a+b-4}{2}+\binom{a+b-6}{2}+\dots}\cdot |SO_{2\ell+1}(\mathbb{F}_q)|_{q'}\prod_{i=1}^{N-\ell+c}(q^{2i}-1)}.$$

Summing the terms (58) then gives a product of the coefficient

$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO_{2\ell+1}(\mathbb{F}_q)|_{q'}\prod_{i=1}^{N-\ell+c}(q^{2i}-1)}dim(\begin{pmatrix}\lambda_1 < \dots < \lambda_a\\ \mu_1 < \dots < \mu_b\end{pmatrix}),$$

with the factor

$$\frac{q^{\binom{a+b-2}{2}+\binom{a+b-4}{2}+\binom{a+b-6}{2}+\dots}}{q^{\binom{a+b-1}{2}+\binom{a+b-3}{2}+\binom{a+b-5}{2}+\dots}} = \frac{1}{q^{\sum_{i=0}^{c-1}(2(c-i)-1)}} = \frac{1}{q^{c^2}},$$

with the sum

$$\prod_{i=1}^{a}(q^{N-\ell+c}-q^{\lambda_i})\prod_{i=1}^{b}(q^{N-\ell+c}+q^{\mu_i})+ \tag{60}$$
$$\prod_{i=1}^{a}(q^{N-\ell+c}+q^{\lambda_i})\prod_{i=1}^{b}(q^{N-\ell+c}-q^{\mu_i}).$$

Since the defect $a-b$ is odd, when multiplying out the factors (60) as a sum of powers of $q^{N-\ell+c}$ (with lesser coefficients, not involving N), we find that only odd powers $q^{(2k+1)(N-\ell+c)}$ have non-zero coefficient (for $k=0,\dots,c$). Explicitly, it is

$$2q^{N-\ell+c}(\sum_{k=0}^{c} q^{2k(N-\ell+c)}\cdot \sum (-1)^r q^{\sum_{s=1}^{r}\lambda_{i_s}+\sum_{s=1}^{2(c-k)-r}\mu_{j_s}})$$

where the second sum runs over all choices of r and $1 \leq i_1 < \cdots < i_r \leq a$, $1 \leq j_1 < \cdots < j_{2(c-k)-r} \leq b$. Consider

$$2q^{N-\ell+c} = q^c((q^{N-\ell}-1)+(q^{N-\ell}+1)) =$$

$$q^c(\frac{|Sp_{2(N-\ell)}(\mathbb{F}_q)|_{q'}}{|SO^+_{2(N-\ell)}(\mathbb{F}_q)|_{q'}} + \frac{|Sp_{2(N-\ell)}(\mathbb{F}_q)|_{q'}}{|SO^-_{2(N-\ell)}(\mathbb{F}_q)|_{q'}}).$$

Redistributing terms, this can be re-expressed as the product of

$$(\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO_{2\ell+1}(\mathbb{F}_q) \times SO^+_{2(N-\ell)}(\mathbb{F}_q)|_{q'}} + \frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO_{2\ell+1}(\mathbb{F}_q) \times SO^-_{2(N-\ell)}(\mathbb{F}_q)|_{q'}}).$$

$$dim(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix})$$

with the fraction

$$\frac{q^c \cdot \displaystyle\sum_{k=0}^{c} q^{2k(N-\ell+c)} \cdot \sum (-1)^r q^{\sum_{s=1}^r \lambda_{i_s} + \sum_{s=1}^{2(c-k)-r} \mu_{j_s}}}{q^{c^2} \cdot \displaystyle\prod_{i=1}^{c} (q^{2(N-\ell+i)}-1)} =$$

(61)
$$\frac{\displaystyle\sum_{k=0}^{c} q^{2k(N-\ell+c)} \cdot \sum (-1)^r q^{\sum_{s=1}^r \lambda_{i_s} + \sum_{s=1}^{2(c-k)-r} \mu_{j_s}}}{\displaystyle\prod_{i=0}^{c-1} (q^{2(N-\ell+c)} - q^{2i})}.$$

In particular, the top degree of q in both the numerator and demoniator of (61) is $2c(N-\ell+c)$. Finally, therefore (61) reduces as 1 (contributing the claimed main term), summed with

$$\frac{\displaystyle\sum_{k=0}^{c-1} q^{2k(N-\ell+c)} \cdot (\sum (-1)^r q^{\sum_{s=1}^r \lambda_{i_s} + \sum_{s=1}^{2(c-k)-r} \mu_{j_s}} - \binom{c}{k}_{q^2})}{\displaystyle\prod_{i=0}^{c-1} (q^{2(N-\ell+c)} - q^{2i})},$$

recalling

$$\sum_{0 \leq \ell_1 < \ldots \ell_{c-k} \leq c-1} q^{2(\ell_1+\cdots+\ell_{c-k})} = \binom{c}{k}_{q^2}.$$

$\square$

The previous terms arise from spillover from previous levele ℓ' corresponding to representations arising from semisimple elements at stage ℓ' with -1 an eigenvalue of multiplicity $2(\ell-\ell')$. Again, summing obtains a full sum of sqares of representations of $Sp_{2(N-(m-\ell))}(\mathbb{F}_q)$.

Summing these error terms then gives

$$(-1)^\ell \frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|Sp_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}} \frac{|Sp_{2m}(\mathbb{F}_q)|_{q'}}{Sp_{2\ell}(\mathbb{F}_q)\times Sp_{2(m-\ell)}(\mathbb{F}_q)|_{q'}}$$

$$\frac{|Sp_{2(m-\ell)}(\mathbb{F}_q)|}{|Sp_{2(m-\ell)}(\mathbb{F}_q)|_{q'}} q^{N-(m-\ell)} \frac{|Sp_{2\ell}(\mathbb{F}_q)|}{|Sp_{2\ell}(\mathbb{F}_q)|_{q'}}$$

which equals the ℓth term of (52). □

4.2. **Modifications for even-dimensional orthogonal spaces.** In the two cases of W with even dimension $dim(W)=2m$, similar arguments for Theorem 13 apply, with the following modifications:

Case 1: B **is totally split** In this case, the orders of the parabolic quotients of $O(W,B)=O^+_{2m}(\mathbb{F}_q)$ are

$$|O^+_{2m}(\mathbb{F}_q)/P_{B,k}| = \binom{m}{k}_q \cdot \prod_{j=m-k}^{m-1}(q^j+1)$$

for $k=0,\dots,m$, again writing $P_{B,k}$ for the parabolic subgroup of $O(W,B)$ with Levi subgroup $O^+_{2(m-k)}(\mathbb{F}_q)\times GL_k(\mathbb{F}_q)$. Again, the dimension of can be directly computed by taking the dimension of (3) and recursively computing. The analogue of (47) then is

$$dim(\omega[V\otimes W]^{\text{top}}) =$$
$$\sum_{i=0}^{m}(-1)^{m-i}q^{\binom{m-i}{2}}\binom{m}{i}_q \cdot \prod_{j=i}^{m-1}(q^j+1)\cdot q^{2iN}$$

The second step of processing the dimension of the top part of the oscillator representation, analogous to Proposition 15, is

(62)
$$dim(\omega[V\otimes W]^{\text{top}}) =$$
$$\sum_{\ell=0}^{m}(-1)^\ell q^{\ell(\ell-1)+(m-\ell)(m-\ell-1)}\binom{m}{\ell}_{q^2}\frac{(q^{m-\ell}+q^\ell)}{(q^m+1)}\prod_{j=0}^{m-\ell-1}(q^{2(N-j)}-1).$$

