Howe duality for finite fields: The endomorphism algebra method

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Introduction

The goal of this monograph is to tell the story of a certain approach to the representation theory, say, of finite groups, over the complex numbers: the structures carried by an object's endomorphism algebra. On some level, the idea of understanding a representation through its endomorphism algebra is quite familar. One of the most basic tools of the subject is, of course, Schur's Lemma, which allows us to use the dimension of a representation's endomorphism space to determine whether it is irreducible or not. But this only takes into account the endomorphisms as a vector space, when they naturally carry quite a bit more structure. We may, for example, further consider the algebra arising from the composition operation on endomorphisms. We can also consider the tensor products of endomorphisms in a tensor power of a given representation and partial traces: this gives rise to a structure called T-algebra, which can be used to study not just representations, but also, in greater generality, symmetric tensor categories with strong duality.

Back to the endomorphism algebra of a representation of a finite group, by decomposing it into matrix algebras, we obtain information such as multiplicities of the non-isomorphic irreducible summands appearing in its decomposition. Idempotent endomorphisms allow us to pick out individual summands. The trace operation on the endomorphism algebra, therefore, also yields information about the dimensions of each irreducible summand in the decomposition. Additionally, certain contexts allow us to consider several kinds of endomorphisms at the same time. For example, we may consider the endomorphisms of a certain representation which may only be equivariant of a subgroup of the original acting group, or we may consider endomorphisms of a tensor power of the original representation. Approaching a representation from these various perspectives allows us to learn deep information about its structure, through the relatively elementary steps of understanding an algebra structure.

This method, while quite intuitive, may be applied to solve interesting and non-trivial questions. In this monograph, we present, as a kind of a case study, an application of the endomorphism method to the finite field Howe duality problem [**27**] for finite groups of Lie type with focus on type B, C, D (type A being simpler and more classically understood). This question concerns the behavior of certain representations called the *oscillator* or *Weil representations* [**57**] of symplectic groups over finite fields. Specifically, one considers products of pairs of certain subgroups embedded into a given finite symplectic group and the decomposition of the restriction of an oscillator representation to these products. The oscillator representation is itself quite special, since it tensor generates every representation of the symplectic group, while also being "miraculously small." The question of finite field Howe duality asks whether we can obtain a relationship between the representation theories of the fixed subgroups by examining the tensor products of each's irreducible representations appearing in the restricted oscillator representation. We recall some history of this problem now:

The question was first introduced by R. Howe in [**27**] and was discussed in the papers [**24, 25**] by S. Gurevich and Howe, bringing to light the finite field case of the Howe duality conjecture [**30**] for locally compact fields. In rough terms, the correspondence concerns the decomposition of the restriction of the oscillator representation of a finite symplectic group to a product of a symplectic and an orthogonal group which are each other's centralizers. More broadly, this can be discussed for general reductive dual pairs, consisting of any pair of subgroups which are each other's centralizers in a finite symplectic group.

Early progress on this program for the case of finite fields was made by J. Adams and A. Moy [**1**], who described the behavior of unipotent cuspidal representations under the Howe correspondence between symplectic and even orthogonal groups, using previous results of S. Kudla [**39**] for the theta correspondence on local fields, which used the Jacquet functor. Another important advance was the paper [**3**] by A.-M. Aubert, J. Michel, and R. Rouquier, who conjectured the behavior of the Howe correspondence on general *unipotent representations* (i.e. those appearing in the Deligne-Lusztig induction of a trivial character), and proved their conjecture for unitary and general linear groups (i.e. the "type II case" of Howe duality). Their conjecture for *type I dual pairs*, i.e. symplectic and orthogonal groups, was proved by S.-Y. Pan [**50**] using the concept of a *uniform projection* (a character is called uniform if it is a linear combination of virtual Deligne-Lusztig characters). This geometric approach was also used by D. Liu and Z. Wang in [**42**] to extend the results of Adams and Moy to the case of reductive dual pairs involving odd orthogonal groups. Pan [**51**] went further and, by proving suitable compatibility results, was able to determine which pairs of irreducible representations occur in the type I Howe duality correspondence with non-zero multiplicity.

