

HOWE DUALITY OVER FINITE FIELDS I: THE TWO STABLE RANGES

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ABSTRACT. This is the first in a series of papers on type I Howe duality for finite fields, concerning the restriction of an oscillator representation of a symplectic group to the product of a symplectic subgroup and an orthogonal subgroup which are each other's centralizers. The goal of the series is to describe this restriction completely explicitly. Applications (described in the third paper of the series) include demonstrating that the tensor pairs previously calculated by S.-Y. Pan to occur with non-zero multiplicity in fact occur with multiplicity 1, proving the type C case of the Gurevich-Howe rank conjecture, and giving a recursive formula for the characters of cuspidal unipotent representations. In this first paper, we construct the correspondence in the two so-called stable ranges, where the rank of one of the subgroup factors is large enough with respect to the other.

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1. INTRODUCTION

This is the first paper in a series dedicated to the problem of Howe correspondence for finite fields. The question was first introduced by R.

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Howe in [16] and was discussed in the papers [13, 14] by S. Gurevich and Howe, bringing to light the finite field case of the Howe duality conjecture [19] for locally compact fields. In rough terms, the correspondence concerns the decomposition of the restriction of the oscillator representation of a finite symplectic group to a product of a symplectic and an orthogonal group which are each other's centralizers. More broadly, this can be discussed for general reductive dual pairs, consisting of any pair of subgroups which are each other's centralizers in a finite symplectic group.

Early progress on this program for the case of finite fields was made by J. Adams and A. Moy [1], who described the behavior of unipotent cuspidal representations under the Howe correspondence between symplectic and even orthogonal groups, applying previous results of S. Kudla [30] for the theta correspondence on local fields, which used the Jacquet functor. Another important advance was the paper [3] by A.-M. Aubert, J. Michel, and R. Rouquier, who conjectured the behavior of the Howe correspondence on general *unipotent representations* (i.e. those appearing in the Deligne-Lusztig induction of a trivial character), and proved their conjecture for unitary and general linear groups (i.e. the “type II case” of Howe duality). Their conjecture for *type I dual pairs*, i.e. symplectic and orthogonal groups, was proved by S.-Y. Pan [37] using the concept of a *uniform projection* (a character is called uniform if it is a linear combination of virtual Deligne-Lusztig characters). This geometric approach was also used by D. Liu and Z. Wang in [33] to extend the results of Adams and Moy to the case of reductive dual pairs involving odd orthogonal groups. Pan [38] went further and, by proving suitable compatibility results, was able to determine which pairs of irreducible representations occur in the type I Howe duality correspondence with non-zero multiplicity.

In the present series of papers, we approach this problem using a different method. We study the Howe duality correspondence directly and explicitly by examining the endomorphism algebras of tensor products of oscillator representations over finite fields. In these terms, we describe explicitly the so-called η -*correspondence* defined by Gurevich and Howe [13, 14], and also define and similarly describe a complementary “ ζ -correspondence.” Having a hands-on description of the irreducible representations occurring in the Howe duality correspondence gives us more precise information. For example, as a payoff, we are able to explicitly demonstrate that the pairs identified by Pan [38] always occur with multiplicity 1. Using the organization of the oscillator representation's decomposition in terms of the eta and zeta correspondences,

we obtain a recursive formula for the characters of the unipotent cuspidal representations, which were still quite mysterious up to this point. We also prove the type C case of the Gurevich-Howe rank conjecture (Conjecture 0.3.8 of [14]), the type A case of which was proved R. M. Guralnick, M. Larsen, and P. H. Tiep [12], and the type B and D cases of which were proved by Larsen and Tiep in [32]. These applications will be discussed in the third paper of the series.

The present paper constructs the correspondence in the two so-called “stable ranges,” in which all irreducible representations of either the symplectic or the orthogonal group occur in the restricted oscillator representation. We precisely construct the summands. (In the symplectic stable range, the summands were previously described by F. Montealegre-Mora and D. Gross [35] in a less explicit way.) In the next paper, we will make these summand explicit using the Lusztig classification of representations of finite groups of Lie type, (extending the results of S.-Y. Pan [37, 38]). In the last paper of the series, we extend that descriptions beyond the stable ranges, to cover all cases of the Howe correspondence.

For a symplectic vector space $\mathbf{V}$ over a field k , given a non-trivial additive character

$$\psi : k \rightarrow \mathbb{C},$$

one may consider the associated *oscillator* or *Weil-Shale* representation $\omega_\psi[\mathbf{V}]$ of the symplectic group $\mathrm{Sp}(\mathbf{V})$ (when, for example, $k = \mathbb{C}$ or $\mathbb{F}_q$), or the metaplectic group $\mathrm{Mp}(\mathbf{V})$ (when $k = \mathbb{R}$ or $\mathbb{Q}_p$). In the case of $k = \mathbb{F}_q$, we fix an identification of the non-trivial additive characters of $\mathbb{F}_q$ with the elements of $\mathbb{F}_q^\times$, writing ψ_a for the character corresponding to $a \in \mathbb{F}_q^\times$. We write $\omega_a[\mathbf{V}]$ for the oscillator representation associated to ψ_a . We also have $\omega_a[\mathbf{V}] \cong \omega_{ab^2}[\mathbf{V}]$, so we may index only by $a \in \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$. Let us write $\omega[\mathbf{V}] = \omega_1[\mathbf{V}]$.

The question of Howe duality asks about the decomposition of the restriction of $\omega[\mathbf{V}]$ to the product of a reductive dual pair of subgroups of $\mathrm{Sp}(\mathbf{V})$ or $\mathrm{Mp}(\mathbf{V})$. Denote this reductive dual pair by G, H . Consider the decomposition

$$(1) \quad \mathrm{Res}_{G \times H}(\omega) = \bigoplus_{\rho \in \widehat{G}, \pi \in \widehat{H}} \mu(\rho, \pi) \cdot (\rho \otimes \pi)$$

for some multiplicities

$$\mu : \widehat{G} \times \widehat{H} \rightarrow \mathbb{N}_0.$$

(In this paper, for a group G , $\widehat{G}$ denotes the set of irreducible unitary G -representations.)

For fields $k = \mathbb{C}, \mathbb{R}$, or $\mathbb{Q}_p$, there in fact exist subsets $\mathcal{S}_G$ and $\mathcal{S}_H$ of the irreducible representations of G and H , and a bijective correspondence (called the *theta correspondence*)

$$\theta : \mathcal{S}_G \rightarrow \mathcal{S}_H,$$

such that

$$(2) \quad \text{Res}_{G \times H}(\omega) = \int_{\rho \in \mathcal{S}_G} \rho \otimes \theta(\rho) = \int_{\pi \in \mathcal{S}_H} \theta^{-1}(\pi) \otimes \pi.$$

This result, referred to as *Howe duality*, is the culmination of a long history of work, (see, for example, R. Howe, [19] for the case of $k = \mathbb{R}$ and W.T. Gan, S. Takeda, [9] for $k = \mathbb{Q}_p$), and has many deep applications in arithmetic geometry, number theory, and representation theory. It is an interesting and difficult question to find its appropriate analogue in the case when k is a finite field. This case, where $k = \mathbb{F}_q$ (for q a power of an odd prime), is the main topic of the present work.

Due to the large isotropic subspaces of symmetric bilinear forms over a finite field, a bijective correspondence θ giving a decomposition (2) is not the correct formalism. Still, one can study the decomposition of $\text{Res}_{\text{Sp}(V) \times \text{O}(W, B)}(\omega[V \otimes W])$, which exhibits interesting relationships between irreducible representations of $\text{Sp}(V)$ and $\text{O}(W, B)$, and many meaningful patterns regarding which pairs $\rho \otimes \pi$ appear in the decomposition (1) have been constructed and studied (see for example [2, 3, 7, 13, 14, 37, 38]).

As in the context of an infinite locally compact field, over a finite field, we may still choose to either consider the decomposition of the restricted oscillator representation to be indexed by the irreducible representations of $\text{Sp}(V)$

$$(3) \quad \text{Res}_{\text{Sp}(V) \times \text{O}(W, B)}(\omega[W \otimes V]) = \bigoplus_{\rho \in \widehat{\text{Sp}(V)}} \rho \otimes \Psi(\rho)$$

or by the irreducible representations of $\text{O}(W)$

$$(4) \quad \text{Res}_{\text{Sp}(V) \times \text{O}(W, B)}(\omega[W \otimes V]) = \bigoplus_{\rho \in \widehat{\text{O}(W)}} \Phi(\rho) \otimes \rho,$$

for some (not necessarily irreducible or non-zero) $\text{Sp}(V)$ - and $\text{O}(W, B)$ -representations $\Phi(\rho)$ and $\Psi(\rho)$. We investigate (3) or (4) by examining the $\text{Sp}(V)$ - or $\text{O}(W, B)$ -equivariant endomorphism algebras

$$\text{End}_{\text{Sp}(V)}(\omega[V \otimes W]), \quad \text{End}_{\text{O}(W, B)}(\omega[V \otimes W]),$$

respectively (note that in this notation, the choice of B is assumed to specify the restriction of $\omega[V \otimes W]$ to $\mathrm{Sp}(V)$, but is hidden in the notation). It intuitively makes sense to consider (3) (resp. (4)) if in the reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ the symplectic group is “much larger” (resp. “much smaller”) than the orthogonal group.

The case of reductive dual pairs $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ where V has a large enough dimension compared to W (specifically, where $\dim(V)/2 \geq \dim(W)$), which we will refer to as the *symplectic stable case*, has been especially studied. S. Gurevich and R. Howe [13] proved that in this case, every irreducible representation ρ of $\mathrm{O}(W, B)$ appears in the restriction of $\mathrm{Res}_{\mathrm{Sp}(V) \times \mathrm{O}(W, B)}(\omega[V \otimes W])$ to the single factor $\mathrm{O}(W, B)$, and therefore, there exists a non-zero representation of $\mathrm{Sp}(V)$ whose tensor product with ρ is a summand of $\mathrm{Res}_{\mathrm{Sp}(V) \times \mathrm{O}(W, B)}(\omega[V \otimes W])$. Further, there is a unique “top” irreducible piece of this $\mathrm{Sp}(V)$ -representation denoted by $\eta(\rho)$, which was constructed by Gurevich and Howe to specifically have highest *U-rank* (see [13, 14] for more details). The resulting function η from $\widehat{\mathrm{O}(W, B)}$ into $\widehat{\mathrm{Sp}(V)}$ is called the *eta correspondence*. Here, we write

$$\eta_{W, B}^V : \widehat{\mathrm{O}(W, B)} \hookrightarrow \widehat{\mathrm{Sp}(V)}$$

for the eta correspondence (omitting the subscript and superscript when they are implied).

Examining the structure of the endomorphism algebras of the oscillator representations (and their tensor powers) can be used to investigate $\eta_{W, B}^V$ and derive an explicit decomposition of the restricted oscillator representation in terms of it. From this perspective, however the roles of the symplectic and orthogonal groups can be switched. In this paper, we find that for an opposite *orthogonal stable range* (where $\dim(V)$ is less than or equal to the dimension of the maximal isotropic subspace of W), there exists a system of injections

$$\zeta_V^{W, B} : \widehat{\mathrm{Sp}(V)} \hookrightarrow \widehat{\mathrm{O}(W, B)}$$

we call the *zeta correspondence*, along with an accompanying explicit decomposition of the restricted oscillator representations. Like $\eta_{W, B}^V$ in the symplectic stable range, $\zeta_V^{W, B}$ all have disjoint images in the orthogonal stable range, each giving a new range of $\mathrm{O}(W, B)$ -representations. To state this concretely, we introduce some notation.

Consider an n -dimensional orthogonal space and symmetric bilinear form (W, B) , and a $2N$ -dimensional symplectic space and form (V, S) . The maximal dimension of an isotropic subspace of V with respect to S is N . We denote by h_W the maximum dimension of an isotropic

subspace of W with respect to B . Over a finite field, in the case when $n = 2m$ is even, we either have $h_W = m - 1$ or m (giving rise to the two orthogonal groups $O_{2m}^+(\mathbb{F}_q)$ and $O_{2m}^-(\mathbb{F}_q)$, respectively). In the case when $n = 2m + 1$ is odd, then we must have $h_W = m$. Then B can be expressed as a direct sum of h_W copies of the 2-dimensional hyperbolic symmetric bilinear form and a final anisotropic part of dimension 0, 1, or 2.

For k the dimension of an isotropic space Z in V , we denote by $(V[-k], S[-k])$ the symplectic subspace of dimension $2N - 2k$ and its accompanying non-degenerate symplectic form obtained by projecting away from Z and its dual in V with respect to S . Similarly, for k the dimension of an isotropic space Z in W , we write $(W[-k], B[-k])$ for the orthogonal subspace obtained by projecting away from Z and its dual in W with respect to B .

Let us also denote by P_k^V (resp. P_k^B) the parabolic subgroups of $\mathrm{Sp}(V)$ (resp. $\mathrm{O}(W, B)$) corresponding to an k -dimensional isotropic subspace of V (resp. W) with Levi subgroup

$$(5) \quad \mathrm{Sp}(V[-k], S[-k]) \times \mathrm{GL}_k(\mathbb{F}_q) = \mathrm{Sp}_{2N-2k}(\mathbb{F}_q) \times \mathrm{GL}_k(\mathbb{F}_q)$$

(resp.

$$(6) \quad \mathrm{O}(W[-k], B[-k]) \times \mathrm{GL}_k(\mathbb{F}_q) = \mathrm{O}_{n-2k}(\mathbb{F}_q) \times \mathrm{GL}_k(\mathbb{F}_q)).$$

We may write $\mathrm{Ind}_{P_k^V}$ or $\mathrm{Ind}_{P_k^B}$ for the Harish-Chandra induction of a representation ρ of (5) or (6), which we recall is defined by first inflating ρ to a representation of P_k^V or P_k^B by taking elements of the parabolic's unipotent radical to act trivially, and then inducing that inflation to $\mathrm{Sp}(V)$ or $\mathrm{O}(W, B)$. We will often consider this in the case of a representation

$$\rho \otimes \epsilon(\det)$$

of (5) or (6) where ρ is an irreducible representation of $\mathrm{Sp}(V[-k], S[-k])$ or $\mathrm{O}(W[-k], B[-k])$, and $\epsilon(\det)$ denotes the 1-dimensional representation of $\mathrm{GL}_k(\mathbb{F}_q)$ (writing ϵ for the non-trivial quadratic character on $\mathbb{F}_q^\times$).

Convention: In this series of papers, we shall adhere to the standard convention in this topic, which is, for a subgroup $H \subseteq G$, to write Ind_H^G for induction from H to G and Res_H^G for restriction from G to H . If one of the variables is understood from the context, we sometimes omit it.

The stable ranges are characterized by the smaller of the symplectic and orthogonal spaces being of a lesser or equal dimension to the maximal dimension of an isotropic subspace of the larger:

Definition 1. Consider a reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ of subgroups of $\mathrm{Sp}(V \otimes W)$.

- (1) We say $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the symplectic stable range if $\dim(W) \leq \dim(V)/2$.
- (2) We say $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the orthogonal stable range if $\dim(V) \leq h_W$.