The significance of the coefficients in (62) (similar to (55)) is that for each ℓ,

$$\binom{m}{\ell}_{q^2} \frac{(q^{m-\ell}+q^{\ell})}{(q^m+1)} \prod_{i=0}^{m-\ell-1} (q^{2(N-j)}-1) =$$

$$\frac{1}{2}\left(\frac{|SO^+_{2m}(\mathbb{F}_q)|_{q'}\cdot|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO^+_{2\ell}(\mathbb{F}_q)|_{q'}\cdot|SO^+_{2(m-\ell)}(\mathbb{F}_q)|_{q'}\cdot|Sp_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}-\right.$$

$$\left.\frac{|SO^+_{2m}(\mathbb{F}_q)|_{q'}\cdot|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO^-_{2\ell}(\mathbb{F}_q)|_{q'}\cdot|SO^-_{2(m-\ell)}(\mathbb{F}_q)|_{q'}\cdot|Sp_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}\right).$$

Case 2: B is not totally split. In this case, the order of the parabolic quotients of $O(W,B)=O^-_{2m}(\mathbb{F}_q)$ are

$$|O^-_{2m}(\mathbb{F}_q)/P_{B,k}| = \binom{m-1}{k}_q \cdot \prod_{j=m-k+1}^{m} (q^j+1),$$

for $k=0,\dots,m-1$, again writing $P_{B,k}$ for the parabolic subgroup of $O(W,B)$ with Levi subgroup $O^-_{2(m-k)}(\mathbb{F}_q)\times GL_k(\mathbb{F}_q)$. The analogue of (47) then is

$$dim(\omega[V\otimes W]^{top}) =$$
$$\sum_{i=1}^{m}(-1)^{m-i}q^{\binom{m-i}{2}}\binom{m-1}{i}_q \cdot \prod_{j=i+1}^{m}(q^j+1)\cdot q^{2iN}$$

Then the second step re-expresses (52) as

(63)
$$dim(\omega[V\otimes W]^{top}) =$$
$$\sum_{\ell=0}^{m-1}(-1)^{\ell}q^{\ell(\ell-1)+(m-\ell)(m-\ell-1)}\binom{m}{\ell}_{q^2}\frac{(q^{m-\ell}-q^{\ell})}{(q^m-1)}\prod_{j=0}^{m-\ell-1}(q^{2(N-j)}-1)$$

Similarly as in the non-split case, the ℓth factor of (63) can be interpreted by

$$\binom{m}{\ell}_{q^2} \frac{(q^{m-\ell}-q^{\ell})}{(q^m-1)} \prod_{i=0}^{m-\ell-1} (q^{2(N-j)}-1) =$$

$$\frac{1}{2}\left(\frac{|SO^-_{2m}(\mathbb{F}_q)|_{q'} \cdot |Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO^-_{2(m-\ell)}(\mathbb{F}_q)|_{q'} \cdot |SO^+_{2\ell}(\mathbb{F}_q)|_{q'} \cdot |Sp_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}} - \right.$$

$$\left. \frac{|SO^-_{2m}(\mathbb{F}_q)|_{q'} \cdot |Sp_{2N}(\mathbb{F}_q)|_{q'}}{|SO^+_{2(m-\ell)}(\mathbb{F}_q)|_{q'} \cdot |SO^-_{2\ell}(\mathbb{F}_q)|_{q'} \cdot |Sp_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}\right).$$

4.3. **The case of the odd orthogonal stable range.** The same argument as in the previous subsections also work for a choice of reductive dual pair $(Sp(V), O(W, B))$ in the orthogonal stable range. The same calculation as in Proposition 14 also holds in this case.

Proposition 17. *Consider symplectic and orthogonal spaces V and (W, B) whose dimensions are in the orthogonal stable range. The dimension of the top part of $\omega[V \otimes W]$ is*

$$dim(\omega[V \otimes W]^{top}) =$$

$$\sum_{i=0}^{N} (-1)^{N-i} \cdot q^{\binom{N-i}{2}} \cdot \binom{N}{i}_q \cdot \prod_{j=i+1}^{N} (q^j+1) \cdot q^{i \cdot dim\ (W)}.$$

(Note again that nothing in the statement or proof of Proposition 17 uses the parity of the dimension of W.)

Again, we process this further:

Proposition 18. *Consider symplectic and orthogonal spaces V and (W, B) The dimension of the top part of the oscillator representation $\omega[V \otimes W]^{top}$ is*

$$dim(\omega[V \otimes W]^{top}) =$$

$$\sum_{\ell=0}^{N} (-1)^{\ell} \cdot q^{(N-\ell)^2+\ell(\ell-1)} \cdot \binom{N}{\ell}_{q^2} \cdot \prod_{i=0}^{N-\ell-1} (q^{2(m-i)}-1). \tag{64}$$

Denote the ℓth term of (64) by

$$X_\ell(N, m) := (-1)^{\ell} \cdot q^{(N-\ell)^2+\ell(\ell-1)} \cdot \binom{N}{\ell}_{q^2} \cdot \prod_{i=0}^{N-\ell-1} (q^{2(m-i)}-1). \tag{65}$$

In particular, note that

$$X_\ell(\ell, m) = (-1)^\ell \cdot q^{\ell(\ell-1)} \tag{66}$$

does not depend on m. Recalling that, for any rank r, the order of the symplectic and odd special orthogonal group is

$$|Sp_{2r}(\mathbb{F}_q)| = |SO_{2r+1}(\mathbb{F}_q)| = q^{r^2}\prod_{i=1}^{r}(q^{2i}-1),$$

we in fact find that
(67)

$$q^{(N-\ell)^2} \cdot \binom{N}{\ell}_{q^2} \cdot \prod_{i=0}^{N-\ell-1}(q^{2(m-i)}-1) = |Sp_{2(N-\ell)}(\mathbb{F}_q)| \cdot$$

$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|Sp_{2(N-\ell)}(\mathbb{F}_q) \times Sp_{2\ell}(\mathbb{F}_q)|_{q'}} \cdot \frac{|SO_{2m+1}(\mathbb{F}_q)|_{q'}}{|SO_{2(m-N+\ell)+1}(\mathbb{F}_q) \times SO_{2(N-\ell)+1}(\mathbb{F}_q)|_{q'}}.$$

In particular, using (66), we find that
(68)

$$X_\ell(N, m) = X_\ell(\ell, m-N+\ell) \cdot |Sp_{2(N-\ell)}(\mathbb{F}_q)| \cdot$$

$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|Sp_{2(N-\ell)}(\mathbb{F}_q) \times Sp_{2\ell}(\mathbb{F}_q)|_{q'}} \cdot \frac{|SO_{2m+1}(\mathbb{F}_q)|_{q'}}{|SO_{2(m-N+\ell)+1}(\mathbb{F}_q) \times SO_{2(N-\ell)+1}(\mathbb{F}_q)|_{q'}}.$$

It remains to produce the terms $X_\ell(N, m)$ from the summands on the right hand side of (46).

We will recursively compute

$$\sum_{\rho \in \widehat{Sp(V)}} \dim(\rho) \cdot \dim(\psi_V^{W,B}(\rho)) \tag{69}$$

using a series of N increasingly accurate approximations. For $\ell = 0, \dots, N$, the "level ℓ" approximation will be equal to

$$X_0(N, m) + X_1(N, m) + \cdots + X_\ell(N, m),$$

and will correctly count the terms

$$\dim(\rho) \cdot \dim(\psi_{(V,W,B)}(\rho)) \tag{70}$$

for ρ with Lusztig data consisting of a conjugacy class of a semisimple element $s \in SO_{2N+1}(\mathbb{F}_q)$ with eigenvalue -1 occuring with multiplicity less than or equal to 2ℓ.