In the present book, we approach this problem using a different method. We study the Howe duality correspondence directly and explicitly by examining the endomorphism algebras of tensor products of oscillator representations over finite fields. In these terms, we describe explicitly the so-called η*-correspondence* defined by Gurevich and Howe [**24, 25**], and also define and similarly describe a complementary "ζ-correspondence." Having a hands-on description of the irreducible representations occuring in the Howe duality correspondence gives us more precise information. For example, as a payoff, we are able to explicitly demonstrate that the pairs identified by Pan [**51**] always occur with multiplicity 1. Using the organization of the oscillator representation's decomposition in terms of the eta and zeta correspondences, we obtain a recursive formula for the characters of the unipotent cuspidal representations, which were still quite mysterious up to this point. We also prove the type C case of the Gurevich-Howe rank conjecture (Conjecture 0.3.8 of [**25**]), the type A case of which was proved R. M. Guralnick, M. Larsen, and P. H. Tiep [**22**], and the type B and D cases of which were proved by Larsen and Tiep in [**41**].

This approach hinges on the fact that the problem of finite field Howe duality is particularly suited for endomorphism algebra methods, since the oscillator representations have very nice endomorphism algebras and the question fundamentally is a decomposition question about their restrictions. The endomorphism algebra method allows us to solve this question quite completely and concretely, thus exhibiting a successful application of the method. By using this more hands-on method not truly passing through geometry, we gain new understanding of not only the finite field Howe duality problem, but the structure of irreducible representations of symplectic and orthogonal groups at large. For example, our results giving a recursive construction of the unipotent "cuspidal" representations of a symplectic or orthogonal group in terms of oscillator representation characters (which can be fully computed using the Schrödinger model).

As is the case with many mathematical areas of study, an enormous amount of background information on this topic is known at this point, which may not be easy to find by a general audience reader. In writing this book, I aimed to introduce the backgound material in as self-contained a way as possible, while adding new observations made by our method. Such background topics include the oscillator representation in the form used here [**27**] and Lusztig's classification [**44, 45**] of irreducible representations of finite groups of Lie type, coming from Deligne-Lusztig theory [**9**].

We recall the construction of the oscillator representations, and some of their special properties in the finite field context, in Chapter 1. We especially focus on their duality and the structure of their endomorphism algebras. We also recall R. Howe's theory of reductive dual pairs and begin considering what a "Howe duality-type result" will look like in the finite field context, and why it must be more refined than the other forms of Howe duality which were previously considered. For the remainder of the book past this point, we primarily focus on the case of type I reductive dual pairs, consisting of symplectic and orthogonal groups.

Chapter 2 is primarily focused on "stable range" results, meaning they consider type I reductive dual pairs of the form $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ where either V or W is of dimension less than or equal to the maximal dimension of an isotropic subspace of the other. We call these stable ranges the orthogonal and symplectic stable ranges, respectively. In these ranges, we observe certain "stability" effects in the endomorphism algebra structures we began computing in Chapter 1. Examining these endomorphism algebras and their invariants allows us to formulate our first statements about the decomposition of oscillator representations restricted to reductive dual pairs. These decompositions can be completely stated in terms of certain correspondences between the irreducible representations of symplectic and orthogonal groups and parabolic inductions. In the case of pairs in the symplectic stable range, meaning $dim(W) \leq dim(V)/2$, we recover the eta correspondence of R. Howe and S. Gurevich from $\mathrm{Sp}(V)$-representations to $\mathrm{O}(W,B)$-representations, while in the mirror orthogonal stable range, we find a new system of injective correspondences in the opposite direction - producing irreducible orthogonal group representations from irreductible symplectic group representations. Let us call this the "zeta correspondence."

In Chapter 3, we recall the statements of Lusztig's classification of irreducible representations as they apply concretely to representations of orthogonal and symplectic groups over finite fields. In Chapter 4, we use Lusztig's classification to give an explicit description of the eta and zeta correspondences in the stable range, obtaining a full explicit decomposition of restricted oscillator representations to stable reductive dual pairs. However, there is a significant gap of possible reductive dual pairs which fit into neither of the stable ranges. These "unstable" pairs are the focus of Chapters 5 through 7.