The purpose of this paper is then to prove the following

Theorem 2. For the two stable ranges of type I reductive dual pairs, there are mirroring results on the corresponding endomorphism algebra of the oscillator representation, giving a decomposition of its restriction:

- (A) For $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the symplectic stable range, the $\mathrm{Sp}(V)$ -equivariant endomorphism algebra of $\omega[V \otimes W]$ is isomorphic to

$$\bigoplus_{k=0}^{h_W} M_{|\mathrm{O}(W, B)/P_k^B|}(\mathbb{CO}(W[-k], B[-k]))$$

giving an explicit isomorphism of $\mathrm{Res}_{\mathrm{Sp}(V) \times \mathrm{O}(W, B)}(\omega[V \otimes W])$ with

$$(7) \quad \bigoplus_{k=0}^{h_W} \bigoplus_{\rho \in \widehat{\mathrm{O}(W[-k], B[-k])}} \eta^V(\rho) \otimes \mathrm{Ind}_{P_k^B}(\rho \otimes \epsilon(\det)).$$

- (B) For $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the orthogonal stable range, the $\mathrm{O}(W, B)$ -equivariant endomorphism algebra of $\omega[V \otimes W]$ is isomorphic to

$$\bigoplus_{k=0}^N M_{|\mathrm{Sp}(V)/P_k^V|}(\mathbb{C}\mathrm{Sp}(V[-k])),$$

and in fact, there therefore exists a system of injective maps $\zeta_V^{W, B} : \widehat{\mathrm{Sp}(V)} \hookrightarrow \widehat{\mathrm{O}(W, B)}$ with disjoint images (for fixed (W, B) and differing V), such that $\mathrm{Res}_{\mathrm{Sp}(V) \times \mathrm{O}(W, B)}(\omega[V \otimes W])$ is isomorphic to

$$(8) \quad \bigoplus_{k=0}^N \bigoplus_{\rho \in \widehat{\mathrm{Sp}(V[-k])}} \mathrm{Ind}_{P_k^V}(\rho \otimes \epsilon(\det)) \otimes \zeta_V^{W, B}(\rho).$$

Comment: The existence of the isomorphism between the restriction of $\omega[V \otimes W]$ and (7) in the symplectic stable range was established from the point of view of mathematical physics by F. Montealegre-Mora and

D. Gross [35]. In this paper, we follow a different approach which can be applied to both the symplectic and orthogonal stable ranges, and which gives an explicit combinatorial formula for the decomposition.

In the stable ranges, we can use these results to find an explicit description of $\eta_{W,B}^V$ and $\zeta_V^{W,B}$ in terms of Lusztig's classification of irreducible representations of finite groups of Lie type, using information about the possible dimensions of irreducible representations of $\mathrm{Sp}(V)$ and $\mathrm{O}(W, B)$. This is done in the next part of this series [28], (extending the results of [37, 38]).

These results obtain explicit decompositions of $\mathrm{Res}_{\mathrm{Sp}(V) \times \mathrm{O}(W, B)}(\omega[V \otimes W])$ in the symplectic and orthogonal stable ranges, covering about half of the general cases of type I reductive dual pairs $(\mathrm{Sp}(V), \mathrm{O}(W, B))$. One may ask if Theorem 2 and the following results in [28] can be extended into the unstable ranges, and specifically the cases up to and including the “middle” where the ranks of $\mathrm{Sp}(V)$ and $\mathrm{O}(W, B)$ are equal or close. This is possible using *interpolated category theory*, and this result is concluded in the third part of this series [29] (completing a full description of $\mathrm{Res}_{\mathrm{Sp}(V) \times \mathrm{O}(W, B)}(\omega[V \otimes W])$ for a general case of $(\mathrm{Sp}(V), \mathrm{O}(W, B))$).

The present paper is organized as follows: In Section 2, we discuss the endomorphism algebras of $\omega[V \otimes W]$ over $\mathrm{Sp}(V)$ and $\mathrm{O}(W, B)$. We interpret the vector spaces of endomorphisms as spaces of fixed points in $\mathbb{C}V \otimes W$ and introduce an operation $\star$ corresponding to composition. Using this description, we can elementarily calculate the dimensions of $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$ and $\mathrm{End}_{\mathrm{O}(W, B)}(\omega[V \otimes W])$ and find elements corresponding to generators of the group algebras of orthogonal and symplectic groups, respectively. In Section 3, we prove a combinatorial identity verifying that the dimensions the $\mathrm{Sp}(V)$ - and $\mathrm{O}(W, B)$ -equivariant endomorphism algebras of $\omega[V \otimes W]$ match the dimensions of the proposed sums of matrix algebras in Theorem 2. Combining this with the work of the previous section, we are able to prove the endomorphism algebras decompose as claimed in Theorem 2. We do this in Section 4, where we also use an inductive argument and information about Hom-spaces between oscillator representations (and their tensor powers) to deduce the decomposition of the restricted oscillator representation, concluding the proof of Theorem 2.

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2. THE ENDOMORPHISM ALGEBRA OF AN OSCILLATOR REPRESENTATION

In this section, we set-up some key results on the structure of the endomorphism algebras of an oscillator representation. In Subsection 2.1, for a general symplectic space $\mathbf{V}$, we consider the algebra of endomorphisms of an oscillator representation $\omega[\mathbf{V}]$ equivariant with respect to any subgroup $G \subseteq \mathrm{Sp}(\mathbf{V})$. We find that it is isomorphic to the fixed point space $\mathbb{C}\mathbf{V}^G$ with respect to a certain algebra operation $\star_{\mathbf{V}}$. In Subsection 2.2, we return to the setting of a reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ and use this re-expression to compute the dimensions of the endomorphism algebras of $\omega[V \otimes W]$ over $\mathrm{Sp}(V)$ and $\mathrm{O}(W, B)$ in the symplectic and orthogonal stable ranges, respectively. In Subsection 2.3, we discuss the “top part” of these endomorphism algebras. We find the group algebra $\mathbb{C}\mathrm{O}(W, B)$ in the top part of $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$, in the case of $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the symplectic stable range, in Subsection 2.4. Finally, in Subsection 2.5, we find the group algebra $\mathbb{C}\mathrm{Sp}(V)$ in the top part of $\mathrm{End}_{\mathrm{O}(W, B)}(\omega[V \otimes W])$ in the case of the orthogonal stable range.

2.1. Endomorphisms and the Schrödinger model. In this subsection, consider a general symplectic space and form $(\mathbf{V}, \mathbf{S})$. Recall that for an oscillator representation $\omega_a[\mathbf{V}]$, its dual is the oscillator representation of opposite character $\omega_{-a}[\mathbf{V}]$ (see [16]). Their tensor product gives the vector representation $\mathbb{C}\mathbf{V}$ where $\mathrm{Sp}(\mathbf{V})$ acts geometrically. For a vector $v \in \mathbf{V}$, we write $[v]$ for the corresponding basis element of $\mathbb{C}\mathbf{V}$.

In particular, for a subgroup $G \subseteq \mathrm{Sp}(\mathbf{V})$, the endomorphism algebra of the restriction of an oscillator representation to G is, as a vector space, isomorphic to the G -fixed point subspace of $\mathbb{C}\mathbf{V}$

$$\mathrm{End}_G(\omega_a[\mathbf{V}]) \cong \mathrm{Hom}_G(1, \omega_a[\mathbf{V}] \otimes \omega_{-a}[\mathbf{V}]) \cong \mathbb{C}\mathbf{V}^G,$$

which can also be considered as the $\mathbb{C}$ -vector space with a basis indexed by G -orbits in $\mathbf{V}$. In particular, for G trivial, the vector endomorphism algebra of $\omega_a[\mathbf{V}]$ is isomorphic, as a vector space, with $\mathbb{C}\mathbf{V}$ itself.

Now we construct an operation $\star_{\mathbf{V}}$ on $\mathbb{C}\mathbf{V}$ corresponding to the composition operation in the vector endomorphism algebra of $\omega_a[\mathbf{V}]$, so that

$$(9) \quad (\mathrm{End}_{\mathrm{Vect}}(\omega_a[\mathbf{V}]), \circ) \cong (\mathbb{C}\mathbf{V}, \star_{\mathbf{V}}),$$

and for subgroups $G \subseteq \mathrm{Sp}(\mathbf{V})$, $\mathrm{End}_G(\omega_a[\mathbf{V}])$ is the subalgebra of G -fixed points in (9). We define this operation $\star_{\mathbf{V}}$ on $\mathbb{C}\mathbf{V}$, by putting, for

$u, v \in \mathbf{V}$,

$$(10) \quad [u] \star_{\mathbf{V}} [v] = \psi_a\left(\frac{1}{2}\mathbf{S}(u, v)\right) [u + v]$$

(and extending $\mathbb{C}$ -linearly). This can be considered as an “untwisted” variant of the algebra operation arising from the *Schrödinger model* of the oscillator representation.

Recall that for a decomposition of a symplectic space $\mathbf{V}$ into Lagrangians $\Lambda_+ \oplus \Lambda_-$, we may identify the oscillator representation as $\omega_a[\mathbf{V}] = \mathbb{C}\Lambda_-$. For a vector $x \in \Lambda_-$, let us write $\{x\}$ for the corresponding basis element of $\mathbb{C}\Lambda_-$. The action of an element (v, c) of the Heisenberg group $\mathbb{H}$ (for $v \in \mathbf{V}$, $c \in \mathbb{F}_q$) on a generator $x \in \Lambda_-$ is given by

$$(v, c)\{x\} = \psi_a\left(c + \frac{1}{2}\mathbf{S}(v_+, x)\right) \{v_- + x\},$$

where $v = v_+ + v_-$ is the unique decomposition of v into vectors $v_+ \in \Lambda_+$, $v_- \in \Lambda_-$. This gives $\omega_a[\mathbf{V}]$ the structure of a *Weil-Shale representation*. The action of $\mathrm{Sp}(\mathbf{V})$ making it an oscillator representation arises from the uniqueness of the Weil-Shale representation for each central character.

Arising from the action of the Heisenberg group, we then see a natural action of an algebra $(\mathbb{C}\mathbf{V}, \star_{\mathbf{V}})$ on $\omega_a[\mathbf{V}] = \mathbb{C}\Lambda_-$, for the operation $\star_{\mathbf{V}}$ given by acting on basis elements, for $u = u_+ + u_-$, $v = v_+ + v_- \in \mathbf{V}$ with $u_{\pm}, v_{\pm} \in \Lambda_{\pm}$, by

$$[u] \star_{\mathbf{V}} [v] = \psi_a(\mathbf{S}(u_+, v_-)) [u + v]$$

and extending $\mathbb{C}$ -linearly (see also Remark 7.2.6 of [13], identifying a vector generator $[v]$ with its corresponding indicator function). Applied to a basis element $x \in \Lambda_-$ of the oscillator representation, a vector $v \in \mathbf{V}$, viewed as a basis element of $(\mathbb{C}\mathbf{V}, \star_{\mathbf{V}})$, acts by

$$(11) \quad [v](\{x\}) = \psi_a(\mathbf{S}(v_+, x)) \{v_- + x\}.$$

The model $(\mathrm{End}_{\mathrm{Vect}}(\omega_a[\mathbf{V}]), \circ) = (\mathbb{C}\mathbf{V}, \star_{\mathbf{V}})$ is the standard form of the endomorphism algebra of the oscillator representation arising from the Schrödinger model.

Now we can see our untwisted choice of algebra $(\mathbb{C}\mathbf{V}, \star_{\mathbf{V}})$ defined by the operation (10) is isomorphic to the standard model of the algebra $\mathrm{End}_{\mathrm{Vect}}(\omega_a[\mathbf{V}])$ along the map

$$\begin{aligned} (\mathbb{C}\mathbf{V}, \star_{\mathbf{V}}) &\rightarrow (\mathbb{C}\mathbf{V}, \star_{\mathbf{V}}) \\ [v] &\mapsto \psi_a\left(\frac{1}{2}\mathbf{S}(v_+, v_-)\right) [v] \end{aligned}$$

for $v = v_+ + v_- \in \mathbf{V}$ with $v_{\pm} \in \mathbf{\Lambda}_{\pm}$. Comparing with (11), we therefore see that a vector $v \in \mathbf{V}$, considered as a basis element of $(\mathbb{C}\mathbf{V}, \star_{\mathbf{V}})$, acts on a basis element $x \in \mathbf{\Lambda}_-$ of the oscillator representation $\omega_a[\mathbf{V}]$ by

$$(12) \quad [v](\{x\}) = \psi_a(\mathbf{S}(v_+, x) + \frac{1}{2}\mathbf{S}(v_+, v_-)) \{v_- + x\}$$

for $v = v_+ + v_-$ with $v_{\pm} \in \mathbf{\Lambda}_{\pm}$.

In particular, note that in this model, $[0]$ is the identity endomorphism of $\omega[\mathbf{V}]$, and the trace of a vector $v \in \mathbf{V}$ considered as an endomorphism in $\mathbb{C}\mathbf{V} = \text{End}_{\text{Vect}}(\omega_a[\mathbf{V}])$ is

$$\text{tr}([v]) = \begin{cases} 0, & \text{for } v \neq 0 \\ |\mathbf{V}| & \text{for } v = 0 \end{cases}$$

where $|\mathbf{V}|$ denotes the set order of $\mathbf{V}$.

2.2. Counting orbits. For the remainder of this section, we return to the context of Howe duality for a reductive dual pair $(\text{Sp}(V), \text{O}(W, B))$ in $\text{Sp}(V \otimes W)$. We may apply the results of the above subsection to this situation by considering $\mathbf{V} = V \otimes W$, $\mathbf{S} = S \otimes B$ (writing S for V 's symplectic form), so that we have $(\text{End}_{\text{Vect}}(\omega[V \otimes W]), \circ) \cong (\mathbb{C}V \otimes W, \star_{V \otimes W})$. The operation is defined by acting on basis elements, for $v_1, v_2 \in V$, $w_1, w_2 \in W$, as

$$(13) \quad [v_1 \otimes w_1] \star_{V \otimes W} [v_2 \otimes w_2] = \psi\left(\frac{S(v_1, v_2) \cdot B(w_1, w_2)}{2}\right) [v_1 \otimes w_1 + v_2 \otimes w_2]$$

(and extending $\mathbb{C}$ -linearly). Since the ground space will remain $V \otimes W$ from now on, we omit the subscript from $\star$. The relevant cases of G are $\text{Sp}(V)$ and $\text{O}(W, B)$, in which case the equivariant endomorphism algebras on $\omega[V \otimes W]$ can be expressed as

$$(14) \quad (\text{End}_{\text{Sp}(V)}(\omega[V \otimes W]), \circ) \cong (\mathbb{C}V \otimes W^{\text{Sp}(V)}, \star)$$

$$(15) \quad (\text{End}_{\text{O}(W, B)}(\omega[V \otimes W]), \circ) \cong (\mathbb{C}V \otimes W^{\text{O}(W, B)}, \star)$$

Now, the purpose of the present subsection is to calculate the dimensions

$$\begin{aligned} \dim(\text{End}_{\text{Sp}(V)}(\omega[V \otimes W])) &= \dim(\mathbb{C}V \otimes W^{\text{Sp}(V)}), \\ \dim(\text{End}_{\text{O}(W, B)}(\omega[V \otimes W])) &= \dim(\mathbb{C}V \otimes W^{\text{O}(W, B)}) \end{aligned}$$

by counting the number of orbits of $\mathrm{Sp}(V)$ on $V \otimes W = V^{\oplus n}$ and the number of orbits of $\mathrm{O}(W, B)$ on $V \otimes W = W^{\oplus 2N}$ in the case of the symplectic and orthogonal stable range, respectively.