Definition 19. *Say a representation ρ of a finite group of Lie type* occurs at level ℓ *if the conjugacy class (s) of a semisimple element in its Lusztig data has eigenvalue -1 with multiplicity 2ℓ.*

The level ℓ approximation of (69) will also generate some error terms that must be accounted for in approximations at later levels. At level $\ell = N$, we will have used all previous levels' errors, and correctly counted the contribution of every $\rho \in \widehat{Sp_{2N}(\mathbb{F}_q)}$.

First, we describe the level 0 approximation of (69). Consider irreducible representations $\rho_{[s,u]}$ where s is a conjugacy class of a semisimple element with no -1 eigenvalues. We then have

$$Z_{SO_{2m+1}(\mathbb{F}_q)}(\psi(s)) = (Z_{Sp_{2N}(\mathbb{F}_q)}(s))^D \times Sp_{2(m-N)}(\mathbb{F}_q), \tag{71}$$

$\psi(u) = \widetilde{u} \otimes 1$ (where 1 denotes the trivial representation of $Sp_{2(m-N)}(\mathbb{F}_q)$). Therefore,

$$\dim(\psi_V^{W,B)}(\rho)) = \frac{|Sp_{2m}(\mathbb{F}_q)|_{q'}}{|Sp_{2(m-N)}(\mathbb{F}_q) \times Sp_{2N}(\mathbb{F}_q)|_{q'}} \cdot \dim(\rho). \tag{72}$$

We define the level 0 approximation of (69), by imagining that (72) holds for every $\rho \in \widehat{Sp_{2N}(\mathbb{F}_q)}$, giving

$$\sum_{\rho \in \widehat{Sp_{2N}(\mathbb{F}_q)}} \frac{|Sp_{2m}(\mathbb{F}_q)|_{q'}}{|Sp_{2(m-N)}(\mathbb{F}_q) \times Sp_{2N}(\mathbb{F}_q)|_{q'}} \cdot \dim(\rho)^2.$$

We can see that this is

$$\frac{|Sp_{2m}(\mathbb{F}_q)|_{q'}}{|Sp_{2(m-N)}(\mathbb{F}_q) \times Sp_{2N}(\mathbb{F}_q)|_{q'}} \cdot |Sp_{2N}(\mathbb{F}_q)| = X_N(0, N).$$

The error of the level 0 approximation consits of two kinds of contributions for ρ occuring at level $1 \leq \ell \leq N$: the "true terms" (70), and the negative of the "faked terms added at level 0," which are precisely

$$-\frac{|Sp_{2m}(\mathbb{F}_q)|_{q'}}{|Sp_{2(m-N)}(\mathbb{F}_q) \times Sp_{2N}(\mathbb{F}_q)|_{q'}} \cdot \dim(\rho)^2. \tag{73}$$

Now let us consider the level ℓ approximation for $1 \leq \ell \leq N$. For a representation $\rho_{[s,u,\pm 1]}$ occuring at level ℓ, we have

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO_{2\ell}^{\pm}(\mathbb{F}_q),$$

where we may consider H as the centralizer of a semisimple element s', which is conjugate to the diagonalization of s restricted away from the 2ℓ coordinates with eigenvalues -1, in $SO_{2(N-\ell)+1}(\mathbb{F}_q)$:

$$H = Z_{SO_{2(N-\ell)+1}(\mathbb{F}_q)}(s'). \tag{74}$$

The centralizers of semisimple elements of $SO_{2(N-\ell)+1}(\mathbb{F}_q)$ which appear as (74) are precisely those with no factors of type D or ${}^2D$ (since s' by definition has no -1 eigenvalues). Write a unipotent representation u of $H \times SO^{\pm}_{2\ell}(\mathbb{F}_q)$ as

$$u = u_H \otimes u_{SO^{\pm}_{2\ell}(\mathbb{F}_q)},$$

for $u_H \in \widehat{H}_u$, $u_{SO^{\pm}_{2\ell}(\mathbb{F}_q)} \in \widehat{SO^{\pm}_{2\ell}(\mathbb{F}_q)}_u$. Then the sum of the true terms contributed by $\rho_{[s,u,1]}$ and $\rho_{[s,u,-1]}$ is the product of the "induction factor"

$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'} \cdot |Sp_{2m}(\mathbb{F}_q)|_{q'}}{|H \times SO^{\pm}_{2\ell}(\mathbb{F}_q)|_{q'} \cdot |H \times SO_{2(m-n+\ell)+1}(\mathbb{F}_q)|_{q'}} \dim(u_H)^2 \tag{75}$$

with

$$\frac{\dim(u_{SO^{\pm}_{2\ell}(\mathbb{F}_q)})}{2} \cdot (\dim(\psi^{+1}(u_{SO^{\pm}_{2\ell}(\mathbb{F}_q)}) + \psi^{-1}(u_{SO^{\pm}_{2\ell}(\mathbb{F}_q)})).$$

Now (75) can be re-written as

$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'} \cdot |Sp_{2m}(\mathbb{F}_q)|_{q'}}{|SO_{2(N-\ell)+1}(\mathbb{F}_q) \times SO^{\pm}_{2\ell}(\mathbb{F}_q)|_{q'} \cdot |SO_{2(N-\ell)+1}(\mathbb{F}_q) \times SO_{2(m-N+\ell)+1}(\mathbb{F}_q)|_{q'}} \tag{76}$$

$$\dim(\rho_{[s',u_H]})^2,$$

where $\rho_{[s',u_H]}$ denotes the irreducible representation of $SO_{2(N-\ell)+1}(\mathbb{F}_q)$ associated to the Lusztig classification data of $s \in SO_{2(N-\ell)+1}(\mathbb{F}_q)$, $u_H \in \widehat{Z_{SO_{2(N-\ell)+1}(\mathbb{F}_q)}(s)}_u$. We introduce "faked terms occuring at level ℓ" which consist of a product of (76) with $X_\ell(\ell, N)$.

Hence, by induction on N, this reduces (46) to checking the "highest level" of singularity, i.e. find terms matching the Nth term. The "true" new representations obtained at level N arise from Lusztig data

$$[\sigma^{\pm}_m, \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}, \pm 1],$$

recalling the description of $\sigma^{\pm}_m$ defined by (34) above, where $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ denotes a symbol specifying a unipotent representation of $SO^{\pm}_{2N}(\mathbb{F}_q)$.