In Chapter 5, we digress into the theory of "interpolated" representation categories, which can be used as a tool to extend certain structural results within a collection of categories with certain structure ressembling representation theory (e.g. endomorphism algebra structure, closed polynomial formulas for dimensions in terms of a running parameter, etc.). Started by P. Deligne [**7, 8**], and continued by many other authors [**26, 34, 35**], interpolation of tensor categories is a large topic of major independent interest. In the context of the present book, we use a certain aspect of interpolation theory. In particular, we form an interpolation category encoding representation theory of a symplectic group generated by oscillator representations, with a complex running parameter taken to be the dimension of one of the oscillator representations.

In Chapter 6, we see how the explicit stable results of Chapters 2 and 4 correspond to statements in this interpolated category. We use these interpolated categories to extend the symplectic and orthogonal stable results to larger ranges which we call the symplectic and orthogonal metastable ranges. These ranges cover every choice of type I reductive dual pair, and are roughly separated by the "middle case" where the rank of the symplectic group is equal to the rank of the orthogonal group (the exact range condition is technical, dependent on whether the orthogonal group is split or non-split). We obtain a decomposition of a restricted oscillator representation involving certain alternating sums of parabolic inductions, tensored with the eta or zeta correspondence. In Chapter 7, we resolve these alternating sums. We also note that the correspondences give an explicit computation of the irreducible unipotent cuspidal representations of symplectic and orthogonal groups, constructing them through iterated applications of the eta and zeta correspondences.

Finally, in Chapter 8, we describe an application of our decomposition statements to the representation theory of finite groups of Lie type. Our explicit metastable computation of the eta correspondence allows us to prove the Gurevich-Howe rank conjecture for finite symplectic groups.

I would like to thank P. Deligne, S. Gurevich, R. Howe, P. Sarnak, P. H. Tiep, and J. Zou for much encouragement, inspiration, and helpful conversations.

CHAPTER 1

Background: The oscillator representation and its basic properties

The oscillator representation (also known as the Weil representation [**57**]) is the basic building block of the objects we will study. The purpose of this chapter is to introduce the oscillator representations of finite symplectic groups and set up the structural results about those representations we will need to apply endomorphism algebra methods. We assume that the reader is familiar with the basic theory of complex representations of finite groups. A good supplementary reference is [**53**].

The basic idea of the oscillator representation is to first look at the Heisenberg group $\mathbb{H}_N(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ for a prime power $q = p^n$ for $p \neq 2$ (which for $q = p$ is one of the two types of extraspecial groups). The Heisenberg group, whose center is $\mathbb{F}_q$, has precisely $q-1$ irreducible representations of order q^N, given by different non-zero additive characters of the center $\mathbb{F}_q$. (In this book, by a representation, we will always mean a finite-dimensional representation over the complex numbers $\mathbb{C}$.)

Now the symplectic group $\mathrm{Sp}_{2N+2}(\mathbb{F}_q) \subset \mathrm{GL}_{2N+2}(\mathbb{F}_q)$ acts on the vector space $\mathbb{F}_q^{2N+2}$, and the isotropy (or stabilizer) group of a non-zero vector is a semidirect product $\mathbb{H}_N(\mathbb{F}_q) \rtimes \mathrm{Sp}_{2N}(\mathbb{F}_q)$, giving an action of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ on $\mathbb{H}_N(\mathbb{F}_q)$. On the other hand, $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ acts trivially on the characters of $\mathbb{F}_q$. Therefore, it acts projectively on each of the $q-1$ irreducible representation of $\mathbb{H}_N(\mathbb{F}_q)$. As it turns out, in the case of a finite field, these projective representations are genuine representations (i.e. their cocycle is trivial) and moreover, oscillator representations arising from characters related by multiplication by an element of $(\mathbb{F}_q^\times)^2$ are isomorphic. Thus, there are only two non-isomorphic oscillator representations.

The purpose of this chapter is to gather the basic facts about the oscillator representation we will need.