Lemma 3. *Consider a symplectic space V of dimension $2N$ and an space with symmetric bilinear form (W, B) of dimension n with maximal dimension of an isotropic subspace denote by h_W .*

(a) *If $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ for a reductive dual pair in the symplectic stable range (i.e. $n \leq N$), then*

$$(16) \quad \dim((\mathbb{C}V^{\oplus n})^{\mathrm{Sp}(V)}) = 2(q+1) \dots (q^{n-1} + 1)$$

(b) *If $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ for a reductive dual pair in the orthogonal stable range (i.e. $2N \leq h_W$), then*

$$(17) \quad \dim((\mathbb{C}W^{\oplus 2N})^{\mathrm{O}(W, B)}) = (q+1)(q^2+1) \dots (q^{2N}+1)$$

Proof. We begin with the proof of Part (a). We write S for the symplectic form on V . For (16), we need to compute the number of $\mathrm{Sp}(V)$ -orbits on $V \otimes W = V^{\oplus n}$ (with $\mathrm{Sp}(V)$ acting diagonally). First recall that orbits of this $\mathrm{Sp}(V)$ action on n -tuples of vectors in V correspond to the data of

- A $d \times n$ matrix M over $\mathbb{F}_q$ in reduced row echelon form for some $0 \leq d \leq n$.
- A choice of $d(d-1)/2$ scalars $a_{i < j}$ corresponding to each pair of indices $i < j \in \{1, \dots, d\}$.

Given this data, its corresponding $\mathrm{Sp}(V)$ -orbit is

$$(18) \quad \{(v_1, \dots, v_d) \cdot M \mid v_1, \dots, v_d \in V \text{ linearly independent,} \\ \text{for every } i < j \quad S(v_i, v_j) = a_{i < j}\}$$

(where $(v_1, \dots, v_d) \cdot M$ is computed by treating $(v_1, \dots, v_d)$ as a $1 \times d$ matrix, so that the result is a $1 \times n$ matrix of vectors in V , i.e. is an element of $V^{\oplus n}$). Now the symplectic stable range condition $n \leq N$ precisely ensures that every choice of data $(M, a_{i < j})$ gives a non-empty orbit (18), since (18) is non-empty if and only if

$$\dim(V) \geq d + \dim(\mathrm{Ker}(S|_{\langle v_1, \dots, v_d \rangle}))$$

where $S|_{\langle v_1, \dots, v_d \rangle}$ denotes the (possibly degenerate) symplectic form obtained from restricting S to the span of $v_1, \dots, v_d$.

Therefore, it suffices to count all choices of data $(M, a_{i < j})$. Given M , the number of choices of scalars $(a_{i < j})$ is $q^{d(d-1)/2}$. Now the number of

$d \times n$ matrices in reduced row echelon form is the Gaussian binomial coefficient $\binom{n}{d}_q$ (see (55)), giving

$$(19) \quad \dim(\text{End}_{\text{Sp}(V)}(\omega[V \otimes W])) = \sum_{d=0}^n \binom{n}{d}_q q^{\frac{d(d-1)}{2}}.$$

To see the simplification to (16) elementarily, it is actually easier to naively write down the number of reduced row echelon $d \times n$ matrices as

$$\binom{n}{d}_q = \sum_{1 \leq \ell_1 < \dots < \ell_d \leq n} q^{dn - (\ell_1 + \dots + \ell_d) - \frac{d(d-1)}{2}},$$

taking ℓ_i to be the length of each row from the left up to an including the pivot, which leaves exactly $dn - (\ell_1 + \dots + \ell_d) - \frac{d(d-1)}{2}$ un-determined entries (the $\frac{d(d-1)}{2} = 1 + 2 + \dots + (d-1)$ term arises from the entries directly above a pivot being 0). Using this in (19) gives

$$\begin{aligned} \dim(\text{End}_{\text{Sp}(V)}(\omega[V \otimes W])) &= \sum_{1 \leq \ell_1 < \dots < \ell_d \leq n} q^{dn - (\ell_1 + \dots + \ell_d) - \frac{d(d-1)}{2}} \\ &= \sum_{1 \leq \ell_1 < \dots < \ell_d \leq n} \prod_{i=1}^d q^{n - \ell_i} = \prod_{j=0}^{n-1} (q^j + 1), \end{aligned}$$

as claimed.

The proof of Part (b) follows similarly: As in the symplectic stable range case, the orbits of $O(W, B)$ on $V \otimes W = W^{\oplus 2N}$ correspond to the data of

- A $d \times 2N$ matrix M over $\mathbb{F}_q$ in reduced row echelon form for some $0 \leq d \leq 2N$.
- A choice of $d(d+1)/2$ scalars $a_{i \leq j}$ corresponding to each pair of indices $i \leq j \in \{1, \dots, d\}$.

The orbit corresponding to $(M, a_{i \leq j})$ is

$$(20) \quad \{(w_1, \dots, w_d) \cdot M \mid w_1, \dots, w_d \in W \text{ linearly independent,} \\ \text{for every } i \leq j \quad B(w_i, w_j) = a_{i \leq j}\}$$

(again, treating $(w_1, \dots, w_d)$ as a $1 \times d$ matrix). As before, the orthogonal stable range condition precisely ensures that each set (20) corresponding to any choice of $(M, a_{i \leq j})$ is non-empty. (We note that the cardinality of (20), like the orthogonal stable range condition, depends on the form of B when n is even.) Therefore, by the above

argument, we have

$$\dim(\text{End}_{\text{O}(W,B)}(\omega[V \otimes W])) = \sum_{d=0}^{2N} \binom{2N}{d}_q q^{\frac{d(d+1)}{2}},$$

which can be simplified using the naive expression for $\binom{2N}{d}_q$ to

$$\begin{aligned} \dim(\text{End}_{\text{O}(W,B)}(\omega[V \otimes W])) &= \sum_{1 \leq \ell_1 < \dots < \ell_d \leq 2N} q^{d(2N+1) - (\ell_1 + \dots + \ell_d)} \\ &= \sum_{1 \leq \ell_1 < \dots < \ell_d \leq 2N} \prod_{i=1}^d q^{2N+1-\ell_i} = \prod_{j=1}^{2N} (q^j + 1), \end{aligned}$$

as claimed. $\square$

2.3. The top subalgebras. To continue examining the $\text{Sp}(V)$ - and $\text{O}(W, B)$ -equivariant endomorphism algebras of $\omega[V \otimes W]$, note that

$$(21) \quad \text{Res}_{\text{Sp}(V)}(\omega[V \otimes W]) \cong \omega_{a_1}[V] \otimes \dots \otimes \omega_{a_n}[V]$$

where $a_1, \dots, a_n$ are the eigenvalues of B . Similarly, we may express

$$(22) \quad \text{Res}_{\text{O}(W,B)}(\omega[V \otimes W]) = \text{Res}_{\text{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W])^{\otimes N},$$

where we consider $\omega[\mathbb{F}_q^2 \otimes W]$ to be the oscillator representation on $\mathbb{F}_q^2$ (with respect to the standard symplectic form) tensored with (W, B) and we take its restriction along the inclusion

$$\text{O}(W, B) \hookrightarrow \text{SL}_2(\mathbb{F}_q) \times \text{O}(W, B)$$

(writing $\text{SL}_2(\mathbb{F}_q) = \text{Sp}(\mathbb{F}_q^2)$). In fact, this restriction is actually isomorphic to the (twisted) vector representation of $\text{O}(W, B)$

$$(23) \quad \text{Res}_{\text{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]) \cong \mathbb{C}W^-,$$

where $-$ indicates the sign character of $\mathbb{Z}/2 = \text{O}(W, B)/\text{SO}(W, B)$, (this can be seen by considering first the restriction of $\omega[\mathbb{F}_q^2 \otimes W]$ to $\text{GL}(W)$).

Now to approach both the symplectic and orthogonal stable ranges, we define the *top subalgebras*

$$\text{End}_{\text{Sp}(V)}^{\text{top}}(\omega[V \otimes W]) \subseteq \text{End}_{\text{Sp}(V)}(\omega[V \otimes W]),$$

$$\text{End}_{\text{O}(W,B)}^{\text{top}}(\omega[V \otimes W]) \subseteq \text{End}_{\text{O}(W,B)}(\omega[V \otimes W]),$$

consisting of endomorphisms of the restricted oscillator representations (21) or (22) which do not factor through a lower degree tensor product of factors of the form $\omega_a[V]$ or $\text{Res}_{\text{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W])$ (respectively). In the case of the symplectic stable range, in the language of Gurevich

and Howe, $\text{End}_{\text{Sp}(V)}^{\text{top}}(\omega[V \otimes W])$ is precisely the endomorphism algebra of the part of $\text{Res}_{\text{Sp}(V)}(\omega[V \otimes W])$ with highest $\otimes$ -rank.

The purpose of the remaining two subsections in Section 2 is to prove the following

Proposition 4. *Suppose V is a symplectic space of dimension $2N$ and W is an n -dimensional space with symmetric bilinear form B . Write h_W for the maximal dimension of an isotropic subspace of W with respect to B .*

(a) *If $(\text{Sp}(V), \text{O}(W, B))$ is in the symplectic stable range, then*

$$(24) \quad \mathbb{C}\text{O}(W, B) \subseteq \text{End}_{\text{Sp}(V)}^{\text{top}}(\omega[V \otimes W])$$

(b) *If $(\text{Sp}(V), \text{O}(W, B))$ is in the orthogonal stable range, then*

$$(25) \quad \mathbb{C}\text{Sp}(V) \subseteq \text{End}_{\text{O}(W, B)}^{\text{top}}(\omega[V \otimes W]),$$

In fact, in both cases, this group algebra will act on $\omega[V \otimes W]$ precisely according to the representation-theoretic action of the corresponding factor of $\text{Sp}(V) \times \text{O}(W, B) \subseteq \text{Sp}(V \otimes W)$.

2.4. Generators of the orthogonal group. In this subsection, we will prove Part (a) of Proposition 4, for $(\text{Sp}(V), \text{O}(W, B))$ in the symplectic stable range. The case is simpler since the orthogonal group has a simpler set of generators than the symplectic group. Due to the complexity of the generators of $\text{Sp}(V)$ and their corresponding elements in $\text{End}_{\text{O}(W, B)}(\omega[V \otimes W])$ as the fixed point space $\mathbb{C}V \otimes W^{\text{O}(W, B)}$, we will treat Part (b) separately in Subsection 2.5 below.

Proof of Proposition 4, Part (a). In the orthogonal group $\text{O}(W, B)$, consider the set of generators consisting of reflections across hyperspaces orthogonal to each anisotropic line in W . We index the lines in $W = \mathbb{F}_q^n$ by $1 \times n$ matrices over $\mathbb{F}_q$ in reduced row echelon form $\lambda = (\lambda_1, \dots, \lambda_n)$ (which we can also consider as vectors in W). In particular, we must find an element of $\text{End}_{\text{Sp}(V)}^{\text{top}}(\omega[V \otimes W])$ representing the reflection generator of $\text{O}(W, B)$ corresponding to each such λ with $B(\lambda, \lambda) \neq 0$.

Let us consider, for $\lambda = (\lambda_1, \dots, \lambda_n)$ a $1 \times n$ matrix over $\mathbb{F}_q$ in reduced row echelon form, the element

$$f_\lambda := \sum_{v \in V} [(\lambda_1 v, \dots, \lambda_n v)] = \sum_{v \in V} [v \otimes \lambda] \in \mathbb{C}V \otimes W^{\text{Sp}(V)}.$$

Lemma 5. *For $\lambda = (\lambda_1, \dots, \lambda_n)$ a $1 \times n$ matrix in reduced row echelon form such that $B(\lambda, \lambda) \neq 0$, the element f_λ/q^N is a reflection*

$$(26) \quad (f_\lambda/q^N) \star (f_\lambda/q^N) = [0] = Id_{\omega[V \otimes W]}$$

Proof. Suppose λ is a $1 \times n$ matrix in reduced row echelon form. Then, applying (13), we can calculate

$$(27) \quad \begin{aligned} f_\lambda \star f_\lambda &= \sum_{u, v \in V} [u \otimes \lambda] \star [v \otimes \lambda] = \\ &= \sum_{u, v \in V} \psi\left(\frac{S(u, v) \cdot B(\lambda, \lambda)}{2}\right) [(u + v) \otimes \lambda], \end{aligned}$$

which by substituting $u' = u + v$ reduces to

$$f_\lambda \star f_\lambda = \sum_{u, v \in V} \psi\left(\frac{S(u', v) \cdot B(\lambda, \lambda)}{2}\right) [u' \otimes \lambda].$$

Collecting terms, since $B(\lambda, \lambda) \neq 0$, for each non-zero $u' \in V$, the coefficient of $[u' \otimes \lambda]$ is a linear sum of characters, giving 0. At $u' = 0$, each $v \in V$ (of which we have q^{2N}) contributes a term with coefficient $\psi(0) = 1$. Hence,

$$f_\lambda \star f_\lambda = q^{2N} [0] \in \mathbb{C}V \otimes W^{\text{Sp}(V)},$$

giving (26). □

Now we verify that these proposed generators f_λ/q^N for λ satisfying $B(\lambda, \lambda) \neq 0$ really do act as the reflection elements of $O(W, B)$ across the orthogonal hyperspaces in W to each λ . To see this we use (12) to calculate the action of each f_λ/q^N on the oscillator representation $\omega[V \otimes W]$, considering its Schrödinger model $\mathbb{C}\Lambda_- \otimes W$ (corresponding to decomposing $V \otimes W$ into Lagrangians $(\Lambda_+ \otimes W) \oplus (\Lambda_- \otimes W)$ according to a decomposition of V into Lagrangians $\Lambda_+ \oplus \Lambda_-$).