Proposition 20. *The sum of the "true" level N terms*

$$\sum_{u \in \widehat{SO^+_{2N}(\mathbb{F}_q)}_u} dim(\rho_{[\sigma^+_N,u,\pm 1]}) \cdot dim(\psi_{V,W}(\rho_{[\sigma^+_N,u,\pm 1]}))+$$
$$\sum_{u \in \widehat{SO^-_{2N}(\mathbb{F}_q)}_u} dim(\rho_{[\sigma^-_N,u,\pm 1]}) \cdot dim(\psi_{V,W}(\rho_{[\sigma^-_N,u,\pm 1]}))$$

(where we sum over both central signs where left ambiguous) and every level ℓ error contribution for $1 \leq \ell \leq N-1$ to the Nth level

$$(-1)^{\ell+1} \cdot \left(\sum_{u \in \widehat{SO^+_{2(N-\ell)}(\mathbb{F}_q)}_u} dim(u)^2 + \sum_{u \in \widehat{SO^-_{2(N-\ell)}(\mathbb{F}_q)}_u} dim(u)^2 \right) \cdot X_N(\ell, m)$$

is equal to

$$X_N(N,m) = q^{N(N-1)}.$$

Proof. First, suppose $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ is a non-degenerate symbol of $SO^{\pm}_{2N}(\mathbb{F}_q)$. Let us write

$$x := N - m + \frac{a+b}{2}$$

Then the sum of dimensions

$$\dim_{SO_{2N+1}(\mathbb{F}_q)}(\binom{\lambda_1 < \cdots < \lambda_a < x}{\mu_1 < \cdots < \mu_b})+$$

$$\dim_{SO_{2N+1}(\mathbb{F}_q)}(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b < x})$$

is equal to the product of

$$(q^N \pm 1) \cdot \dim_{SO^{\pm}_{2N}(\mathbb{F}_q)}(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}) \tag{77}$$

with the factor

$$\frac{\prod_{i=1}^{a}(q^x - q^{\lambda_i}) \prod_{j=1}^{b}(q^x + q^{\mu_j}) + \prod_{i=1}^{a}(q^x + q^{\lambda_i}) \prod_{j=1}^{b}(q^x - q^{\mu_j})}{\prod_{i=1}^{(a+b)/2} (q^x - q^i)(q^x + q^i)}. \tag{78}$$

The top q-degrees of the numerator and denominator of (78) clearly match, and are equal to $x(a+b)$, suggesting a cancellation with the corresponding "level 0" error term. Our goal is to re-express the numerator of (78) in terms of the previous levels' error terms. To do this, we proceed inductively, replacing each error term's $X_N(\ell, m)$ factor with the induction hypothesis for $X_\ell(\ell, m)$, multiplied by (67). This will give

$$\frac{(-1)^N}{2} \cdot \left(\frac{\sum_{\rho \in \widehat{SO^+_{2N}(\mathbb{F}_q)}} dim(\rho)^2}{|SO^+_{2N}(\mathbb{F}_q)|_{q'}} + \frac{\sum_{\rho \in \widehat{SO^-_{2N}(\mathbb{F}_q)}} dim(\rho)^2}{|SO^-_{2N}(\mathbb{F}_q)|_{q'}} \right),$$

which is $(-1)^N q^{N(N-1)}$ since the q part of the order of $|SO^{\pm}_{2N}(\mathbb{F}_q)|$ is $q^{N(N-1)}$.

□

4.4. **Modifications for even orthogonal groups.** Now consider orthogonal spaces W of even dimension $\dim(W) = 2m$. First, note that there is no distinction in the dimension of the top part depending on whether the symmetric bilinear form on W is completely split or not. Our replacement for the calculation of the dimension of the top part is

Proposition 21. *Suppose $dim(W) = 2m$, $dim\ (V) = 2N$. Then*

$$\omega[V \otimes W]^{top} =$$

$$\sum_{\ell=0}^{N} q^{\ell^2+(N-\ell)(N-\ell-1)} \binom{N}{\ell}_{q^2} \cdot \prod_{i=1}^{N-\ell} (q^{2(m-N+\ell+i)} - 1) \tag{79}$$

Write $Y_\ell(N, m)$ for the ℓth term of (79), replacing (64).

Let us suppose that the symmetric bilinear form on W is completely split, i.e. $O(W, B) = O^+_{2m}(\mathbb{F}_q)$. (Again the non-split even case follows similarly.) Consider a semisimple element s of $SO_{2N+1}(\mathbb{F}_q)$ with 1 as an eigenvalue of total multiplicity $2\ell + 1$. Then its centralizer is of the form

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(s) = H \times SO_{2\ell+1}(\mathbb{F}_q), \tag{80}$$

for H now denoting a centralizer of a semisimple element with no 1 eigenvalues in an even special orthogonal group (of either parity) $SO^{\pm}_{2(N-\ell)}(\mathbb{F}_q)$. Let us write $H \subseteq SO^{\epsilon}_{2(N-\ell}(\mathbb{F}_q)$. Then

$$Z_{SO^+_{2m}(\mathbb{F}_q)}(\psi(s)) = H^D \times SO^{\epsilon}_{2(m-N+\ell)}(\mathbb{F}_q). \tag{81}$$

Hence, inductively, the level ℓ approximation in this case has terms equal to $Y_\ell(\ell, m)$, multiplied by "inductive factor" equal to half of the sum of

$$|SO^{\epsilon}_{2(N-\ell)}(\mathbb{F}_q)|\frac{|SO_{2N+1}(\mathbb{F}_q)|_{q'}}{|SO^{\epsilon}_{2(N-\ell)}(\mathbb{F}_q)\times SO_{2\ell+1}(\mathbb{F}_q)|_{q'}}$$

$$\frac{|SO^{+}_{2m}(\mathbb{F}_q)|_{q'}}{|SO^{\epsilon}_{2(N-\ell)}(\mathbb{F}_q)\times SO^{\epsilon}_{2(m-N+\ell)}(\mathbb{F}_q)|_{q'}} \tag{82}$$

over the two choices of $\epsilon=\pm$. Now (82) can be simplified as

$$q^{(N-\ell)(N-\ell-1)}\cdot\binom{N}{\ell}_{q^2}\cdot(q^m-1)\cdot\prod_{i=1}^{N-\ell-1}(q^{2(m-N+\ell+i)}-1)\cdot$$

$$(q^{N-\ell}+\epsilon 1)\cdot(q^{m-N+\ell}+\epsilon 1), \tag{83}$$

and we further have

$$\frac{1}{2}((q^{N-\ell}+1)\cdot(q^{m-N+\ell}+1)+(q^{N-\ell}-1)\cdot(q^{m-N+\ell}-1))=q^m+1.$$

Therefore, the average of (83) over the two choices of parity $\epsilon=\pm$ is

$$q^{(N-\ell)(N-\ell-1)}\cdot\binom{N}{\ell}_{q^2}\prod_{i=1}^{N-\ell}(q^{2(m-N+\ell+i)}-1). \tag{84}$$

Hence, considering (79), it remains to find

$$Y_\ell(\ell,m)=q^{\ell^2}. \tag{85}$$

Finding these terms proceeds exactly similarly to in the case of odd-dimensional W, since it is the q-part of the order of $SO_{2\ell+1}(\mathbb{F}_q)$.

In the case when the symmetric bilinear form on W is not completely split, i.e. $O(W,B)=O^-_{2m}(\mathbb{F}_q)$, if we have (80), then instead of (81), we have

$$Z_{SO^-_{2m}(\mathbb{F}_q)}(\psi(s))=H^D\times SO^{-\epsilon}_{2(m-N+\ell)}(\mathbb{F}_q)$$

and therefore the inductive factor (82) is replaced by

$$|SO^{\epsilon}_{2(N-\ell)}(\mathbb{F}_q)|\frac{|SO_{2N+1}(\mathbb{F}_q)|_{q'}}{|SO^{\epsilon}_{2(N-\ell)}(\mathbb{F}_q)\times SO_{2\ell+1}(\mathbb{F}_q)|_{q'}}$$

$$\frac{|SO^{-}_{2m}(\mathbb{F}_q)|_{q'}}{|SO^{\epsilon}_{2(N-\ell)}(\mathbb{F}_q)\times SO^{-\epsilon}_{2(m-N+\ell)}(\mathbb{F}_q)|_{q'}}, \tag{86}$$

which is simplified as

$$q^{(N-\ell)(N-\ell-1)} \cdot \binom{N}{\ell}_{q^2} \cdot (q^m+1) \cdot \prod_{i=1}^{N-\ell-1} (q^{2(m-N+\ell+i)}-1) \cdot$$

$$(q^{N-\ell}+\epsilon 1) \cdot (q^{m-N+\ell}-\epsilon 1).$$

Now we have

$$\frac{1}{2}((q^{N-\ell}+1) \cdot (q^{m-N+\ell}-1) + (q^{N-\ell}-1) \cdot (q^{m-N+\ell}+1)) = q^m - 1,$$

again simplifying the average of terms for different parities $\epsilon = \pm$ into (84), meaning that it remains to find the same terms (85).