We introduce the oscillator representation in more detail in Subsection 1.1. We will study its endomorphism algebra in Subsection 1.2

In Subsection 1.3, we will describe concretely how the oscillator representation generates all representations of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$. In Subsection 1.4, we introduce "Howe duality," the basic object of study we focus on in this book.

1.1. The construction of the oscillator representation. In this subsection, we introduce the Heisenberg group for symplectic $\mathbb{F}_q$-vector spaces and recall the construction of the oscillator representations.

Let us fix from here on a finite field $\mathbb{F}_q$ for an odd prime power q. Consider an $\mathbb{F}_q$-vector space V with a symplectic form S. We write $\mathrm{Sp}(V, S)$ for the symplectic group of S-preserving linear transformations on V. We typically omit S from the notation (since all symplectic forms are equivalent), and write $\mathrm{Sp}(V)$ for the symplectic group. Additionally, in this chapter, it will be convenient to keep track of the dimension of a symplectic $\mathbb{F}_q$-space within its notation. We write V_N for a symplectic $\mathbb{F}_q$-space of dimension $2N$.

The construction of the oscillator representations of the symplectic group $\mathrm{Sp}(V_N) = \mathrm{Sp}_{2N}(\mathbb{F}_q)$ proceeds through the Heisenberg group $\mathbb{H}_N(\mathbb{F}_q)$. First, we consider classification of irreducible representations of $\mathbb{H}_N(\mathbb{F}_q)$ according to additive characters of $\mathbb{F}_q$ by the Stone-von Neumann Theorem. Next, by considering the action of the symplectic group $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ on $\mathbb{H}_N(\mathbb{F}_q)$, we obtain a projective representations of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, which turns out to form a genuine representation due to us working over a finite field $\mathbb{F}_q$. These are the *oscillator* or *Weil representations*. We will also consider the *Weil-Shale representations* of the semiproduct $\mathrm{Sp}(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$, by considering the actions of the Heisenberg and symplectic groups simultaneously. Broadly, this story follows [**57, 28, 32**].

Convention: We shall use the convention, for a subgroup $H \subseteq G$, to write Ind_H^G for induction from H to G and Res_H^G for restriction from G to H. If one of the variables is understood from the context, we sometimes omit it.

We begin by recalling the definition of the Heisenberg group:

1.1.1. DEFINITION. *Consider a natural number N and symplectic space and form (V_N, S_N) over $\mathbb{F}_q$. The* Heisenberg group *of V_N which we denote by $\mathbb{H}(V_N)$ or $\mathbb{H}_N(\mathbb{F}_q)$ is, as a set, defined as the product*

$$V_N \times \mathbb{F}_q.$$

The group operation $*$ *is defined by putting, for choices of* $v, w \in V_N$, $a, b \in \mathbb{F}_q$,

$$(v,a)*(w,b) = (v+w, a+b+S_N(v,w)). \tag{1.1.1}$$

(and therefore $0 \times \mathbb{F}_q$ *is the center of* $\mathbb{H}_N(\mathbb{F}_q)$*).*

In particular, we notice one immediate feature of the Heisenberg group is that its center is isomorphic to $\mathbb{F}_q$, consisting exactly of the subgroup on pairs $(0,a) \in 0 \times \mathbb{F}_q$. Additionally, we note the action of the symplectic group $\mathrm{Sp}(V_N)$ acting on $\mathbb{H}_N(\mathbb{F}_q)$ by acting linearly on the factor of V_N and trivially on the center. We may consider the semidirect product

$$\mathrm{Sp}(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q).$$

For an additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$, let us consider the "regular $\mathbb{H}_N(\mathbb{F}_q)$-representation" which as a vector space is $\mathbb{C}V_N$, taken with the action of $(v,a) \in \mathbb{H}_N(\mathbb{F}_q)$ on a generator (w) of $\mathbb{C}V_N$ by

$$(v,a)(w) = \psi(a + S_N(v,w)) \cdot (v+w) \tag{1.1.2}$$

1.1.2. Theorem (Stone-von Neumann). *For a non-trivial additive character*

$$\psi_a : \mathbb{F}_q \to \mathbb{C}^\times,$$

there is a unique q^N*-dimensional irreducible* $\mathbb{H}_N(\mathbb{F}_q)$*-representation with the center* $\mathbb{F}_q \cong Z(\mathbb{H}_N(\mathbb{F}_q))$ *acting* ψ_a*-isotypically.*