Claim 1. *Consider a $\lambda = (\lambda_1, \dots, \lambda_n)$ in reduced row echelon form such that $B(\lambda, \lambda) \neq 0$. For a vector $\sum_{i=1}^r x_i \otimes w_i \in \Lambda_- \otimes W$, for $x_i \in \Lambda_-$, $w_i \in W$, considering $[\sum_{i=1}^r x_i \otimes w_i]$ as a basis element of $\omega[V \otimes W] = \mathbb{C}\Lambda_- \otimes W$, the element f_λ/q^N acts by*

$$(28) \quad \frac{f_\lambda}{q^N} \left(\left\{ \sum_{i=1}^r x_i \otimes w_i \right\} \right) = \left\{ \sum_{i=1}^r x_i \otimes \left(w_i - 2 \frac{B(w_i, \lambda)}{B(\lambda, \lambda)} \lambda \right) \right\}.$$

Proof of Claim 1. Fix a λ in reduced row echelon form with $B(\lambda, \lambda) \neq 0$. Consider a basis element $\{\sum_{i=1}^r x_i \otimes w_i\}$ of $\mathbb{C}\Lambda_- \otimes W$ for $x_i \in \Lambda_-$, $w_i \in W$. Writing out f_λ , we can apply (12) to calculate the output $f_\lambda(\{\sum_{i=1}^r x_i \otimes w_i\})$, which upon being reduced can be re-written as

$$(29) \quad \begin{aligned} f_\lambda(\{\sum_{i=1}^r x_i \otimes w_i\}) &= \sum_{v \in V} [v \otimes \lambda](\{\sum_{i=1}^r x_i \otimes w_i\}) = \\ &\sum_{v_\pm \in \Lambda_\pm} \psi(S(v_+, \sum_{i=1}^r B(\lambda, w_i)x_i + \frac{B(\lambda, \lambda)}{2}v_-)) \{v_- \otimes \lambda + \sum_{i=1}^r x_i \otimes w_i\}. \end{aligned}$$

Collecting terms, we see that if the coordinate $\sum_{i=1}^r B(\lambda, w_i)x_i + \frac{B(\lambda, \lambda)}{2}v_-$ appearing with v_+ in S is non-zero, then the total coefficient of the basis element $\{v_- \otimes \lambda + \sum_{i=1}^r x_i \otimes w_i\}$ (which does not depend on v_+) is a linear sum of characters, canceling to 0. The only surviving term is then for $\sum_{i=1}^r B(\lambda, w_i)x_i + \frac{B(\lambda, \lambda)}{2}v_- = 0$, meaning

$$v_- = -\frac{2}{B(\lambda, \lambda)} \sum_{i=1}^r B(w_i, \lambda)x_i.$$

Each of the q^N choices of $v_+ \in \Lambda_+$ contributes a term with coefficient $\psi(0) = 1$ for this case of the vector

$$v_- \otimes \lambda + \sum_{i=1}^r x_i \otimes w_i = -\frac{2}{B(\lambda, \lambda)} \sum_{i=1}^r B(w_i, \lambda)x_i \otimes \lambda + \sum_{i=1}^r x_i \otimes w_i.$$

Reducing, we therefore obtain

$$f_\lambda(\{\sum_{i=1}^r x_i \otimes w_i\}) = q^N \left\{ \sum_{i=1}^r x_i \otimes (w_i - 2 \frac{B(w_i, \lambda)}{B(\lambda, \lambda)} \lambda) \right\},$$

giving (28). $\square$

Since these reflections generate $O(W, B)$, this claim therefore implies that for any $\phi \in O(W, B)$, there is a corresponding element of the endomorphism algebra acting on a basis element $\{\sum_{i=1}^r x_i \otimes w_i\} \in \mathbb{C}\Lambda_- \otimes W$ by

$$\{\sum_{i=1}^r x_i \otimes w_i\} \mapsto \{\sum_{i=1}^r x_i \otimes \phi(w_i)\},$$

which is precisely its action as an element of $O(W, B) \subseteq \text{Sp}(V) \times O(W, B) \subseteq \text{Sp}(V \otimes W)$. Therefore the subalgebra generated by the

elements f_λ for $\lambda \in W$ in reduced row echelon form such that $B(\lambda, \lambda) \neq 0$ is isomorphic to the group algebra

$$\text{CO}(W, B) \subseteq \text{End}_{\text{Sp}(V)}(\omega[V \otimes W])$$

with each generating group element acting according to the geometric representation action on $\omega[V \otimes W]$. In particular, every generating element of $\text{O}(W, B)$ is an automorphism of $\omega[V \otimes W]$, and therefore cannot factor through any smaller tensor power of oscillator representations $\omega_a[V]$. Hence, it is in the top part of the endomorphism algebra and we have (24). $\square$

Remark: One may ask what the significance of an element f_λ is for $\lambda \in W$ in reduced row echelon form such that $B(\lambda, \lambda) = 0$. These degenerate elements give idempotents precisely identifying lower tensor powers of oscillator representations:

Proposition 6. *Suppose $\lambda_1, \dots, \lambda_k \in W$ are $1 \times n$ matrices in reduced row echelon form which are linearly independent and such that their span $\langle \lambda_1, \dots, \lambda_k \rangle \subseteq W$ is an isotropic subspace with respect to B . Then f_{λ_i}/q^{2N} are all commuting idempotents and*

$$(30) \quad \text{Im}\left(\frac{f_{\lambda_1} \star \dots \star f_{\lambda_k}}{q^{2Nk}}\right) \cong \text{Res}_{\text{Sp}(V)}(\omega[V \otimes W[-k]])$$

considering the restriction of $\omega[V \otimes W[-k]]$ to $\text{Sp}(V)$ along the inclusion

$$\text{Sp}(V) \hookrightarrow \text{Sp}(V) \times \text{O}(W[-k], B[-k]) \hookrightarrow \text{Sp}(V \otimes W[-k]).$$

Proof. First note that the calculation (27) still holds, but in the case of $B(\lambda, \lambda) = 0$, the coefficient of every term is trivial. Substituting $u' = u + v$ therefore gives

$$f_\lambda \star f_\lambda = \sum_{u', v \in V} [u' \otimes \lambda] = q^{2N} f_\lambda,$$

Hence, for every λ in reduced row echelon form with $B(\lambda, \lambda) = 0$, f_λ/q^{2N} is an idempotent.

Now for two vectors $\lambda, \mu \in \mathbb{F}_q^n$ considered as $1 \times n$ matrices in reduced row echelon form, we have

$$\begin{aligned} f_\lambda \star f_\mu &= \sum_{u, v \in V} [u \otimes \lambda] \star [v \otimes \mu] = \\ &= \sum_{u, v \in V} \psi\left(\frac{S(v, w) \cdot B(\lambda, \mu)}{2}\right) [u \otimes \lambda + v \otimes \mu]. \end{aligned}$$

In particular, this immediately gives that for λ, μ such that $\langle \lambda, \mu \rangle$ is isotropic with respect to B , the endomorphisms f_λ and f_μ commute.

It remains to prove that (30) holds. Fix $k \leq h_W$. For simplicity, without loss of generality, let us write

$$(31) \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\oplus k} \oplus B[-k], \quad B[-k] = \begin{pmatrix} a_{2k+1} & 0 & \dots & 0 \\ 0 & a_{2k+2} & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}$$

(i.e. B is diagonal, and the first $2k$ diagonal entries are alternating ± 1). Then (21) gives

$$(32) \quad \begin{aligned} & \text{Res}_{\text{Sp}(V)}(\omega[V \otimes W]) = \\ & (\omega[V] \otimes \omega_{-1}[V])^{\otimes k} \otimes \text{Res}_{\text{Sp}(V)}(\omega[V \otimes W[-k]]). \end{aligned}$$

Let us consider an individual $\omega[V] \otimes \omega_{-1}[V]$ factor, i.e. consider the case of $n = 2$ and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case, $\lambda = (1, 1)$ gives an idempotent

$$f_{(1,1)}/q^{2N} = \frac{1}{q^{2N}} \sum_{v \in V} [(v, v)].$$

The only term of $f_{(1,1)}/q^{2N}$ contributing to its trace is $v = 0$, giving

$$\dim(\text{Im}(f_{(1,1)}/q^{2N})) = \text{tr}(f_{(1,1)}/q^{2N}) = 1,$$

and hence $\text{Im}(f_{(1,1)}/q^{2N})$ is the trivial representation of $\text{Sp}(V)$ since that is the only one-dimensional representation.

Now let us return to the general case of n and B of the form (31). Consider the special case where λ_i is the n -tuple with all entries 0 except for the $(2i-1)$ th and $2i$ th which are put to be 1. Then as an element of (32), each f_{λ_i} is a tensor product of $\text{Id}_{\omega[V] \otimes \omega_{-1}[V]}$ factors, except for the i th one, which is replaced by a factor $f_{(1,1)} \in \text{End}_{\text{Sp}(V)}(\omega[V] \otimes \omega_{-1}[V])$, with $\text{Id}_{\omega[V \otimes W[-k]]}$. Therefore, we may consider

$$\frac{f_{\lambda_1} \star \dots \star f_{\lambda_k}}{q^{2Nk}} = (f_{(1,1)})^{\otimes k} \otimes \text{Id}_{\omega[V \otimes W[-k]]},$$

which then has image

$$1^{\otimes k} \otimes \text{Res}_{\text{Sp}(V)}(\omega[V \otimes W[-k]]) = \text{Res}_{\text{Sp}(V)}(\omega[V \otimes W[-k]]).$$

To get the statement in the general case of $\lambda_1, \dots, \lambda_k$, note that the images of idempotents $(f_{\lambda_1} \star \dots \star f_{\lambda_k})/q^{2Nk}$ for $\lambda_i \in W$ generating a k -dimensional isotropic subspace are isomorphic, since any basis of any

k -dimensional isotropic subspace can be transformed into any other using an orthogonal group element, by Witt's Theorem. Hence, all images of such $(f_{\lambda_1} \star \cdots \star f_{\lambda_k})/q^{2Nk}$ are isomorphic. $\square$

2.5. Generators of the symplectic group. The purpose of this subsection is to prove Proposition 4, Part (b), which we recall corresponds to the case of a reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in $\mathrm{Sp}(V \otimes W)$ which is in the orthogonal stable range.

Proof of Proposition 4, Part (b). We will use the same method as in the proof of Part (a), finding elements of $\mathbb{C}V \otimes W^{\mathrm{O}(W, B)}$ which act on $\omega[V \otimes W]$ as the generators of $\mathrm{Sp}(V)$.

We will again use the Schrödinger model of $\omega[V \otimes W] = \mathbb{C}\Lambda_- \otimes W$, corresponding to a decomposition of $V \otimes W$ into Lagrangians $(\Lambda_+ \otimes W) \oplus (\Lambda_- \otimes W)$ obtained from a decomposition of V into Lagrangians $\Lambda_+ \oplus \Lambda_-$. Further, let us fix bases $e_1^+, \dots, e_N^+$ and $e_1^-, \dots, e_N^-$ of the Lagrangians Λ_+, Λ_- such that the matrix corresponding to the symplectic form S with respect to the basis $e_1^+, \dots, e_N^+, e_1^-, \dots, e_N^-$ of V is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Then, writing an element of $V \otimes W = W^{\oplus 2N}$ as $(z_1^+, z_1^-, \dots, z_N^+, z_N^-)$ for $z_i^\pm \in W = \mathbb{F}_q\{e_i^\pm\} \otimes W$, and an element of $\Lambda_- \otimes W = W^{\oplus N}$ as $(u_1, \dots, u_N)$ for $u_i \in W = \mathbb{F}_q\{e_i^-\} \otimes W$, the action (12) is rewritten as

$$(33) \quad \begin{aligned} & [(z_1^+, z_1^-, \dots, z_N^+, z_N^-)](\{ (u_1, \dots, u_N) \}) = \\ & \psi\left(\sum_{i=1}^N B(z_i^+, u_i) + \frac{B(z_i^+, z_i^-)}{2}\right) \{ (u_1 + z_1^-, \dots, u_N + z_N^-) \} \end{aligned}$$

This expression will allow us to pick elements of $\mathbb{C}V \otimes W^{\mathrm{O}(W, B)}$ designed to act as generators of $\mathrm{Sp}(V)$.

Let us first consider $\dim(V) = 2$. On the one hand, in this case, (33) gives

$$(34) \quad [(z^+, z^-)](\{u\}) = \psi\left(B(z^+, u) + \frac{B(z^+, z^-)}{2}\right) \{u + z^-\}.$$

On the other hand, let us consider the action of the matrices in $\mathrm{SL}_2(\mathbb{F}_q)$ on the Schrödinger model of $\omega[\mathbb{F}_q^2 \otimes W]$, which as a vector space, is $\mathbb{C}W$. The matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

act on $\omega[\mathbb{F}_q^2 \otimes W] = \mathbb{C}W$ by transforming a basis element corresponding to a vector $u \in W$ by

$$(35) \quad \{u\} \mapsto \frac{1}{(-q)^{n/2}} \sum_{w \in W} \psi(B(-u, w)) \{w\}, \quad \{u\} \mapsto \psi\left(\frac{s}{2}B(u, u)\right) \{u\}$$

respectively. The matrices

$$\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$$

corresponding to $t \in \mathbb{F}_q^\times$ act on $\omega[\mathbb{F}_q^2 \otimes W] = \mathbb{C}W$ by transforming a basis element corresponding to $u \in W$ by

$$(36) \quad \{u\} \mapsto \epsilon_q(t) \{tu\}.$$

Returning to the action of the algebra $\text{End}_{O(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]) = (\mathbb{C}\mathbb{F}_q^2 \otimes W^{O(W,B)}, \star)$ on the oscillator representation, let us pick endomorphism elements

$$g_t = \frac{\epsilon_q(t)}{(-q)^{n/2}} \sum_{z \in W} \psi\left(\frac{tB(z, z)}{2}\right) [(z, tz)] \in \mathbb{C}\mathbb{F}_q^2 \otimes W^{O(W,B)}$$

corresponding to each $t \in \mathbb{F}_q^\times$. According to (34), these endomorphisms act on the basis element of the Schrödinger model $\omega[\mathbb{F}_q^2 \otimes W] = \mathbb{C}W$ corresponding to a vector $u \in W$ by

$$g_t(\{u\}) = \frac{\epsilon_q(t)}{(-q)^{n/2}} \sum_{z \in W} \psi(B(z, u) + B(z, tz)) \{u + tz\},$$

which, replacing $w = u + tz$, can be simplified to

$$\begin{aligned} & \frac{\epsilon_q(t)}{(-q)^{n/2}} \sum_{w \in W} \psi\left(B\left(\frac{w-u}{t}, w\right)\right) \{w\} = \\ & \frac{\epsilon_q(t)}{(-q)^{n/2}} \sum_{w \in W} \psi\left(\frac{B(w, w)}{t}\right) \psi\left(B\left(-\frac{u}{t}, w\right)\right) \{v\}. \end{aligned}$$

By comparing with the oscillator representation actions of the $\text{SL}_2(\mathbb{F}_q)$ elements given in (35) and (36), we therefore can identify that each g_t is precisely the endomorphism corresponding to the group action of the matrix

$$(37) \quad \begin{pmatrix} 1 & 0 \\ 2/t & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & t \\ -1/t & 2 \end{pmatrix}$$

on $\omega[W \otimes V]$ (note that $\epsilon_q(t) = \epsilon_q(1/t)$). Since the matrices (37) generate $\text{SL}_2(\mathbb{F}_q)$, we have therefore identified a subalgebra $\mathbb{C}\text{SL}_2(\mathbb{F}_q) \subseteq \text{End}_{O(W,B)}(\omega[\mathbb{F}_q^2 \otimes W])$, which must lie in the top part, since every

group element acts representation theoretically, and in particular, corresponds to an automorphism and cannot factor through the lower (degree 0) tensor power $1 = \omega[\mathbb{F}_q^2 \otimes W]^{\otimes 0}$.

Now consider the case of a general V , $\dim(V) = 2N$. We may find endomorphism algebra elements analogous to the g_t generating each subgroup $\mathrm{SL}_2(\mathbb{F}_q)$ corresponding to a choice of isotropic 1-dimensional subspace of V . These elements generate the full symplectic group $\mathrm{Sp}(V)$, and therefore we have identified a subalgebra

$$(38) \quad \mathbb{C}\mathrm{Sp}(V) \subseteq \mathrm{End}_{\mathrm{O}(W,B)}(\omega[V \otimes W]).$$

Again, since these endomorphisms encode the representation action of $\mathrm{Sp}(V) \subseteq \mathrm{Sp}(W \otimes V)$ on the oscillator representation $\omega[V \otimes W]$ and are, in particular, automorphisms, they are inexpressible as compositions factoring through a lower degree tensor power of $\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]) \cong \mathbb{C}W^-$. Hence, (38) in fact sits in the top part $\mathrm{End}_{\mathrm{O}(W,B)}^{\mathrm{top}}(\omega[V \otimes W])$.