5. An inductive argument

In this section, we conclude the statement of Theorem 2. First, we note that the toral characters of the eta and zeta correspondence are determined inductively, by examining the restriction of the oscillator representations to finite general linear groups. This confirms that the semisimple and cetnral sign data of $\eta^V_{W,B}(\rho)$ (resp. $\zeta^{W,B}_V(\rho)$) matches that of $\phi^V_{W,B}(\rho)$ (resp. $\psi^{W,B}_V(\rho)$). This is treated in Subsection 5.1.

It then remains in all cases to confirm the unipotent part of $\eta^V_{W,B}(\rho)$ (resp. $\zeta^{W,B}_V(\rho)$) matches that of $\phi^V_{W,B}(\rho)$ (resp. $\psi^{W,B}_V(\rho)$). First, we prove Proposition 3, and conclude that for $N >> n$, we have

$$dim(\eta^V_{W,B}(\rho)) = dim(\phi^V_{W,B}(\rho)) \tag{87}$$

(and similarly, for $n >> N$, we have

$$dim(\zeta^V_{W,B}(\rho)) = dim(\psi^V_{W,B}(\rho))). \tag{88}$$

We may view these dimensions as polynomials of q^N (resp. q^n). The results of [12] can be used to see that in either stable range, the idempotent in the endomorphism algebra picking out any summand of the eta (resp. zeta) correspondence does not depend on N (resp. n). Therefore, we can apply the description from [12] to see that (96) and (88) both hold for any choice of N, n in the symplectic and orthogonal stable ranges. Therefore, since each unipotent representation corresponding to a different symbol has a different dimension, we find that our claimed construction is the only possible choice. Hence, we conclude Theorem 2.

For the remainder of this section, we restrict attention to the case of the eta correspondence and $\phi^V_{W,B}$, since the case of the zeta correspondence and $\psi^{W,B}_V$ can be done completely similarly.

5.1. **Determining the semisimple and sign data.** The purpose of this subsection is to prove that the semisimple part (and sign data) of the $Sp(V)$-Lusztig classification data of the representation obtained by applying an eta correspondence $\eta^V_{W,B}(\rho)$ matches the semisimple part (and sign data) of our constructed representation $\phi^V_{W,B}(\rho)$ (and the similar statement for $\zeta_V^{W,B}$ and $\psi_V^{W,B}$).

Broadly, this can be concluded since, considering $GL_N(\mathbb{F}_q) \subseteq Sp(V)$, the restriction of the oscillator representation is

$$Res_{GL_N(\mathbb{F}_q)}(\omega[V]) \cong \epsilon(det) \otimes \mathbb{C}\mathbb{F}_q^N.$$

Now we also have the restriction

$$Res_{GL(V)}(\omega[V \otimes W]) \cong (Res_{GL(V)}(\omega[V]))^{\otimes W}$$

where $\otimes W$ denotes a degree $\dim(W)$ tensor product of oscillator representations $\omega[V]$. Since characters are matched exactly in the premutation representation factors, for example in the case of odd-dimensional W, we know the underlying toral character and the sign data. We now restrict attention to the case of comparing $\eta^V_{W,B}$ and $\phi^V_{W,B}$, for $(Sp(V), O(W,B))$ in the symplectic stable range. The case of comparing $\zeta_V^{W,B}$ and $\psi_V^{W,B}$ for the orthogonal stable range is similar.

Proposition 22. *Suppose $(Sp(V), O(W,B))$ is in the symplectic stable range. If $dim(W) = 2m+1$ is odd, for ρ an irreducible representation of $SO_{2m+1}(\mathbb{F}_q)$ arising from the conjugacy class of a semisimple element $s \in SO_{2m+1}(\mathbb{F}_q)$ and a unipotent representation u of its centralizer, then in the Lusztig classification data of $\eta^V_{W,B}((\pm 1) \otimes \rho)$, its semisimple part is*

$$(\phi^\pm(s)) = (s \oplus \sigma^\pm_{N-m}).$$

If $dim(W) = 2m$ is even, for ρ an irreducible representation of $O^\pm_{2m}(\mathbb{F}_q)$ arising from an $SO^\pm_{2m}(\mathbb{F}_q)$-representation corresponding to the conjugacy class of a semisimple element $s \in SO^\pm_{2m}(\mathbb{F}_q)$ and a unipotent representation u of its centralizer, then in the Jordan decomposition of $\eta^V_{W,B}(\rho)$, its semisimple part is

$$(\phi(s)) = (s \oplus I_{2(N-m)+1}).$$

Proof. Suppose $dim(W) = 2m+1$. Let us begin by considering

$$\underbrace{SO_2^\pm(\mathbb{F}_q) \times \cdots \times SO_2^\pm(\mathbb{F}_q)}_{m} \tag{89}$$

as a torus of $SO(W, B)$. Fix a character

$$\chi_{a_1} \otimes \cdots \otimes \chi_{a_m},$$

corresponding to $a_1, \dots, a_m \in \mu_{q\mp 1} \cong SO_2^{\pm}(\mathbb{F}_q)$. Consider the maximal parabolic subgroup with Levi (89) (i.e. the Borel subgroup) $B(W, B) \subseteq SO(W, B)$. Then, for an irreducible representation ρ with this character, i.e.

$$\rho \subseteq Ind_{O(W,B)}(\chi_{a_1} \otimes \cdots \otimes \chi_{a_m}),$$

we need to prove that $\eta_{W,B}(\rho)$ corresponds to a toral character

$$\chi_{a_1} \otimes \cdots \otimes \chi_{a_m} \otimes (\epsilon)^{\otimes N-m} \tag{90}$$

in $\underbrace{SO_2^{\pm}(\mathbb{F}_q) \times \cdots \times SO_2^{\pm}(\mathbb{F}_q)}_{N} \subseteq Sp_{2N}(\mathbb{F}_q)$ (considering ϵ as the quadratic character of $\mu_{q\mp 1} = SO_2^{\pm}(\mathbb{F}_q)$.