We denote the $\mathbb{H}_N(\mathbb{F}_q)$-representation corresponding to a choice of additive character ψ_a according to the Stone-von Neumann Theorem by $\rho_{\psi_a}[V_N]$.These representations form representations of $\mathrm{Sp}(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ called the *Weil-Shale representation* (a priori projective but an actual representation for a finite field $\mathbb{F}_q$, see [**28**]). Denote this representation by $\boldsymbol{\omega}_{\psi_a}[V_N]$. Restricting to $\mathrm{Sp}(V_N)$, we obtain the *oscillator representation* $\omega_{\psi_a}[V_N]$.

Let us note that we may also reinflate the $\mathrm{Sp}(V)$-representation $\omega_{\psi_a}[V_N]$ to an $\mathrm{Sp}(V) \ltimes \mathbb{H}_N(\mathbb{F}_q)$-representation by letting $\mathbb{H}_N(\mathbb{F}_q)$ act trivially. We write $\overline{\boldsymbol{\omega}}_{\psi_a}[V_N]$ for this representation. We will examine the relationship between the Weil-Shale representations $\boldsymbol{\omega}_{\psi_a}[V_N]$ and these "simplified" Weil-Shale representations $\overline{\boldsymbol{\omega}}_{\psi_a}[V_N]$.

We write ψ for ψ_1, omitting the subscript from the notation. When ψ is not fixed, we may also write ω_a where $a \in \mathbb{F}_q^\times$ is the element corresponding to ψ under a fixed identification of $\mathbb{F}_q^\times$ with its non-trivial multiplicative characters.

Considering the usual inclusion $\mathrm{GL}_N(\mathbb{F}_q) \subset \mathrm{Sp}(V_N)$, the restriction of $\omega_{\psi_a}[V_N]$ to $\mathrm{GL}_N(\mathbb{F}_q)$ gives the signed permutation representation $\mathbb{C}\mathbb{F}_q^N$ where $\mathrm{GL}_N(\mathbb{F}_q)$ acts on $\mathbb{F}_q^N$ by matrix multiplication, tensored with

$$\epsilon(\det) \tag{1.1.3}$$

where $\det : \mathrm{GL}_N(\mathbb{F}_q) \to \mathbb{F}_q^\times$ is the determinant map and $\epsilon : \mathbb{F}_q^\times \to \mathbb{C}^\times$ is the character of order 2.

Considering $\mathbb{C}\mathbb{F}_q^N$ as the underlying vector space of $\omega_{\psi_a}[V_N]$. For a symmetric matrix B, one also has

$$\begin{pmatrix} I & 0 \\ B & I \end{pmatrix} : (u) \mapsto \psi_a(\frac{1}{2}u^T Bu) \cdot (u)$$

and additionally,

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} : (u) \mapsto \frac{1}{(-q)^{N/2}} \sum_{v \in \mathbb{F}_q^N} \psi_a(-v^T u) \cdot (v). \tag{1.1.4}$$

When N is odd, (1.1.4) depends on a choice of $\sqrt{-1} \in \{i, -i\}$ which corresponds to choosing one of the two different oscillator representations.

1.2. Duality and the models of the endomorphism algebra. Since we aim to use the oscillator representation as a basis for applying an endomorphism algebra method to studying a certain representation theory, our next step is to examine the duality of the oscillator representations and their endomorphism algebras. We do this in this subsection, and find an identification of the endomorphism algebra of an oscillator representation with certain algebra structures on the vector representation of the symplectic group.

We note that one advantage of working over a finite field $\mathbb{F}_q$ is that all representations we consider, including the regular and oscillator representations are finite dimensional. In particular, we may consider the dual representations of oscillator representations.