□

Remark: The reason why we use the matrices (37) instead of a more common set of generators of $\mathrm{SL}_2(\mathbb{F}_q)$, such as (39), is because their corresponding elements g_t of $\mathbb{C}\mathbb{F}_q^2 \otimes W^{\mathrm{O}(W,B)}$ are fairly simple to write down. While not directly necessary to the logic of the proof of Proposition 4, it may still be instructive to write down the explicit formulas for the elements of $\mathrm{End}_{\mathrm{O}(W,B)}^{\mathrm{top}}(\omega[\mathbb{F}_q^2 \otimes W])$ which act on the Schrödinger model of $\omega[\mathbb{F}_q^2 \otimes W]$ as the matrices

$$(39) \quad \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q)$$

for $t, s \in \mathbb{F}_q^\times$.

These expressions involve quadratic sums of characters, whose calculation we recall now. We continue to write ψ for the fixed additive character of $\mathbb{F}_q$ defining $\omega[\mathbb{F}_q^2 \otimes W]$. Let us write $q = p^\ell$ for p an odd prime, and $\ell \in \mathbb{N}$. We denote the quadratic characters of $\mathbb{F}_q$ and $\mathbb{F}_p$ by

$$\epsilon_p : \mathbb{F}_p^\times \rightarrow \{\pm 1\}, \quad \epsilon_q : \mathbb{F}_q^\times \rightarrow \{\pm 1\},$$

respectively, which we extend by putting $\epsilon_p(0) = \epsilon_q(0) = 0$. Denoting the norm of the field extension by $N_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q^\times \rightarrow \mathbb{F}_p^\times$ and the trace by $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$, we have

$$(40) \quad \epsilon_p \circ N_{\mathbb{F}_q/\mathbb{F}_p} = \epsilon_q.$$

We may re-write the classical quadratic Gauss sum as

$$(41) \quad \sum_{n \in \mathbb{F}_p} e^{\frac{2\pi i}{p} n^2} = \sum_{m \in \mathbb{F}_p} (1 + \epsilon_p(m)) e^{\frac{2\pi i}{p} m} = \sum_{m \in \mathbb{F}_p} \epsilon_p(m) e^{\frac{2\pi i}{p} m}$$

(since the linear sum of characters is 0, and for each $m \in \mathbb{F}_p$, there are exactly $1 + \epsilon_p(m)$ elements in $\mathbb{F}_p$ whose square is m). It is well-known that

$$(42) \quad \sum_{n \in \mathbb{F}_p} e^{\frac{2\pi i}{p} a n^2} = \epsilon_p(a) \sqrt{\epsilon_p(-1)p}.$$

The same argument as (41) can also be applied to calculate

$$\sum_{x \in \mathbb{F}_q} \psi(x^2) = \sum_{x \in \mathbb{F}_q} e^{\frac{2\pi i}{p} \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x^2)} = \sum_{y \in \mathbb{F}_q} \epsilon_q(y) e^{\frac{2\pi i}{p} \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(y)}.$$

Applying the Hasse-Davenport relation to (42), this reduces to

$$(43) \quad (-1)^{\ell+1} \left(\sqrt{\epsilon_p(-1)p} \right)^\ell = \sum_{x \in \mathbb{F}_q} \psi(x^2) = (-1)^{\ell+1} \sqrt{\epsilon_q(-1)q}.$$

For $c \in \mathbb{F}_q^\times$, we have

$$\sum_{x \in \mathbb{F}_q} \psi(cx^2) = \epsilon_q(c) \sum_{x \in \mathbb{F}_q} \psi(x^2),$$

which, combined with (43), gives

$$\begin{aligned} \sum_{u \in W} \psi(cB(u, u)) &= \epsilon_q(c^n \det(B)) \left(\sum_{x \in \mathbb{F}_q} \psi(x^2) \right)^n = \\ &= (-1)^{n(\ell+1)} \text{disc}(B) \epsilon_q(c)^n (\epsilon_q(-1)q)^{n/2}, \end{aligned}$$

writing $\text{disc}(B) = \epsilon_q(\det(B))$ for the discriminant of B . We denote the constants by

$$(44) \quad \begin{aligned} K(c) &:= \sum_{u \in W} \psi(cB(u, u)) = \\ &= (-1)^{n(\ell+1)} \text{disc}(B) \epsilon_q(c)^n (\epsilon_q(-1)q)^{n/2}. \end{aligned}$$

Proposition 7. *Suppose we are given an orthogonal space (W, B) such that $(\text{SL}_2(\mathbb{F}_q), \text{O}(W, B))$ forms a reductive dual pair in the orthogonal stable range.*

(a) *For $t \neq 1 \in \mathbb{F}_q^\times$, the matrix*

$$\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \in \text{SL}_2(\mathbb{F}_q)$$

acting on the Schrödinger model of $\omega[\mathbb{F}_q^2 \otimes W]$ corresponds to the element

$$\alpha_t := \frac{\epsilon_q(t)}{q^n} \sum_{y^+, y^- \in W} \psi\left(-\frac{t+1}{2(t-1)} B(y^+, y^-)\right) [(y^+, y^-)]$$

of the endomorphism algebra, modeled as $(\mathbb{C}\mathbb{F}_q^2 \otimes W^{O(W,B)}, \star)$.

(b) The matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q)$$

acting on the Schrödinger model of $\omega[\mathbb{F}_q^2 \otimes W]$ corresponds to the element

$$\beta := \frac{1}{K(1)(-q)^{n/2}} \sum_{y^+, y^- \in W} \psi\left(\frac{B(y^+, y^+) + B(y^-, y^-)}{4}\right) [(y^+, y^-)]$$

of the endomorphism algebra, modeled as $(\mathbb{C}\mathbb{F}_q^2 \otimes W^{O(W,B)}, \star)$.

(c) For $s \in \mathbb{F}_q^\times$, the advection matrix

$$\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q)$$

acting on the Schrödinger model of $\omega[\mathbb{F}_q^2 \otimes W]$ corresponds to the element

$$\gamma_s := \frac{1}{K(\frac{-1}{2s})} \sum_{z \in W} \psi\left(-\frac{B(z, z)}{2s}\right) [(z, 0)]$$

of the endomorphism algebra, modeled as $(\mathbb{C}\mathbb{F}_q^2 \otimes W^{O(W,B)}, \star)$.

Proof of Proposition 7. Fix a vector $u \in W$ considered as a basis element of the Schrödinger model $\omega[\mathbb{F}_q^2 \otimes W] = \mathbb{C}W$. To prove each part, it suffices to verify the given endomorphism algebra element acts on $\{u\}$ the same as the proposed corresponding matrix, according to (35) and (36).

We begin with the proof of (a): For $t \in \mathbb{F}_q^\times$, we can calculate the following:

$$\begin{aligned} \alpha_t(\{u\}) &= \\ (45) \quad & \frac{\epsilon_q(t)}{q^n} \sum_{y^+, y^- \in W} \psi\left(\left(\frac{1}{2} - \frac{t+1}{2(t-1)}\right) B(y^+, y^-) + B(y^+, u)\right) \{y^- + u\} = \\ & \frac{\epsilon_q(t)}{q^n} \sum_{y^+, y^- \in W} \psi\left(B(y^+, -\frac{y^-}{t-1} + u)\right) \{y^- + u\}. \end{aligned}$$

The sum runs over arbitrary choices of y^+ , meaning that for fixed $u \in W$ and chosen $y^- \in W$, the coefficient sum

$$(46) \quad \sum_{y^+ \in W} \psi(B(y^+, -\frac{y^-}{t-1} + u))$$

of the vector $\{y^- + u\}$ is a linear sum of characters, and is therefore 0, unless $u = y^-/(t-1)$, in which case (46) is q^n , since each choice of $y^+ \in W$ (of which there are q^N contributes $\psi(0) = 1$. Hence, the only contributing choice of y^- is $y^- = (t-1)u$. Therefore, (45) simplifies as

$$\alpha_t(\{u\}) = \epsilon_q(t) \frac{q^n}{q^n} \{(t-1)u + u\} = \epsilon_q(t) \{tu\},$$

agreeing with the action (36) of the proposed matrix on the oscillator representation.

Now we prove (b): For $u \in W$, at each choice of $y^+, y^- \in W$, applying the corresponding term of the sum in β (disregarding the coefficient, for now) gives

$$\begin{aligned} & \psi\left(\frac{B(y^+, y^+) + B(y^-, y^-)}{4}\right) [(y^+, y^-)](\{u\}) = \\ & \psi\left(\frac{B(y^+, y^+) + 2B(y^+, y^-) + B(y^-, y^-)}{4} + B(y^+, u)\right) \{y^- + u\} = \\ & \psi\left(B\left(\frac{y^+ + y^-}{2}, \frac{y^+ + y^-}{2}\right) + B(y^+, u)\right) \{y^- + u\}. \end{aligned}$$

Therefore, we have

$$(47) \quad \beta(\{u\}) = \frac{1}{K(1) (-q)^{n/2}} \sum_{y^+, y^- \in W} \psi\left(B\left(\frac{y^+ + y^-}{2}, \frac{y^+ + y^-}{2}\right) + B(y^+, u)\right) \{y^- + u\}.$$

Renaming variables using $z = y^- + u$, we may rewrite this as

$$(48) \quad \frac{1}{K(1) (-q)^{n/2}} \sum_{y^+, z \in W} \psi\left(B\left(\frac{y^+ + z - u}{2}, \frac{y^+ + z - u}{2}\right) + B(y^+, u)\right) \{z\}.$$

Now we may also notice that

$$\begin{aligned} & B\left(\frac{y^+ + z - u}{2}, \frac{y^+ + z - u}{2}\right) + B(y^+, u) = \\ & B\left(\frac{y^+ + z + u}{2}, \frac{y^+ + z + u}{2}\right) - B(z, u), \end{aligned}$$

allowing us to rewrite (48) as

$$\frac{1}{K(1) (-q)^{n/2}} \sum_{z, y^+ \in W} \psi \left(B\left(\frac{y^+ + z + u}{2}, \frac{y^+ + z + u}{2}\right) - B(z, u) \right) \{z\}.$$

Renaming variables using $w = (y^+ + z + u)/2$ gives

$$\frac{1}{K(1) (-q)^{n/2}} \sum_{z, w \in W} \psi(B(w, w)) \psi(-B(z, u)) \{z\},$$

which, applying (44), reduces to

$$\beta(\{u\}) = \frac{1}{(-q)^{n/2}} \sum_{z \in W} \psi(-B(z, u)) \{z\},$$

which matches precisely the action of the proposed matrix according to (35).

Finally, we prove (c): For $u \in W$, $s \in \mathbb{F}_q^\times$,

$$(49) \quad \begin{aligned} \gamma_s(\{u\}) &= \frac{1}{K(\frac{-1}{2s})} \sum_{z \in W} \psi\left(-\frac{B(z, z)}{2s}\right) [(z, 0)](\{u\}) = \\ &= \frac{1}{K(\frac{-1}{2s})} \sum_{z \in W} \psi\left(-\frac{B(z, z)}{2s} + B(z, u)\right) \{u\}. \end{aligned}$$

Now we may notice that

$$\begin{aligned} B(z, u) - \frac{B(z, z)}{2s} &= -\frac{1}{2s}(-2s B(z, u) + B(z, z)) = \\ &= -\frac{1}{2s} (B(su - z, su - z) - s^2 B(u, u)) = \\ &= -\frac{1}{2s} B(su - z, su - z) + \frac{s}{2} B(u, u). \end{aligned}$$

Therefore, substituting $w = su - z$, we can rewrite (49) as

$$(50) \quad \gamma_s(\{u\}) = \frac{1}{K(\frac{-1}{2s})} \sum_{w \in W} \psi\left(-\frac{B(w, w)}{2s}\right) \psi\left(\frac{s}{2} B(u, u)\right) \{u\},$$

which by the definition (44) of the constants K then reduces to

$$\gamma_s(\{u\}) = \psi\left(\frac{s}{2} B(u, u)\right) \{u\},$$

agreeing precisely with the action of the proposed matrix according to (35). □

It may also be instructive to compute some examples of $\star$ applied to these elements, and see how it recovers matrix multiplication (especially to see the relationship between g_t , α_t , β , and $\gamma_{2/t}$). We do an example of such a computation in the Appendix.

3. COMBINATORICS

In the previous section, we were able to identify the group algebra of the orthogonal group $O(W, B)$ (resp. the symplectic group $Sp(V)$) in the top part of the $Sp(V)$ -equivariant (resp. $O(W, B)$ -equivariant) endomorphism algebra of $\omega[V \otimes W]$, for a dual pair $(Sp(V), O(W, B))$ in the symplectic (resp. orthogonal) stable range. Further, we note that what we consider to be the “lower” part of the endomorphism algebra of $\omega[V \otimes W]$ consists of maps factoring through a lower tensor power of $\omega[V]$ (i.e. the restriction of $\omega[V \otimes W[-\ell]]$ to $Sp(V)$ for some $0 \leq \ell \leq h_W$) in the case of the symplectic stable range, or through a lower tensor power of $\mathbb{C}W^-$ (i.e. the restriction of $\omega[V[-\ell] \otimes W]$ to $O(W, B)$ for $0 \leq \ell \leq \dim(V)/2$) in the case of the orthogonal stable range. This suggests that by proceeding inductively, fixing a choice of V and increasing the dimension of (W, B) within the symplectic stable range condition (resp. fixing a choice of (W, B) and increasing the dimension of V within the orthogonal stable range condition), we should be able to prove the proposed sum of matrix algebras in Part (A) (resp. Part (B)) of Theorem 2 is a subalgebra of $\text{End}_{Sp(V)}(\omega[V \otimes W])$ (resp. $\text{End}_{O(W, B)}(\omega[V \otimes W])$), with the top part corresponding to the $k = 0$ summand.

However, to complete this induction and prove that the proposed sum of matrix algebras in Parts (A), (B) match the claimed endomorphism algebras of $\omega[V \otimes W]$, we will use some combinatorial identities about the relationship between $\omega[V \otimes W]$ and lower $\omega[V \otimes W[-\ell]]$ (resp. $\omega[V[-\ell] \otimes W]$) for the symplectic (resp. orthogonal) stable statements. We will also see that the dimensions of the endomorphism algebras of the restricted oscillator representations match the dimensions of their claimed decompositions into matrix algebras of group algebras. Setting up these results is the purpose of this section.