Consider the inclusion of the product of this torus with $Sp(V)$

$$\begin{aligned}\underbrace{SO_2^{\pm}(\mathbb{F}_q) \times \cdots \times SO_2^{\pm}(\mathbb{F}_q)}_{m} \times Sp(V) &\subseteq SO(W, B) \times Sp(V) \\ &\subseteq Sp(V \otimes W).\end{aligned} \tag{91}$$

Pick the ith factor $SO_2^{\pm}(\mathbb{F}_q)$ in (91), taking the inclusion

$$SO_2^{\pm}(\mathbb{F}_q) \times Sp(V) \subseteq Sp(V \otimes W) \tag{92}$$

Restricting $\omega[V \otimes W]$ along (92) gives a restriction

$$Res_{SO_2^{\pm}(\mathbb{F}_q)\times Sp(V)}(\omega[V \otimes \mathbb{F}_q^2]) \otimes \mathbb{C}^{q^{(2m-1)N}} \tag{93}$$

considering $\mathbb{F}_q^2$ with the split and non-split symmetric bilinear form, respectively, (and taking the trivial action on $\mathbb{C}^{q^{(2m-1)N}}$. Recalling the results of [12], in each factor (93), it decomposes as a $SO_2^{\pm}(\mathbb{F}_q) \times Sp(V)$-representation pairing every χ_{a_i}-type $SO_2^{\pm}(\mathbb{F}_q)$-representation with a representation $Sp(V)$ in the induction

$$Ind^{SO_2^{\pm}(\mathbb{F}_q)}(\chi_{a_i}),$$

considering $SO_2^{\pm}(\mathbb{F}_q)$ as a factor of a torus in $Sp(V)$ Since this holds for every i, it also holds in the restriction of $\omega[V \otimes W]$ along (91): in

$$Res_{SO_2^{\pm}(\mathbb{F}_q)\times\cdots\times SO_2^{\pm}(\mathbb{F}_q)\times Sp(V)}(\omega[V \otimes W]),$$

the character $\chi_{a_1} \otimes \cdots \otimes \chi_{a_m}$ as a representation of $SO_2^{\pm}(\mathbb{F}_q) \times \cdots \times SO_2^{\pm}(\mathbb{F}_q)$ is paired with a representation of $Sp(V)$ in that character's induction, viewing the copies of $SO_2^{\pm}(\mathbb{F}_q)$'s as blocks in a torus of $Sp(V)$.

The remaining factors of ϵ in (90) corresponding to the remaining $N-m$ factors in a torus of $Sp(V)$ arise since the restriction of $Sp(V)$ to a representation of

$$GL_{N-m}(\mathbb{F}_q) \subseteq GL(\Lambda) \subseteq Sp(V)$$

is $\epsilon(det)$ tensored with a permutation representation.

A similar argument applies to both even-dimensional cases.

□

5.2. **The proof of Propostion 3.** The purpose of thie subsection is to prove Propostion 3 by induction. Again, we restrict attention to the case of $N >> n$, since the case of $n >> N$ is completely similar.

First, we begin by observing the following

Lemma 23. *Fix n, and consider $N >> n$. Every irreducible representation of $Sp_{2N}(\mathbb{F}_q)$ with N-rank n is constructed by applying $\phi^V_{W,B}$ to an irreducible representation of $O(W,B)$ for n-dimensional orthogonal space (W,B).*

Proof. First suppose $dim(W) = n = 2m+1$. Writing out the definition of $\phi_{W,B}$, we find that the statement is equivalent to the claim that every irreducible representation of $Sp_{2N}(\mathbb{F}_q)$ of N-rank $2m+1$ arises from a conjugacy class (s) of a semisimple element of $SO_{2N+1}(\mathbb{F}_q)$ with centralizer

$$\prod_{i=1}^{r} U^+_{j_i}(\mathbb{F}_q) \times \prod_{i=1}^{t} U^-_{k_i}(\mathbb{F}_q) \times SO_{2\ell+1}(\mathbb{F}_q) \times SO^{\pm}_{2(N-m+p)}(\mathbb{F}_q) \tag{94}$$

and a unipotent representation u, whose $SO^{\pm}_{2(N-m+p)}(\mathbb{F}_q)$-representation tensor factor $u_{SO^{\pm}_{2(N-m+p)}}$ corresponds to a symbol

$$\begin{pmatrix} \alpha_1 < \cdots < \alpha_a \\ \beta_1 < \cdots < \beta_b \end{pmatrix}$$

such that either $\alpha_a = N - m + p + \frac{a+b-1}{2}$ or $\beta_b = N - m + p + \frac{a+b-1}{2}$.

First note the prime to q part of the group orders

$$|SO_{2N+1}(\mathbb{F}_q)|_{q'} = \prod_{i=1}^{N}(q^{2i}-1), \quad |SO_{2\ell+1}(\mathbb{F}_q)|_{q'} = \prod_{i=1}^{\ell}(q^{2i}-1),$$

for the groups of type B

$$|SO^{\pm}_{2(N-m+p)}(\mathbb{F}_q)|_{q'} = (q^{N-m+p} \mp 1) \prod_{i=1}^{N-m+p-1} (q^{2i}-1),$$

for the froup of type D, and

$$|U^{+}_{j_i}(\mathbb{F}_q)|_{q'} = \prod_{u=1}^{j_i}(q^u-1) \text{ for } i = 1, \ldots, r$$

$$|U^{-}_{k_i}(\mathbb{F}_q)|_{q'} = \prod_{u=1}^{k_i}(q^u-(-1)^u) \text{ for } i = 1, \ldots, t.$$

Therefore, the total top degree of q in the quotient of prime to q parts of group orders (30) is

$$\sum_{i=1}^{N} 2i - (\sum_{i=1}^{\ell} 2i + (N-m+p) + \sum_{i=1}^{N-m+p-1} 2i + \sum_{i=1}^{r}\sum_{u=1}^{j_i} u + \sum_{i=1}^{t}\sum_{u=1}^{k_i} u),$$

which can be simplified as

$$(95)\quad N(N+1)-(\ell(\ell+1)+(N-m+p)^2+\sum_{i=1}^{r}\frac{j_i(j_i+1)}{2}+\sum_{i=1}^{t}\frac{k_i(k_i+1)}{2}).$$

The terms not involving N (arising from $SO_{2\ell+1}(\mathbb{F}_q)$ and the unitary groups) do not affect the N-rank of the final $Sp_{2N}(\mathbb{F}_q)$-representation, since

$$\ell + \sum_{i=1}^{r} j_i + \sum_{i=1}^{t} k_i \le m < \frac{N}{2}.$$

The remaining terms of (95) are

$$N \cdot (1-2(m-p)) + (m-p)^2.$$

Therefore, no smaller factor of type D can occur than those allowed by (94).

The condition on the symbol arises since otherwise the factor (24) contributes additional copies of N, unless it is cancelled by the denominator of (18), which can only occur if the rank $N-m+p+(a+b-1)/2$ occurs as an entry in the symbol itself.

A similar argument applies to even cases of $n = dim(W)$.

□

The case of Proposition 3 for $N >> n$ then follows by induction.

Proof of Proposition 3, part (1). First we consider the case of W with odd dimensions, and proceed by induction. Suppose for every $m' < m$, we know that the disjoint union of the images of the two eta correspondences $\eta^V_{W,B}$ such that $dim(W) = 2m' + 1$ is exactly the set of all irreducible representations of $Sp_{2N}(\mathbb{F}_q)$ with N-rank $2m' + 1$, for $N >> m$.

Suppose (W, B) forms an orthogonal space of dimension $2m+1$. By the definition of $\eta^V_{W,B}$, the sum

$$\bigoplus_{\rho \in \widehat{O(W,B)}} \rho \otimes \eta^V_{W,B}(\rho)$$

is the top summand of $\omega[V \otimes W]$. In particular, its dimension less than or equal to

$$dim(\omega) = q^{(2m+1)N},$$

so all $Sp_{2N}(\mathbb{F}_q)$-representations of higher N-rank cannot occur in the image of $\eta^V_{W,B}$. Additionally, the images of the different η-correspondences are all disjoint. Therefore, by the induction hypothesis, no irreducible representations of lesser odd N-rank may occur in the image of $\eta_{W,B}$.