Let us write

$$\Omega := \omega \otimes (\omega^\vee).$$

We first claim the following

1.2.1. LEMMA. *For $N > 1$, as a representation of $Sp(V_N) \ltimes V_N$, Ω is isomorphic to the space of functions on V_N:*

$$\mathrm{Res}^{\mathrm{Sp}(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)}_{\mathrm{Sp}(V_N) \ltimes V_N}(\Omega) \cong \{f : V_N \to \mathbb{C}\}$$

where an element $(A, w) \in Sp(V_N) \ltimes V_N$ *acts on a function* $f : V_N \to \mathbb{C}$ *by sending it to the function*

$$(A, w)[f] : V_N \to \mathbb{C}$$

where for $v \in V_N$

$$((A, w)[f])(v) = \psi(S_N(v, w)) \cdot f(A(v)).$$

Proof. First note that we may write

$$\Omega = \bigoplus_{v \in V_N} \Omega_v$$

for lines Ω_v such that an element $w \in V_N = V_N \times \{0\} \subset \mathbb{H}_N(\mathbb{F}_q)$ preserves each Ω_v and acts by multiplication by the character

$$x \mapsto \psi(S_N(v, w)) \cdot x.$$

Ω can then be considered as the space of global sections of an $\mathrm{Sp}(V_N)$-equivariant line bundle Ω_v over V_N (as a discrete set) where the action of $\mathrm{Sp}(V_N)$ on Ω induces an action of $\mathrm{Sp}(V_N)$ on the line bungle, i.e. for $\gamma \in \mathrm{Sp}(V_N)$,

$$\gamma(\Omega_v) = \Omega_{\gamma(v)}.$$

However, Ω_v forms a trivial $\mathrm{Sp}(V_N)$-equivariant line bundle, meaning that for every $v \in V_N$, the stabilizer subgroup $\mathrm{Sp}(V_N)^v \subseteq \mathrm{Sp}(V_N)$ fixing v acts trivially on Ω_v: At $v = 0$, for $N > 0$, $\mathrm{Sp}(V_N)^0 = \mathrm{Sp}(V_N)$, which is a perfect group (meaning that it has no non-trivial abelian quotients), and therefore acts trivially on Ω_0. For $v \neq 0$, taking W_v to be the quotient of the orthogonal space $V_N^{\perp v}$ of vectors perpendicular to v by $\mathbb{F}_q$-multiples of v,

$$\mathrm{Sp}(V_N)^v = \mathrm{Sp}(W_v),$$

which again is a perfect group (for $N > 0$).

Therefore, Ω is the space of global sections of the trivial $\mathrm{Sp}(V_N)$-equivariant line bundle, i.e. a space of functions

$$\Omega = \{f : V_N \to \mathbb{C}\}, \tag{1.2.1}$$

and the action of $\mathrm{Sp}(V_N) \ltimes V_N \subset \mathrm{Sp}(V_N) \ltimes \mathbb{H}_n(\mathbb{F}_q)$ on a function f in (1.2.1) is given by $\mathrm{Sp}(V_N)$ acting by composition, and $w \in V_N$ acting by sending f to the function

$$V_N \to \mathbb{C}$$
$$v \mapsto \psi(S_N(v, w)) \cdot f(v)$$

□

Now we consider the algebra structure of $\mathrm{End}_{\mathrm{Sp}(V)}(\omega_{a_1} \otimes \cdots \otimes \omega_{a_n})$. First, let us again consider the $n = 1$ case. Defining an algebra structure

$$(v) \star_a (w) = \psi_a(S(w, v)) \cdot (v + w)$$

on $\mathbb{C}V$, we have that as $\mathbb{C}$-algebras, the endomorphism algebra of ω_a is isomorphic to the subalgebra of $(\mathbb{C}V, \star_a)$ generated by $\mathrm{Sp}(V)$-fixed points in $\mathbb{C}V$

$$(\mathrm{End}_{\mathrm{Sp}(V)}(\omega_a), \circ) \cong ((\mathbb{C}V)^{\mathrm{Sp}(V)}, \star_a) \subseteq (\mathbb{C}V, \star_a). \tag{1.2.2}$$