The combinatorial result we need can be stated as the following:

Proposition 8. *Consider a reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$.*

- (1) *If $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the symplectic stable range, then for every choice of $0 \leq \ell \leq h_W$,*

$$(51) \quad \dim(\mathrm{Hom}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]], \omega[V \otimes W])) = \sum_{k=0}^{h_W - \ell} \frac{|\mathrm{O}(W[-\ell], B[-\ell])|}{|P_k^{B[-\ell]}|} \frac{|\mathrm{O}(W, B)|}{|P_{k+\ell}^B|} |\mathrm{O}(W[-k-\ell], B[-k-\ell])|.$$

- (2) *If $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is a reductive dual pair in the orthogonal stable range, then for every choice of $0 \leq \ell \leq \dim(V)/2$*

$$(52) \quad \dim(\mathrm{Hom}_{\mathrm{O}(W, B)}(\omega[V[-\ell] \otimes W], \omega[V \otimes W])) = \sum_{k=0}^{\dim(V)/2 - \ell} \frac{|\mathrm{Sp}(V[-\ell])|}{|P_k^{V[-\ell]}|} \frac{|\mathrm{Sp}(V)|}{|P_{k+\ell}^V|} |\mathrm{Sp}(V[-k-\ell])|.$$

In particular, in the case of $\ell = 0$, this statement reduces to the following

Corollary 1. *Consider a reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$.*

- (1) *If $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the symplectic stable range, then*

$$(53) \quad \dim(\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W])) = \sum_{k=0}^{h_W} |\mathrm{O}(W, B)/P_k^B|^2 |\mathrm{O}(W[-k], B[-k])|$$

- (2) *If $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the orthogonal stable range, then*

$$(54) \quad \dim(\mathrm{End}_{\mathrm{O}(W, B)}(\omega[V \otimes W])) = \sum_{k=0}^{\dim(V)/2} |\mathrm{Sp}(V)/P_k^V|^2 |\mathrm{Sp}(V[-k])|$$

This directly verifies that the dimensions of $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$ and $\mathrm{End}_{\mathrm{O}(W, B)}(\omega[V \otimes W])$ match the dimensions of the sums of matrix algebras proposed in Parts (A) and (B) of Theorem 2, in the case of the corresponding stable range.

The proof of Proposition 8 is the purpose of the remainder of this section. In Subsection 3.1, we state a combinatorial lemma on Gaussian binomial coefficients and use it to derive Proposition 8. In Subsection 3.2, we prove the lemma.

3.1. A key lemma. Let us write $Q_n = q^n + 1$. Recall the Gaussian binomial coefficients

$$(55) \quad \binom{a}{b}_q := \frac{(q^a - 1)(q^{a-1} - 1) \dots (q^{a-b+1} - 1)}{(q^b - 1)(q^{b-1} - 1) \dots (q - 1)}.$$

The purpose of this section is to prove the following

Lemma 9. *For every $p \in \mathbb{N}_0$, for every $r > b \in \mathbb{N}_0$,*

$$(56) \quad \prod_{i=b}^r Q_{i+p} = \sum_{a=0}^{\min(p, r-b+1)} q^{a(b+a-1)} \binom{r-b+1}{a}_q \prod_{i=p-a+1}^p (q^i - 1) \prod_{j=b+a}^r Q_j.$$

Assume Lemma 9 holds.

Proof of Proposition 8. Fix a symplectic space V and an orthogonal space (W, B) . We write $\dim(V) = 2N$, $\dim(W) = n$.

First, let us process the left hand sides of (51) and (52). Recalling (21) and that the tensor product of an oscillator representation with its dual oscillator representation is the permutation representation, for $0 \leq \ell \leq h_W$, we get

$$\text{Res}_{\text{Sp}(V)}(\omega[V \otimes W]) = \text{Res}_{\text{Sp}(V)}(\omega[V \otimes W[-\ell]]) \otimes (\mathbb{C}V)^{\otimes \ell}.$$

Therefore, for $0 \leq \ell \leq h_W$, we have

$$\text{Hom}_{\text{Sp}(V)}(\omega[V \otimes W[-\ell]], \omega[V \otimes W]) \cong \text{Hom}_{\text{Sp}(V)}(1, \mathbb{C}V^{\oplus(n-\ell)})$$

Similarly, recalling (22) and (23), for $0 \leq \ell \leq N$, we have

$$\text{Hom}_{\text{O}(W, B)}(\omega[V[-\ell] \otimes W], \omega[V \otimes W]) \cong \text{Hom}_{\text{O}(W, B)}(1, \mathbb{C}W^{\oplus 2N-\ell}).$$

Therefore, by Lemma 3, in the case of the symplectic stable range, we may rewrite the left hand side of (51) as

$$(57) \quad \dim(\text{Hom}_{\text{Sp}(V)}(\omega[V \otimes W[-\ell]], \omega[V \otimes W])) = 2 \prod_{i=1}^{n-\ell-1} Q_i,$$

and, in the case of the orthogonal stable range, we may rewrite the left hand side of (52) as

$$(58) \quad \dim(\text{Hom}_{\text{O}(W, B)}(\omega[V[-\ell] \otimes W], \omega[V \otimes W])) = \prod_{i=1}^{2N-\ell} Q_i.$$

In particular, also note that (58) is precisely half of (57) in the case of $n = 2N + 1$.

To reduce the right hand sides of (51) and (52), we need expressions for the group orders of finite orthogonal groups, finite symplectic groups, and their parabolic quotients. First, recall that

$$\begin{aligned}
 |\mathrm{O}_{2m+1}(\mathbb{F}_q)| &= 2q^{m^2} \prod_{i=1}^m (q^{2i} - 1) \\
 (59) \quad |\mathrm{O}_{2m}^\pm(\mathbb{F}_q)| &= 2q^{2\binom{m}{2}} (q^m \mp 1) \prod_{i=1}^{m-1} (q^{2i} - 1) \\
 |\mathrm{Sp}_{2N}(\mathbb{F}_q)| &= |\mathrm{O}_{2N+1}(\mathbb{F}_q)|/2 = q^{N^2} \prod_{i=1}^N (q^{2i} - 1)
 \end{aligned}$$

(recalling the notation that for $n = 2m$ even, $\mathrm{O}_{2m}^+(\mathbb{F}_q)$ denotes the orthogonal group $\mathrm{O}(W, B)$ when $h_W = m$ and $\mathrm{O}_{2m}^-(\mathbb{F}_q)$ denotes $\mathrm{O}(W, B)$ when $h_W = m - 1$).

We also recall that one can compute the orders of the parabolic quotients of orthogonal and symplectic groups as the prime to q part of the order of $\mathrm{O}(W, B)$ or $\mathrm{Sp}(V)$ divided by prime to q part of the order of the Levi subgroup corresponding to the parabolic. In the case of odd orthogonal groups $\dim(W) = n = 2m + 1$, writing P_k^{2m+1} for the maximal parabolic corresponding to a k -dimensional isotropic subspace,

$$(60) \quad \frac{|\mathrm{O}_{2m+1}(\mathbb{F}_q)|}{|P_k^{2m+1}|} = \frac{\prod_{i=m-k+1}^m (q^{2i} - 1)}{\prod_{i=1}^k (q^i - 1)} = \binom{m}{k}_q \prod_{j=m-k+1}^m Q_j.$$

In the case of even orthogonal groups $\dim(W) = 2m$, $\mathrm{O}(W, B) = \mathrm{O}_{2m}^\pm(\mathbb{F}_q)$, writing $P_k^{2m, \pm}$ for the maximal parabolic corresponding to a k -dimensional isotropic subspace,

$$(61) \quad \frac{|\mathrm{O}_{2m}^\pm(\mathbb{F}_q)|}{|P_k^{2m, \pm}|} = \frac{(q^m \mp 1)(q^{m-k} \mp 1) \prod_{i=m-k+1}^{m-1} (q^{2i} - 1)}{\prod_{i=1}^k (q^i - 1)}.$$

We can again re-write this in terms of binomial coefficients as

$$(62) \quad \begin{aligned} \frac{|O_{2m}^+(\mathbb{F}_q)|}{|P_k^{2m,+}|} &= \binom{m}{k}_q \prod_{i=m-k}^{m-1} Q_i, \\ \frac{|O_{2m}^-(\mathbb{F}_q)|}{|P_k^{2m,-}|} &= \binom{m-1}{k}_q \prod_{i=m-k+1}^m Q_i. \end{aligned}$$

The symplectic group formula can be similarly computed, and since $|\mathrm{Sp}_{2N}(\mathbb{F}_q)| = |\mathrm{SO}_{2N+1}(\mathbb{F}_q)|$, we have

$$|\mathrm{Sp}_{2N}(\mathbb{F}_q)/P_k^{2N}| = |\mathrm{O}_{2N+1}(\mathbb{F}_q)/P_k^{2N+1}|$$

(where P_k^{2N} on the left hand side denotes the parabolic in $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ corresponding to a k -dimensional isotropic subspace, while P_k^{2N+1} on the right hand side denotes the parabolic in $\mathrm{O}_{2N+1}(\mathbb{F}_q)$ corresponding to a k -dimensional isotropic subspace). In particular, combined with (59), this allows us to rewrite the right hand side of (52) entirely in terms of orders of odd orthogonal groups and their parabolic quotients. In fact, we find that this right hand side of (52) is precisely half of the right hand side of (51) at $n = 2N + 1$. Since we observed the same effect for the left hand side above, we therefore find that (52) is, as an expression, exactly (51) at $n = 2N + 1$ (in which case $N = h_W$, giving the same range of ℓ), multiplied by $1/2$. Thus, it suffices to prove Part (1) of the proposition.

Case 1: n is odd. Let us write $n = 2m + 1$, in which case m must be the number of hyperbolics h_W in B . In this case, for $k = 0, \dots, m - \ell$, we use the Gaussian binomial coefficient form of (60) to write

$$\frac{|O(W, B)|}{|P_{k+\ell}^B|} = \binom{m}{k+\ell}_q \prod_{j=m-k-\ell+1}^m Q_j,$$

and the form of (60) as a fraction to write

$$\frac{|O(W[-\ell], B[-\ell])|}{|P_k^{B[-\ell]}|} = \frac{\prod_{i=m-k-\ell+1}^{m-\ell} (q^i - 1) Q_i}{\prod_{i=1}^k (q^i - 1)}$$

(factoring each $q^{2i} - 1$ in the numerator as $(q^i - 1)Q_i$). Applying (59), again factoring each $q^{2i} - 1$ as $(q^i - 1)Q_i$, we also can write

$$|\mathcal{O}(W[-k-\ell], B[-k-\ell])| = 2q^{(m-k-\ell)^2} \prod_{i=1}^{m-k-\ell} (q^i - 1)Q_i.$$

Replacing each factor of the right hand side of (51) according to the above formulas and re-writing the left hand side of (51) according to (57), the claim of the proposition therefore has been reduced to

$$2 \prod_{i=1}^{2m-\ell} Q_i = \sum_{k=0}^{m-\ell} 2q^{(m-k-\ell)^2} \binom{m}{k+\ell}_q \frac{\prod_{i=1}^{m-\ell} (q^i - 1)Q_i}{\prod_{i=1}^k (q^i - 1)} \prod_{j=m-k-\ell+1}^m Q_j$$

Dividing both sides of this equation by $2Q_1 \dots Q_{m-\ell}$, we see that it is equivalent to the statement

$$\prod_{i=1}^m Q_{m-\ell+i} = \sum_{k=0}^{m-\ell} q^{(m-k-\ell)^2} \binom{m}{k+\ell}_q \prod_{i=k+1}^{m-\ell} (q^i - 1) \prod_{j=m-k-\ell+1}^m Q_j.$$

Replacing $\binom{m}{k+\ell}_q = \binom{m}{m-k-\ell}_q$, this is precisely the claim of Lemma 9 by putting $r = m$, $p = m - \ell$, $b = 1$ and substituting $a = m - k - \ell$.

Case 2: n is even. Let us write $n = 2m$.

Case 2A: Suppose B decomposes completely into hyperbolics, i.e. $h_W = m$. In this case, for $k = 0, \dots, m - \ell$, we apply the split case of (62) to write

$$\frac{|\mathcal{O}(W, B)|}{|P_{k+\ell}^B|} = \binom{m}{k+\ell}_q \prod_{j=m-k-\ell}^{m-1} Q_j.$$

Factoring each $q^{2i} - 1$ appearing in the numerator of the split case of (61) (and shifting indices), we can also write

$$\frac{|\mathcal{O}(W[-\ell], B[-\ell])|}{|P_k^{B[-\ell]}|} = \frac{\prod_{i=m-k-\ell}^{m-\ell-1} (q^{i+1} - 1)Q_i}{\prod_{i=1}^k (q^i - 1)}.$$

Finally, again factoring each $q^{2i} - 1$ (and shifting indices), we can also re-write the group order corresponding to the split case of (59) as

$$|\mathcal{O}(W[-k-\ell], B[-k-\ell])| = 2q^{2\binom{m-k-\ell}{2}}(q^{m-k-\ell} - 1) \prod_{i=1}^{m-k-\ell-1} (q^i - 1)Q_i.$$

Now by re-writing the right hand side of (51) according to the above formulas and re-writing the left hand side of (51) according to (57), we can reduce the claim of the proposition in this case to the equation

$$2 \prod_{i=1}^{2m-\ell-1} Q_i = \sum_{k=0}^{m-\ell} 2q^{2\binom{m-k-\ell}{2}} \binom{m}{k+\ell}_q \frac{(q^{m-\ell} - 1) \prod_{i=1}^{m-\ell-1} (q^i - 1)Q_i}{\prod_{i=1}^k (q^i - 1)} \prod_{j=m-k-\ell}^{m-1} Q_j$$

Dividing both sides of the above expression by $2Q_{m-\ell-1} \dots Q_1$, we find that it is equivalent to

$$\prod_{i=0}^{m-1} Q_{m-\ell+i} = \sum_{k=0}^{m-\ell} q^{2\binom{m-k-\ell}{2}} \binom{m}{k+\ell}_q \prod_{i=k+1}^{m-\ell} (q^i - 1) \prod_{j=m-k-\ell}^{m-1} Q_j.$$

Rewriting $\binom{m}{k+\ell}_q = \binom{m}{m-k-\ell}_q$, this is precisely the claim of Lemma 9 applied to $r = m-1$, $p = m-\ell$, $b = 0$, after substituting $a = m-k-\ell$.

Case 2B: Suppose B has a non-trivial anisotropic part, i.e. $h_W = m-1$. In this case, for $k = 0, \dots, m-1-\ell$, we apply the non-split case of (62) to write

$$\frac{|\mathcal{O}(W, B)|}{|P_{k+\ell}^B|} = \binom{m-1}{k+\ell}_q \prod_{j=m-k-\ell+1}^m Q_j.$$

Factoring each $q^{2i} - 1$ (and shifting indices), we re-express the non-split case of (61) as

$$\frac{|\mathcal{O}(W[-\ell], B[-\ell])|}{|P_k^{B[-\ell]}|} = \frac{\prod_{i=m-k-\ell+1}^{m-\ell} (q^{i-1} - 1)Q_i}{\prod_{i=1}^k (q^i - 1)}.$$

Finally, again by factoring each $q^{2i} - 1$ (and shifting indices), we express the group order corresponding to the non-split case of (59) as

$$|\mathcal{O}(W[-k-\ell], B[-k-\ell])| = 2q^{2\binom{m-k-\ell}{2}} Q_{m-k-\ell} \prod_{i=1}^{m-k-\ell-1} (q^i - 1) Q_i.$$

For a final time, by rewriting the right hand side of (51) according to the above formulas and rewriting the left hand side of (51) according to (57), we can claim of the proposition in this case to the equation

$$2 \prod_{i=1}^{2m-\ell-1} Q_i = \sum_{k=0}^{m-\ell-1} q^{2\binom{m-k-\ell}{2}} \binom{m-1}{k+\ell}_q \frac{Q_{m-\ell} \prod_{i=1}^{m-\ell-1} (q^i - 1) Q_i}{\prod_{i=1}^k (q^i - 1)} \prod_{j=m-k-\ell+1}^m Q_j$$

Similarly as in the previous cases, we may divide both sides of the above expression by $2Q_{m-\ell} \dots Q_1$, giving

$$\prod_{i=2}^m Q_{i+m-\ell-1} = \sum_{k=0}^{m-\ell-1} q^{2\binom{m-k-\ell}{2}} \binom{m-1}{k+\ell}_q \prod_{i=k+1}^{m-\ell-1} (q^i - 1) \prod_{j=m-k-\ell+1}^m Q_j.$$

Rewriting $\binom{m-1}{k+\ell}_q = \binom{m-1}{m-k-\ell-1}_q$, this statement is precisely the claim of Lemma 9 applied to $r = m$, $p = m - \ell - 1$, $b = 2$, after substituting $a = m - k - \ell - 1$. □

3.2. The proof of Lemma 9. The proof of Lemma 9 proceeds by induction. The argument is perhaps slightly unusual due to the fact that our formula does not reduce well at $q \rightarrow 1$ and is therefore not a “quantization” of a classical formula.