To conclude Theorem 2, note that the pairing $\phi_{W,B}$ obtains the maximal possible dimension

$$dim(\bigoplus_{\rho \in O(W,B)} \rho \otimes \phi_{W,B}(\rho)).$$

If a representation of $O(W, B)$ were paired by $\eta_{W,B}$ with a $Sp_{2N}(\mathbb{F}_q)$-representation of lesser N-rank, it would waste dimensions in

$$dim(\bigoplus_{\rho \in \widehat{O(W,B)}} \rho \otimes \eta_{W,B}(\rho)),$$

which would be impossible to get back, by Theorem 13, since no other representations of N-rank $2m + 1$ exist by Proposition 23. $\square$

5.3. **Concluding Theorem 2.** In this subsection, we first conclude that for every $\rho \in \widehat{O(W, B)}$,

$$dim(\eta^V_{W,B}(\rho)) = dim(\phi^V_{W,B}(\rho)). \tag{96}$$

for V of dimension $2N$ and W of dimension n, with $N >> n$. In our construction, for a fixed choice of (W, B) and ρ, for every $N \geq n$, the dimension of our constructed representation $\phi^V_{W,B}(\rho)$ for $dim(V) = 2N$ can be expressed as a polynomial of q^N (see (97) below). On the other hand, we recall the results of [12], which allow us to consider the eta correspondence on the level of idempotents. By the stable description of the endomorphism algebra of an oscillator representation given in [12], we also know the dimensions of $\eta^V_{W,B}(\rho)$ for a fixed ρ and (W,B) must be polynomial in q^N. Therefore, 96 must in fact hold for every $N \geq n$. Combining this with the results of the previous subsection, we conclude that

$$\eta^V_{W,B}(\rho) = \phi^V_{W,B}(\rho),$$

since all symbols have different dimensions.

First, combining Proposition 22, Proposition 3, and Theorem 13 allows us to conclude (96) for $N >> n$: Our construction $\phi^V_{W,B}$ satisfies the condition that, for representations $\rho, \pi \in \widehat{O(W,B)}$ such that $dim(\rho) < dim(\pi)$, we have

$$dim(\phi^V_{W,B}(\rho)) < dim(\phi^V_{W,B}(\pi)).$$

Therefore, $\phi^V_{W,B}$ is an injective correspondence from which maximizes the dimension sum

$$\sum_{\rho \in \widehat{O(W,B)}} dim(\rho) \cdot dim(\phi_{W,B}(\rho)),$$

which we know numerically matches with

$$\sum_{\rho \in \widehat{O(W,B)}} dim(\rho) \cdot dim(\eta_{W,B}(\rho))$$

by Theorem 13. Therefore, for $N >> n$, we must have that the dimensions of $\eta^V_{W,B}(\rho)$ match the dimensions of $\phi^V_{W,B}(\rho)$. It remains to prove that this holds for every $N \geq n$, from which we can conclude that the unipotent parts of their Lusztig classification data agree in general. We do this now, concluding Theorem 2, art (1). The proof of Part (2) is similar, using the analogue of Proposition 22 for the zeta correspondence, and the orthogonal stable cases of Proposition 3, and Theorem 13.

Proof of Theorem 2, part (1). We restrict attention to the case of W odd dimensional. The even dimensional case proceeds similarly. Fix an orthogonal space (W,B) of dimesion $n = 2m+1$, and fix an irreducible representation ρ of $O(W,B)$. Considering $O(W,B) = \mathbb{Z}/2 \times SO_{2m+1}(\mathbb{F}_q)$, write ρ as a tensor product

$$\rho = (\alpha) \otimes \rho_{(s),u}$$

for α denoting a sign specifying a $\mathbb{Z}/2$-action, and $[(s), u]$ denoting the $SO_{2m+1}(\mathbb{F}_q)$-Lusztig classification data corresponding to the restriction of ρ to $SO_{2m+1}(\mathbb{F}_q)$. Let us consider the symbol $\binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ associated to the factor of u corresponding to the -1 eigenvalues of s, as in the construction of $\phi^V_{W,B}(\rho)$. Recall the notation $N'_\rho = N - m + \frac{a+b-1}{2}$. For every V of dimension $2N$ with $N \geq 2m+1$, the dimension of $\phi^V_{W,B}(\rho)$ is then equal to

$$\frac{dim(\rho)\cdot \prod_{i=N'_\rho+1}^{N}(q^{2i}-1)\cdot \prod_{i=1}^{a}(q^{N'_\rho}+\alpha\cdot q^{\lambda_i})\cdot \prod_{i=1}^{b}(q^{N'_\rho}-\alpha\cdot q^{\mu_i})}{2\cdot q^{(a+b-1)(a+b+1)/4}\cdot |SO_{2m+1}(\mathbb{F}_q)|_{q'}}., \tag{97}$$

which is a polynomial expression applied to q^N.

On the other hand, let us consider the values of $dim(\eta^V_{W,B}(\rho))$ for V of dimension $2N$ as a function of N. We recall the description of endomorphism algebra of $\omega[V\otimes W]$ over $Sp(V)$ given in Section 2 of [12]: Considering the Schrödinger model of the oscillator representation, there is an isomorphism between the endomorphism algebra and the space of $Sp(V)$-fixed points in $\mathbb{C}(V\otimes W)$

$$(End_{Sp(V)}(\omega[V\otimes W]),\circ)\cong(\mathbb{C}(V\otimes W)^{Sp(V)},\star), \tag{98}$$

where $\star$ is defined by

$$(v_1\otimes w_1)\star(v_2\otimes w_2)=\psi(\frac{S(v_1,v_2)\cdot B(w_1,w_2)}{2})\cdot(v_1\otimes w_1+v_2\otimes w_2)$$

(here ψ denotes the non-trivial additive character corresponding to $1\in\mathbb{F}_q^\times$, under our identification of $\mathbb{F}_q$ with its Pontrjagin dual). To consider the eta correspondence $\eta^V_{W,B}$, in [12] we consider $\omega[V\otimes W]$ as a degree $dim(W)$ tensor product of oscillator representations $\omega_{a_1}[V]\otimes\dots\otimes\omega_{a_n}[V]$ (considering B to be equivalent to the symmetric bilinear form corresponding to a diagonal matrix with entries $a_1,\dots,a_n$). This essential corresponds to writing out $V\otimes W$ as a direct sum of n copies of V. Therefore we also view (98) as describing

$$End_{Sp(V)}(\omega_{a_1}[V]\otimes\dots\otimes\omega_{a_n}[V])). \tag{99}$$

We note that as long as $N\geq n$, the right hand side of (98), as an algebra, is stable and does not depend on N. Therefore the same linear combination of n-tuples of V vectors in the right hand side of (98) describes the idempotent with image $\eta^V_{W,B}(\rho)$ for any choice of $N\geq n$. In particular, the dimension of $\eta^V_{W,B}(\rho)$ (expressible as the trace of this idempotent in (99) for V of dimension $2N$, is also polynomial in q^N, since, considering one tensor factor at a time, trace of a linear combination of V-vectors (v) as an endomorphism of $\omega_{a_i}[V]$ is computed according to

$$tr((v)) = \begin{cases} 0 \text{ if } v \neq 0 \\ q^N \text{ if } v = 0 \end{cases} .$$

Hence, since this polynomial agrees with the polynomial (97) for infinitely many values i.e., when applied to q^N for N large enough, they must in fact always agree. Therefore, we obtain (96) for every $N \geq n$.