Therefore, for a tensor product $\omega_{a_1} \otimes \cdots \otimes \omega_{a_n}$ we define an algebra operation $\star_{(a_1,\ldots,a_n)}$ on $\mathbb{C}(V^{\oplus n})$ by

$$(v_1, \ldots, v_n) \star_{(a_1,\ldots,a_n)} (w_1, \ldots, w_n) =$$
$$\psi_1(a_1 \cdot S(w_1, v_1) + \cdots + a_n \cdot S(w_n, v_n)) \cdot (v_1 + w_1, \ldots, v_n + w_n).$$

Then, again, we have that the endomorphism algebra of $\omega_{a_1} \otimes \cdots \otimes \omega_{a_n}$ is the sub-algebra of $\mathbb{C}(V^{\oplus n})$ on fixed points

$$(\mathrm{End}_{\mathrm{Sp}(V)}(\omega_{a_1} \otimes \cdots \otimes \omega_{a_n}), \circ) \cong$$
$$((\mathbb{C}(V^{\oplus n}))^{\mathrm{Sp}(V)}, \star_{(a_1,\ldots,a_n)}) \subseteq (\mathbb{C}(V^{\oplus n}), \star_{(a_1,\ldots,a_n)}).$$

For a representation ρ of a group G, let us write ρ^G for the space of G-fixed points in ρ. It is also instructive to see how an element of $(\mathbb{C}V)^{\mathrm{Sp}(V)}$, and more generally $\mathbb{C}V$, acts on an element of ω_a. Consider a decomposition of V into Lagrangians

$$V = \Lambda_+ \oplus \Lambda_- \tag{1.2.3}$$

Recall that the Heisenberg group $\mathbb{H} = (V \times \mathbb{F}_q, *)$,

$$(v_1, x) * (v_2, y) = (v_1 + v_2, x + y + \frac{S(v_2, v_1)}{2})$$

can be considered to act on $\mathbb{C}\Lambda_-$ by, for $(v_+ + v_-, x) \in \mathbb{H}$, with $v_\pm \in \Lambda_\pm$, and $w \in \Lambda_-$,

$$(v, x) \cdot w = \psi_a(x + \frac{S(v_+, w)}{2})(v_- + w)$$

defining the *Schrödinger model* of the Weil-Shale representation. The uniqueness of the Weil-Shale representation with a fixed central character of $\mathbb{F}_q = Z(\mathbb{H})$ determines an action of $\mathrm{Sp}(V)$, giving the oscillator representation ω_a. There is a natural action of $\mathbb{C}V$ with a different algebra structure $\circ$ where, for $u_\pm, v_\pm \in \Lambda_\pm$,

$$(u_+ + u_-) \circ (v_+ + v_-) = \psi_a(S(u_+, v_-))(u_+ + v_+ + u_- + v_-).$$

The algebra $(\mathbb{C}V, \circ)$ acts on $\mathbb{C}\Lambda_-$ by putting, for a generator $w \in \Lambda_-$ of $\omega_a \cong \mathbb{C}\Lambda_-$,

$$(v_+ + v_-)w = \psi_a(S(v_+, w))(v_- + w).$$

To get the action of (1.2.2), we must apply the isomorphism

$$(\mathbb{C}V, \star) \to (\mathbb{C}V, \circ)$$

$$(v_+ + v_-) \mapsto \psi_a(\frac{S(v_+, v_-)}{2}) \cdot (v_+ + v_-).$$

Therefore, the action of (1.2.2) is given by putting, for $v_\pm \in \Lambda_\pm$,

$$(v_+ + v_-)(w) = \psi_a(S(v_+, w) + \frac{S(v_+, v_-)}{2})(v_- + w) \tag{1.2.4}$$

(in fact defining an action of the whole algebra $(\mathbb{C}V, \star_a)$ on ω_a).

1.3. Tensor generation by oscillator representations. Any faithful finite dimensional complex representation ρ of a finite group G generates the whole category $\mathrm{Rep}(G)$ in the sense that every irreducible representation of G is a direct summand of $\rho^{\otimes n}$ for some n. In the case when $G = \mathrm{Sp}_{2N}(\mathbb{F}_q)$ and ρ is the oscillator representation, it will turn out helpful to work this out explicitly.