Proof of Lemma 9. To prove (56), we begin by rewriting

$$(63) \quad \prod_{i=b}^r Q_{i+p} = \prod_{i=b}^r Q_i + \sum_{k=b}^r (Q_{p+k} - Q_k) \prod_{j'=b}^{k-1} Q_{j'+p} \prod_{j=k+1}^r Q_j.$$

Now, for each $b \leq k \leq r$, we have $Q_{p+k} - Q_k = q^k(q^p - 1)$, so we may rewrite (63) as

$$(64) \quad \prod_{i=b}^r Q_{i+p} = \prod_{i=b}^r Q_i + \sum_{k=b}^r q^k(q^p - 1) \prod_{j'=b}^{k-1} Q_{j'+p} \prod_{j=k+1}^r Q_j.$$

Our goal is now to process each term of the right hand side of (64) to convert, step by step, the highest appearing $Q_{j'+p}$ factor into the

next smallest Q_j not yet appearing. In the statement of Lemma 9, all terms consist of multiples of products of the form

$$Q_r Q_{r-1} \dots Q_{b+a+1} Q_{b+a},$$

so we cannot skip any Q_j 's in the process. We make the following

Claim 2. *For each choice of $b \leq k \leq r$, we have*

$$(65) \quad q^k(q^p - 1) \prod_{j'=b}^{k-1} Q_{j'+p} \prod_{j=k+1}^r Q_j = \sum_{a=1}^{k-b+1} \left(\sum_{a+b-1 \leq \ell_1 \leq \dots \leq \ell_a = k} q^{\ell_1 + \dots + \ell_a} \right) \prod_{i=p-a+1}^p (q^i - 1) \prod_{j=b+a}^r Q_j.$$

Proof of Claim 2. We will proceed inductively, step by step, converting each factor $Q_{j'+p}$ in a term of the previous step's reduction of (65), starting with the largest appearing j' , into a sum of the next lower Q_j not yet appearing, with the appropriate error term of q^j multiplied by a factor $(q^{j'+p-j} - 1)$. This process will terminate in $k - b$ steps (we are already done when $k = b$).

The induction hypothesis is that after n steps, we will have reduced (65) to

$$(66) \quad \sum_{a=1}^{n+1} \left(\sum_{a-n+k-1 \leq \ell_1 \leq \dots \leq \ell_a = k} q^{\ell_1 + \dots + \ell_a} \right) \prod_{i=p-a+1}^p (q^i - 1) \prod_{j'=b}^{k-n-1} Q_{j'+p} \prod_{j=k-n+a}^r Q_j$$

Let us describe the first step of this process for (65). The largest appearing j' is $j' = k - 1$. The next lower Q_j factor not yet appearing is for $j = k$. Therefore, this step uses the replacement

$$Q_{k-1+p} = Q_k + q^k(q^{p-1} - 1).$$

This gives

$$\begin{aligned} & q^k(q^p - 1) \prod_{j'=b}^{k-2} Q_{j'+p} \prod_{j=k}^r Q_j + \\ & q^{k+k}(q^p - 1)(q^{p-1} - 1) \prod_{j'=b}^{k-2} Q_{j'+p} \prod_{j=k+1}^r Q_j \end{aligned}$$

which is precisely (66) at Step $n = 1$.

Suppose (66) holds at Step n . We now need to perform Step $(n+1)$. For $1 \leq a \leq n+1$, consider the corresponding term of (66) at Step n . The highest occurring $Q_{j'+p}$ is at $j' = k - n - 1$. The next lower Q_j factor not yet appearing is for $j = k - (n+1) + a$. Therefore, in this term, we must use the replacement

$$Q_{k-n-1+p} = Q_{k-(n+1)+a} + q^{k-(n+1)+a}(q^{p-a} - 1).$$

This replaces the term of (66) at Step n corresponding to a by

$$(67) \quad \begin{aligned} & \left(\sum_{a-n+k-1 \leq \ell_1 \leq \dots \leq \ell_a = k} q^{\ell_1 + \dots + \ell_a} \right) \prod_{i=p-a+1}^p (q^i - 1) \prod_{j'=b}^{k-n-2} Q_{j'+p} \prod_{j=k-(n+1)+a}^r Q_j + \\ & + \left(\sum_{a-n+k-1 = \ell_0 \leq \ell_1 \leq \dots \leq \ell_a = k} q^{\ell_0 + \ell_1 + \dots + \ell_a} \right) \prod_{i=p-a}^p (q^i - 1) \prod_{j'=b}^{k-n-2} Q_{j'+p} \prod_{j=k-n+a}^r Q_j. \end{aligned}$$

These terms appear in the $(n+1)$ th inductive step; the first one occurs in the expression (66) with n replaced by $n+1$ with no re-indexing of a or ℓ_i , and the second one occurs after replacing a by $a+1$ and shifting $\ell_0 \leq \dots \leq \ell_a$ to $\ell_1 \leq \dots \leq \ell_{a+1}$. Therefore, we may proceed inductively, and at Step $n = k - b$, we obtain the reduction (65). $\square$

In particular, we note that the claim implies for every $b \leq k \leq r$,

$$\begin{aligned} & \sum_{i=b}^k q^k (q^p - 1) \prod_{j=k+1}^r Q_j \prod_{j'=b}^{k-1} Q_{j'+p} = \\ & \sum_{a=1}^{k-b+1} \left(\sum_{a+b-1 \leq \ell_1 \leq \dots \leq \ell_a \leq k} q^{\ell_1 + \dots + \ell_a} \right) \prod_{i=p-a+1}^p (q^i - 1) \prod_{j=b+a}^r Q_j. \end{aligned}$$

Applying this in the case of $k = r$ and combining with (64), we obtain

$$\prod_{i=b}^r Q_{i+p} = \sum_{a=0}^{r-b+1} \left(\sum_{a+b-1 \leq \ell_1 \leq \dots \leq \ell_a \leq r} q^{\ell_1 + \dots + \ell_a} \right) \prod_{i=p-a+1}^p (q^i - 1) \prod_{j=b+a}^r Q_j$$

(note that the $a = 0$ term arises from the single term $Q_r \dots Q_b$ in (64)). If $r - b + 1 > p$, we also note that for terms of the above sum corresponding to $a > p$, the product $\prod_{i=p-a+1}^p (q^i - 1)$ will contain a 0 factor corresponding to $i = 0$, and therefore vanish. Hence, the above expression for $\prod_{i=1}^r Q_{i+p}$ can be reduced to

$$(68) \quad \sum_{a=0}^{\min(p, r-b+1)} \left(\sum_{a+b-1 \leq \ell_1 \leq \dots \leq \ell_a \leq r} q^{\ell_1 + \dots + \ell_a} \right) \prod_{i=p-a+1}^p (q^i - 1) \prod_{j=b+a}^r Q_j.$$

Finally, by the Gaussian binomial coefficient theorem, we can re-express the coefficients

$$\sum_{a+b-1 \leq \ell_1 \leq \dots \leq \ell_a \leq r} q^{\ell_1 + \dots + \ell_a} = q^{a(a+b-1)} \sum_{0 \leq \ell_1 \leq \dots \leq \ell_a \leq r-a-b+1} q^{\ell_1 + \dots + \ell_a} = q^{a(a+b-1)} \binom{r-b+1}{a}_q.$$

Plugging this into (68) gives (56). □

4. THE PROOF OF THEOREM 2

In this section, we conclude the proof of Theorem 2 using an inductive argument that proceeds completely similarly in both stable ranges.

Proof. We begin with Part (A). Fixing a symplectic space V , we proceed by induction on h_W (corresponding to (W, B) within the symplectic stable range). Suppose the $\mathrm{Sp}(V)$ -equivariant endomorphism algebra of $\omega[V \otimes W]$ decomposes as claimed for every choice of (W, B) with $h_W < m$. Now consider a choice of (W, B) with $h_W = m$.

For every $1 \leq \ell \leq m$, the induction hypothesis applies to $\omega[V \otimes W[-\ell]]$. Write, for $1 \leq \ell \leq m$,

$$Z_\ell = \bigoplus_{\rho \in \widehat{\mathrm{O}(W[-\ell], B[-\ell])}} \eta_{W[-\ell], B[-\ell]}^V(\rho) \otimes \rho$$

as an $\mathrm{Sp}(V) \times \mathrm{O}(W[-\ell], B[-\ell])$ -representation. The induction hypothesis gives that

$$\mathrm{CO}(W[-\ell], B[-\ell]) = \mathrm{End}_{\mathrm{Sp}(V)}(Z_\ell) = \mathrm{End}_{\mathrm{Sp}(V)}^{\mathrm{top}}(\omega[V \otimes W[-\ell]]),$$

and

$$\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]]) = \bigoplus_{k=0}^{h_W-\ell} \frac{|\mathrm{O}(W[-\ell], B[-\ell])|}{|P_k^{B[-\ell]}|} \mathrm{Res}_{\mathrm{Sp}(V)}(Z_{k+\ell})$$

(where the coefficient denotes a sum of copies).

First, we claim that as $\mathrm{Sp}(V)$ -representations, a sum of $|\mathrm{O}(W, B)/P_\ell^B|$ copies of $\mathrm{Res}_{\mathrm{Sp}(V)}(Z_\ell)$ appears in $\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$. Now this holds since by the induction hypothesis, we have

$$(69) \quad \mathrm{Hom}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]], \omega[V \otimes W]) = \bigoplus_{k=0}^{h_W-\ell} \frac{|\mathrm{O}(W[-\ell], B[-\ell])|}{|P_k^{B[-\ell]}|} \mathrm{Hom}_{\mathrm{Sp}(V)}(Z_{k+\ell}, \omega[V \otimes W])$$

(again, where the coefficient denotes a sum of copies). Applying Proposition 8, the dimension of the left hand side of (69) can be re-expressed as

$$\sum_{k=0}^{h_W - \ell} \frac{|\mathcal{O}(W[-\ell], B[-\ell])|}{|P_k^{B[-\ell]}|} \frac{|\mathcal{O}(W, B)|}{|P_{k+\ell}^B|} |\mathcal{O}(W[-k-\ell], B[-k-\ell])|$$

giving a linear system of equations for the dimensions of the Hom-spaces $\mathrm{Hom}_{\mathrm{Sp}(V)}(Z_{k+\ell}, \omega[V \otimes W])$ whose only solution is

$$\dim(\mathrm{Hom}_{\mathrm{Sp}(V)}(Z_k, \omega[V \otimes W])) = \frac{|\mathcal{O}(W, B)|}{|P_k^B|} |\mathcal{O}(W[-k], B[-k])|.$$

This gives the claim since this Hom-space is a free module over the endomorphism algebra $\mathrm{End}_{\mathrm{Sp}(V)}(Z_k) = \mathbb{C}\mathcal{O}(W[-k], B[-k])$, and we can consider

$$\mathrm{Hom}_{\mathrm{Sp}(V) \times \mathcal{O}(W[-k], B[-k])}(Z_k, \omega[V \otimes W]).$$

Since we have

$$(70) \quad \mathrm{Res}_{\mathrm{Sp}(V) \times \mathcal{O}(W[-k], B[-k])}(\omega[V \otimes W]) = \omega[V \otimes W[-k]] \otimes (\mathbb{C}V)^{\otimes k},$$

where $\mathcal{O}(W[-k], B[-k])$ acts trivially on the $\mathbb{C}V$ factors, the action of $\mathcal{O}(W[-k], B[-k])$ on each copy of Z_k is preserved and therefore, Z_k occurs as a summand of (73) with multiplicity $|\mathcal{O}(W, B)/P_k^B|$. Now, since we know that

$$\mathrm{Res}_{\mathrm{GL}_N(\mathbb{F}_q)}(\omega[V \otimes W]) = (\mathbb{C}\mathbb{F}_q^N) \otimes \epsilon(\det),$$

we get, by adjunction, summands

$$\bigoplus_{k=1}^{h_W} \bigoplus_{\rho \in \widehat{\mathcal{O}(W[-k], B[-k])}} \eta_{W[-k], B[-k]}^V(\rho) \otimes \mathrm{Ind}_{P_k^B}(\rho \otimes \epsilon(\det)),$$

as in the claimed expression (7). From the perspective of endomorphism algebras, we have

$$(71) \quad \bigoplus_{k=1}^{h_W} M_{|\mathcal{O}(W, B)/P_k^B|}(\mathbb{C}\mathcal{O}(W[-k], B[-k])) \subseteq \mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W]),$$

all occurring independently from the top part, since they specifically all arise from terms appearing in $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-k]])$. Combining with Proposition 4 gives the $k = 0$ term as well:

$$\bigoplus_{k=0}^{h_W} M_{|\mathcal{O}(W, B)/P_k^B|}(\mathbb{C}\mathcal{O}(W[-k], B[-k])) \subseteq \mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W]).$$

This must be an equality, however, since the dimensions match by Corollary 1. Therefore, we have exactly enough room for $\omega[V \otimes W]$ to decompose as (71) plus a sum of the form

$$\bigoplus_{\rho \in \widehat{\mathrm{O}(W, B)}} \eta_{W, B}^V(\rho) \otimes \rho,$$

for new irreducible $\mathrm{Sp}(V)$ -representations $\eta_{W, B}^V(\rho)$ not isomorphic to the output of any lower eta correspondence $\eta_{W[-k], B[-k]}^V$. (We know each of these new $\eta_{W, B}^V(\rho)$ is non-zero since this is the only possible way to obtain the full group algebra $\mathbb{CO}(W, B)$ in the top part of the endomorphism algebra.)

Similarly, to prove Part (B), we fix (W, B) and proceed by induction on $N = \dim(V)/2$. Suppose the claimed decomposition of the $\mathrm{O}(W, B)$ -equivariant endomorphism algebra of $\omega[V \otimes W]$ holds for every symplectic space V of dimension $2M$ with $M < N$. Now consider a symplectic space V of dimension $2N$.