Combining this with the results of the previous subsection which confirm that the semisimple and sign parts of the Lusztig classification data for $\eta^V_{W,B}(\rho)$ and $\phi^V_{W,B}(\rho)$ always match, we obtain that the unipotent parts must match also (since every symbol has a different dimension). Therefore, we obtain that

$$\eta^V_{W,B}(\rho) = \phi^V_{W,B}(\rho),$$

by Lusztig's classification of irreducible representations, as claimed. $\square$

6. An Explicit Example: The case of $SL_2(\mathbb{F}_q)$

Consider, for example, the case of $N = 1$ (i.e. $Sp_2(\mathbb{F}_q) = SL_2(\mathbb{F}_q)$), for $n = 2m + 1$. The oscillator representation $\omega[\mathbb{F}_q^2]$ is q-dimensional, and decomposes along the central $\mathbb{Z}/2$-action into pieces

$$\omega[\mathbb{F}_q^2] = \omega^+[\mathbb{F}_q^2] \oplus \omega^-[\mathbb{F}_q^2]$$

of dimension $(q + 1)/2$, $(q - 1)/2$, respectively. Applying Lemma 17 and Proposition 18 above gives that the top part of $\omega[\mathbb{F}_q^2 \otimes W]$ has dimension

$$q^{2m+1} - (q + 1) = q \cdot (q^{2m} - 1) - 1.$$

Consider representations ρ of $SL_2(\mathbb{F}_q)$. The Lusztig classification consists of the data of a conjugacy class of a semisimple element

$$s \in SO_3(\mathbb{F}_q) = SL_2(\mathbb{F}_q)^D,$$

a unipotent representation u of $Z_{SO_3(\mathbb{F}_q)}(u)$, and an additional choice of sign when s has -1 eigenvalues. There are $(q-3)/2$, resp. $(q-1)/2$, conjugacy classes (s) (corresponding to having eigenvalues $\{\lambda, \lambda^{-1}\} \subseteq \mu_{q-1} \smallsetminus \{\pm 1\}$, resp. $\mu_{q+1} \smallsetminus \{\pm 1\}$) with

$$Z_{SO_3(\mathbb{F}_q)}(s) = U_1^+(\mathbb{F}_q), \text{ resp. } U_1^-(\mathbb{F}_q),$$

whose only unipotent representation is trivial, and whose corresponding $SL_2(\mathbb{F}_q)$-representation then has dimension

$$\dim(\rho_{[s,1]}) = q+1, \text{ resp. } q-1.$$

There is a single choice of semisimple conjugacy class $(\sigma_1^\pm)$ each with centralizer $Z_{SO_3(\mathbb{F}_q)}(s) = SO_2^\pm(\mathbb{F}_q)$ (corresponding to having eigenvalue -1 with multiplicity two, with sign determined by the placement of the last eigenvalue 1, depending on the presentation of the form defining $SO_3(\mathbb{F}_q)$), which again has only the trivial unipotent representation, giving representations of dimension

$$\dim(\rho_{[\sigma_1^\pm,1,+1]}) = \dim(\rho_{[\sigma_1^\pm,1,-1]}) = (q\pm 1)/2.$$

Finally, only $(s) = (I)$ has centralizer the full $SO_3(\mathbb{F}_q)$, which has two non-trivial unipotent representations corresponding to symbols $\binom{1}{\emptyset}$, $\binom{0<1}{1}$, of dimesions 1 and q, respectively.

We call the s with no -1 eigenvalues the "level 0" choices. Call the other choices of s the "level 1" choices. For the level 0 choices of s, we have that

$$Z_{Sp_{2m}(\mathbb{F}_q)}(\psi(s)) = Z_{SO_3}(\mathbb{F}_q)^D \times Sp_{2(m-1)}(\mathbb{F}_q),$$

with $\psi(u)$ defined as the representation corresponding to u of the first factor, tensored with the trivial representation of $Sp_{2(m-1)}(\mathbb{F}_q)$. We assign the central sign describing the action of of according to the discriminant of the form on W and the quadratic character This fully describes $\zeta(\rho_{[s,1]})$ for the level 0 s, and we find

$$\dim(\zeta(\rho_{[s,1]})) = \dim(\rho_{[s,1]}) \cdot \frac{|SO_{2m+1}(\mathbb{F}_q)|_{q'}}{|Sp_{2(m-1)}(\mathbb{F}_q)|_{q'} \cdot |SO_3(\mathbb{F}_q)|_{q'}} =$$

$$\dim(\rho_{[s,1]}) \cdot \frac{q^{2m}-1}{q^2-1}$$

For both level 1 choices of s (in this case, precisely $(s) = (\sigma_1^\pm)$), we have

$$Z_{Sp_{2m}(\mathbb{F}_q)}(\psi(s)) = Sp_{2m}(\mathbb{F}_q),$$

and we need to assign two choices of unipotent representations in both cases of the sign. We alter the trivial representation of $SO_2^+(\mathbb{F}_q)$ (corresponding to the symbol $\binom{1}{0}$ of rank 1, type D) by adjoining the coordinate m to obtain the two choices of symbols

$$\binom{1<m}{0}, \quad \binom{1}{0<m},$$

describing unipotent representations of $SO_{2m+1}(\mathbb{F}_q)$. Similarly, we alter the trivial representation of $SO_2^-(\mathbb{F}_q)$ (corresponding to the symbol $\binom{0<1}{\emptyset}$ of rank 1, type ${}^2D$) by adjoining the cooedinate m to obtain the two choices of symbols

$$\begin{pmatrix} 0<1<m \\ \emptyset \end{pmatrix}, \quad \begin{pmatrix} 0<1 \\ m \end{pmatrix}.$$

Therefore, for level 1 representations of $SL_2(\mathbb{F}_q)$, we have

$$\dim(\zeta(\rho_{[\sigma_1^+,1,\pm 1]})) = \frac{(q^m \pm 1)(q^m \mp q)}{2(q-1)}$$

$$\dim(\zeta(\rho_{[\sigma_1^-,1,\pm 1]})) = \frac{(q^m \pm 1)(q^m \pm q)}{2(q+1)}.$$

We may now apply our general combinatorial argument, but this case is small enough to verify directly. Indeed, we can explicitly write out

$$\sum_{\rho \in \widehat{SL_2(\mathbb{F}_q)}} \dim(\rho) \cdot \dim(\zeta(\rho)) =$$

$$\frac{q-3}{2} \cdot \frac{(q+1)^2(q^{2m}-1)}{q^2-1} + \frac{q-1}{2} \cdot \frac{(q-1)^2(q^{2m}-1)}{q^2-1} + \frac{(1+q^2)(q^{2m}-1)}{q^2-1}$$

$$+ \frac{(q+1)(q^m+1)(q^m-q)}{4(q-1)} + \frac{(q+1)(q^m-1)(q^m+q)}{4(q-1)}$$

$$+ \frac{(q-1)(q^m+1)(q^m+q)}{4(q+1)} + \frac{(q-1)(q^m+1)(q^m+q)}{4(q+1)}$$

(the first row corresponds to the level 0 ρ, the second row corresponds to ρ from $(s) = (\sigma_1^+)$, and the third row corresponds to ρ from $(s) = (\sigma_1^-)$), and verify that it equals $q(q^{2m}-1)-1$.

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