For every $1 \leq \ell \leq N$, the induction hypothesis applies to $\omega[V[-\ell] \otimes W]$. Write, for $1 \leq \ell \leq N$,

$$Y_\ell = \bigoplus_{\rho \in \widehat{\mathrm{Sp}(V[-\ell])}} \rho \otimes \zeta_{V[-\ell]}^{W, B}(\rho)$$

as an $\mathrm{Sp}(V[-\ell]) \times \mathrm{O}(W, B)$ -representation. The induction hypothesis gives that

$$\mathbb{C}\mathrm{Sp}(V[-\ell]) = \mathrm{End}_{\mathrm{O}(W, B)}(Y_\ell) = \mathrm{End}_{\mathrm{O}(W, B)}^{\mathrm{top}}(\omega[V[-\ell] \otimes W]),$$

and

$$\mathrm{Res}_{\mathrm{O}(W, B)}(\omega[V[-\ell] \otimes W]) = \bigoplus_{k=0}^{h_W - \ell} \frac{|\mathrm{Sp}(V[-\ell])|}{|P_k^{V[-\ell]}|} \mathrm{Res}_{\mathrm{O}(W, B)}(Y_{k+\ell}).$$

Again, we begin by arguing that as $\mathrm{O}(W, B)$ -representations, a sum of $|\mathrm{Sp}(V)/P_\ell^V|$ copies of $\mathrm{Res}_{\mathrm{O}(W, B)}(Y_\ell)$ appears in $\mathrm{Res}_{\mathrm{O}(W, B)}(\omega[V \otimes W])$, which holds by applying the induction hypothesis to get

$$(72) \quad \begin{aligned} & \mathrm{Hom}_{\mathrm{O}(W, B)}(\omega[V[-\ell] \otimes W], \omega[V \otimes W]) = \\ & \bigoplus_{k=0}^{h_W - \ell} \frac{|\mathrm{Sp}(V[-\ell])|}{|P_k^{V[-\ell]}|} \mathrm{Hom}_{\mathrm{O}(W, B)}(Y_{k+\ell}, \omega[V \otimes W]). \end{aligned}$$

Applying Proposition 8, the dimension of the left hand side of (72) can be re-expressed as

$$\sum_{k=0}^{h_W-\ell} \frac{|\mathrm{Sp}(V[-\ell])|}{|P_k^{V[-\ell]}|} \frac{|\mathrm{Sp}(V)|}{|P_{k+\ell}^V|} |\mathrm{Sp}(V[-k-\ell])|$$

giving a linear system of equations for the dimensions of the Hom-spaces $\mathrm{Hom}_{\mathrm{O}(W,B)}(Y_{k+\ell}, \omega[V \otimes W])$ whose only solution is

$$\dim(\mathrm{Hom}_{\mathrm{O}(W,B)}(Y_\ell, \omega[V \otimes W])) = \frac{|\mathrm{Sp}(V)|}{|P_\ell^V|} |\mathrm{Sp}(V[-\ell])|.$$

We then obtain the claim since this Hom-space is a free module over the endomorphism algebra $\mathrm{End}_{\mathrm{O}(W,B)}(Y_k) = \mathbb{C}\mathrm{Sp}(V[-k])$, and we can consider

$$\mathrm{Hom}_{\mathrm{Sp}(V[-k]) \times \mathrm{O}(W,B)}(Y_k, \omega[V \otimes W]).$$

Since we have

$$(73) \quad \mathrm{Res}_{\mathrm{Sp}(V[-k]) \times \mathrm{O}(W,B)}(\omega[V \otimes W]) = \omega[V[-k] \otimes W] \otimes (\mathbb{C}W)^{\otimes k},$$

where $\mathrm{Sp}(V[-k])$ acts trivially on the $\mathbb{C}W$ factors, the action of $\mathrm{Sp}(V[-k])$ on each copy of Y_k is preserved and therefore, Y_k occurs as a summand of (73) with multiplicity $|\mathrm{Sp}(V)/P_k^V|$. Again, by adjunction, we get summands

$$(74) \quad \bigoplus_{k=1}^N \bigoplus_{\rho \in \widehat{\mathrm{Sp}(V[-k])}} \mathrm{Ind}_{P_k^V}(\rho \otimes \epsilon(\det)) \otimes \zeta_{V[-k]}^{W,B}(\rho),$$

as in the claimed expression (8). From the perspective of endomorphism algebras, we have

$$\bigoplus_{k=1}^N M_{|\mathrm{Sp}(V)/P_k^V|}(\mathbb{C}\mathrm{Sp}(V[-k])) \subseteq \mathrm{End}_{\mathrm{O}(W,B)}(\omega[V \otimes W]),$$

all occurring independently from the top part, since they specifically all arise from terms appearing in $\mathrm{End}_{\mathrm{O}(W,B)}(\omega[V[-k] \otimes W])$. As before, combining with Proposition 4 gives that, in fact,

$$\bigoplus_{k=0}^N M_{|\mathrm{Sp}(V)/P_k^V|}(\mathbb{C}\mathrm{Sp}(V[-k])) \subseteq \mathrm{End}_{\mathrm{O}(W,B)}(\omega[V \otimes W])$$

This must be an equality, again, since the dimensions match by Corollary 1. Finally, then, we have exactly enough room for $\omega[V \otimes W]$ to decompose as (74) plus a sum of the form

$$\bigoplus_{\rho \in \widehat{\mathrm{Sp}(V)}} \rho \otimes \zeta_V^{W,B}(\rho),$$

for new irreducible $O(W, B)$ -representations $\zeta_V^{W,B}(\rho)$ not isomorphic to the output of any lower zeta correspondence $\zeta_{V[-k]}^{W,B}$. (We know each of these new $\zeta_V^{W,B}(\rho)$ is non-zero since this is the only possible way to obtain the full group algebra $\mathbb{C}\text{Sp}(V)$ in the top part of the endomorphism algebra.) $\square$

APPENDIX A. AN EXPLICIT COMPOSITION COMPUTATION

Fix $t \in \mathbb{F}_q^\times$. In this appendix, we return to the context of 2-dimensional symplectic spaces, and verify a relation between the endomorphisms of the oscillator representation implied by their correspondence with elements of the group algebra $\mathbb{C}\text{SL}_2(\mathbb{F}_q)$. Fix a scalar $t \in \mathbb{F}_q^\times$. We have the following relation of matrices

$$\begin{pmatrix} 0 & t \\ -1/t & 2 \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2/t & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in $\text{SL}_2(\mathbb{F}_q)$. The purpose of this appendix is to complete the calculation that the corresponding relation

$$(75) \quad g_t \star \alpha_t = \gamma_{2/t} \star \beta$$

holds in $\text{End}_{O(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]) = (\mathbb{C}\mathbb{F}_q^2 \otimes W^{O(W,B)}, \star)$.

The composition $g_t \star \alpha_t$ is $\frac{\epsilon(t)^2}{q^n(-q)^{n/2}} = \frac{1}{q^n(-q)^{n/2}}$ times the sum over all choices of $y^+, y^-, z \in W$ of terms

$$(76) \quad \psi\left(\frac{tB(z, z)}{2} - \frac{t+1}{2(t-1)}B(y^+, y^-)\right) [(z, tz)] \star [(y^+, y^-)].$$

Writing out

$$[(z, tz)] \star [(y^+, y^-)] = \psi\left(\frac{B(z, y^-) - tB(z, y^+)}{2}\right) [(z + y^+, tz + y^-)],$$

each term (76) can be simplified to the pair of vector $[(y^+ + z, tz + y^-)]$ multiplied by the coefficient

$$\psi\left(\frac{tB(z, z)}{2} - \frac{t+1}{2(t-1)}B(y^+, y^-) + \frac{B(z, y^-)}{2} - \frac{tB(z, y^+)}{2}\right).$$

By considering

$$-\frac{t+1}{2(t-1)} = \frac{1}{2} - \frac{t}{t-1}, \quad -\frac{t}{2} = \frac{t}{2} - t,$$

this can be rewritten as

$$\psi\left(\frac{B(y^+ + z, y^- + tz)}{2} - \frac{t B(y^+, y^- + (t-1)z)}{t-1}\right).$$

Substituting $u = y^+ + z$, $v = y^- + tz$ gives

$$\psi\left(\frac{B(u, v)}{2} - \frac{t B(u - z, v - z)}{t-1}\right).$$

Therefore, we have reduced $g_t \star \alpha_t$ to

$$(77) \quad \frac{1}{q^n(-q)^{n/2}} \sum_{z, u, v \in W} \psi\left(\frac{B(u, v)}{2} - \frac{t B(u - z, v - z)}{t-1}\right) [(u, v)].$$

Writing

$$B(u - z, v - z) = B(u, v) - B(u + v, z) + B(z, z),$$

we may “complete the square” by noticing that

$$\begin{aligned} -B(u + v, z) + B(z, z) &= \\ B\left(z - \frac{u + v}{2}, z - \frac{u + v}{2}\right) - B\left(\frac{u + v}{2}, \frac{u + v}{2}\right). \end{aligned}$$

Substituting variables using $w = z - \frac{u+v}{2}$, putting the terms together, we get

$$\begin{aligned} B(u - z, v - z) &= B(u, v) + B(w, w) - B\left(\frac{u + v}{2}, \frac{u + v}{2}\right) = \\ &B(w, w) - B\left(\frac{u - v}{2}, \frac{u - v}{2}\right). \end{aligned}$$

Therefore, (77) reduces to

$$\begin{aligned} (78) \quad &g_t \star \alpha_t = \\ &\frac{1}{q^n(-q)^{n/2}} \sum_{w, u, v \in W} \psi\left(\frac{B(u, v)}{2} - \frac{t(B(w, w) - B(\frac{u-v}{2}, \frac{u-v}{2}))}{t-1}\right) [(u, v)] = \\ &\frac{K(-t/(t-1))}{q^n(-q)^{n/2}} \sum_{u, v \in W} \psi\left(\frac{B(u, v)}{2} + \frac{t}{t-1} B\left(\frac{u - v}{2}, \frac{u - v}{2}\right)\right) [(u, v)]. \end{aligned}$$

Now let us consider the other side of (75). The composition $\gamma_{2/t} \star \beta$ is $\frac{1}{(-q)^{n/2} K(-t/4) K(1)}$ times the sum over all choices of $y^+, y^-, z \in W$ of terms

$$\psi\left(-\frac{tB(z, z)}{4} + \frac{B(y^+, y^+) + B(y^-, y^-)}{4}\right) [(z, 0)] \star [(y^+, y^-)]$$

Writing out

$$[(z, 0)] \star [(y^+, y^-)] = \psi\left(\frac{B(z, y^-)}{2}\right) [(z + y^+, y^-)],$$

this term is the basis element $[(z + y^+, y^-)]$ multiplied by the coefficient

$$\begin{aligned} & \psi\left(-\frac{tB(z, z)}{4} + \frac{B(y^+, y^+) + B(y^-, y^-)}{4} + \frac{B(z, y^-)}{2}\right) = \\ & \psi\left(-tB\left(\frac{z}{2}, \frac{z}{2}\right) + B\left(\frac{y^+ - y^-}{2}, \frac{y^+ - y^-}{2}\right) + \frac{B(z + y^+, y^-)}{2}\right) \end{aligned}$$

Substituting variables $u = z + y^+$, $v = y^-$, $w = z/2$ we reduce $\gamma_{2/t} \star \beta$ to the coefficient $\frac{1}{(-q)^{n/2} K(-t/4) K(1)}$ times

$$\sum_{u, v, w \in W} \psi\left(-tB(w, w) + B\left(\frac{u - v}{2} - w, \frac{u - v}{2} - w\right) + \frac{B(u, v)}{2}\right) [(u, v)].$$

Writing

$$\begin{aligned} & B\left(\frac{u - v}{2} - w, \frac{u - v}{2} - w\right) = \\ & B\left(\frac{u - v}{2}, \frac{u - v}{2}\right) - 2B\left(\frac{u - v}{2}, w\right) + B(w, w), \end{aligned}$$

we have

$$\begin{aligned} & -tB(w, w) + B\left(\frac{u - v}{2} - w, \frac{u - v}{2} - w\right) + \frac{B(u, v)}{2} = \\ & -(t - 1)B(w, w) - 2B\left(\frac{u - v}{2}, w\right) + B\left(\frac{u - v}{2}, \frac{u - v}{2}\right) + \frac{B(u, v)}{2}. \end{aligned}$$

Completing the square gives

$$\begin{aligned} & -(t - 1)\left(B(w, w) + \frac{2}{t - 1}B\left(\frac{u - v}{2}, w\right)\right) = \\ & -(t - 1)\left(B\left(w + \frac{u - v}{2(t - 1)}, w + \frac{u - v}{2(t - 1)}\right) - \frac{1}{(t - 1)^2}B\left(\frac{u - v}{2}, \frac{u - v}{2}\right)\right). \end{aligned}$$

Therefore, replacing variables $x = w + \frac{1}{2(t - 1)}(u - v)$ gives

$$\begin{aligned} & -tB(w, w) + B\left(\frac{u - v}{2} - w, \frac{u - v}{2} - w\right) + \frac{B(u, v)}{2} = \\ & -(t - 1)B(x, x) + \left(1 + \frac{t - 1}{(t - 1)^2}\right)B\left(\frac{u - v}{2}, \frac{u - v}{2}\right) + \frac{B(u, v)}{2} = \\ & -(t - 1)B(x, x) + \frac{t}{t - 1}B\left(\frac{u - v}{2}, \frac{u - v}{2}\right) + \frac{B(u, v)}{2} \end{aligned}$$

Thus, $\gamma_{2/t} \star \beta$ is the factor $\frac{1}{(-q)^{n/2} K(-t/4) K(1)}$ times

$$\sum_{u,v,x \in W} \psi(-(t-1)B(x,x) + \frac{t}{t-1}B(\frac{u-v}{2}, \frac{u-v}{2}) + \frac{B(u,v)}{2}) [(u,v)] =$$

$$K(-(t-1)) \sum_{u,v \in W} \psi(\frac{t}{t-1}B(\frac{u-v}{2}, \frac{u-v}{2}) + \frac{B(u,v)}{2}) [(u,v)].$$

This agrees with our above calculation of $g_t \star \alpha_t$ in (78), up to a constant. It remains to check that the constants precisely agree, i.e.

$$(79) \quad \frac{K(-(t-1))}{(-q)^{n/2} K(-\frac{t}{4}) K(1)} = \frac{K(-\frac{t}{t-1})}{q^n (-q)^{n/2}}.$$

Recalling (44), first note that since $K(c)$ only depends on $\epsilon_q(c)$, we have $K(-t/4) = K(-t)$. We can therefore simplify (79) to

$$q^n K(-(t-1)) = K(-\frac{t}{t-1}) K(-t) K(1).$$

Next, the signs, i.e. the factors $(-1)^{n(\ell+1)} \text{disc}(B)$ in each K factor will cancel, since both the left and right hand side have an odd number of K factors. Further, collecting factors, both sides have a factor of $q^n (-q)^{n/2}$, which we may factor out. This reduces the claim to

$$\epsilon_q(-(t-1))^n \epsilon_q(-1)^{n/2} = \epsilon_q(-\frac{t}{t-1})^n \epsilon_q(-t)^n \epsilon_q(-1)^{3n/2}.$$

Dividing both sides by $\epsilon_q(-1)^{n/2}$ and collecting terms gives

$$\epsilon_q(-(t-1))^n = \epsilon_q(\frac{-t}{t-1}(-t)(-1))^n,$$

which holds, since $\epsilon_q(-(t-1)) = \epsilon_q(-\frac{1}{t-1})$.

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