In this subsection, we consider the generation of general representations of finite symplectic groups by tensor products of oscillator representations. We examine the relationship between Weil-Shale representations (of the semiproduct of the symplectic group with its Heisenberg group) and oscillator representations.

In the previous subsection, we found that the regular representation $\mathbb{C}V_N$ is generated as a tensor product of two oscillator representations of $\mathrm{Sp}(V_N)$ corresponding to opposite central characters. One thought this brings to mind is that, hence, every irreducible representation of $\mathrm{Sp}(V_N)$ is generated as a summand of a tensor product of a certain number of oscillator representations.

In this subsection, we consider tensor products of oscillator representations. We also consider the significance of the Heisenberg group action in more detail, describing certain facts about the relationship between the Weil-Shale representations and the (inflated) oscillator representations.

1.3.1. Theorem (following P. Deligne). *For $a, b \in \mathbb{F}_q^\times$ such that $a + b \neq 0$, as representations of $Sp(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ for any $N \in \mathbb{N}$ (as long as $V_N \neq \mathbb{F}_3^2$),*

$$\boldsymbol{\omega}_a[V_N] \otimes \boldsymbol{\omega}_b[V_N] \cong \omega_{ab(a+b)}[V_N] \otimes \boldsymbol{\omega}_{a+b}[V_N]. \tag{1.3.1}$$

Proof. N is fixed. To simplify notation, in this proof, we omit the subscript N of V. Let us denote by $S(v,w)$ for $v,w \in V$ the symplectic form of V. Write V_1, V_2 for two copies of V, with symplectic forms S_1, S_2 equivalent to S, and isomorphisms

$$V \xrightarrow{\cong} V_i$$
$$v \mapsto v_i$$

for $i = 1,2$.

It is enough to consider ω_a, ω_b as projective representation of the quotient

$$(\mathrm{Sp}(V) \ltimes \mathbb{H}_N(\mathbb{F}_q))/Z(\mathrm{Sp}(V) \ltimes \mathbb{H}_N(\mathbb{F}_q)),$$

which is the affine symplectic group

$$\mathrm{Sp}(V) \ltimes V.$$

For every $a \in \mathbb{F}_q^\times$, the representation ω_a for (V,S) is isomorphic to ω_1 for $(V, a\cdot S)$ (replacing the symplectic form $S(v,w)$ on V by $a \cdot S(v,w)$). Thus, $\omega_a \otimes \omega_b$ for (V,S) can be considered the pullback of ω_1 for $(V_1 \oplus V_2, a\cdot S_1 + b \cdot S_2)$, using the diagonal embedding

$$\Delta : V \hookrightarrow V_1 \oplus V_2$$

$$v \mapsto (v_1, v_2)$$

Note that the pullback of the symplectic form $a \cdot S_1 + b\cdot S_2$ on $V_1 \oplus V_2$ along Δ is the form $(a+b)\cdot S$ on V.

Now we may also consider an antidiagonal embedding

$$\Delta' : V \hookrightarrow V_1 \oplus V_2$$

$$v \mapsto (bv_1, -av_2),$$

which has image $Im(\Delta')$ orthogonal to $Im(\Delta)$ using the form $a\cdot S_1 + b\cdot S_2$ on $V_1 \oplus V_2$. The pullback of the symplectic form $a\cdot S_1 + b \cdot S_2$ on $V_1 \oplus V_2$ along Δ' is the form $a \cdot b \cdot (a+b)\cdot S$ on V. Reparametrizing using the isomorphism

$$V_1 \oplus V_2 \cong Im(\Delta) \oplus Im(\Delta'),$$

we get an isomorphism between ω_1 for $(V_1 \oplus V_2, a\cdot S_1 + b\cdot S_2)$ and a tensor product of the pullbacks of ω_1 from $Im(\Delta)$ and $Im(\Delta')$. However, the embedding

$$\begin{array}{ccc} \mathrm{Sp}(V) \ltimes V & \longrightarrow & \mathrm{Sp}(V_1 \oplus V_2) \ltimes (V_1 \oplus V_2) \\ & & \downarrow \\ & & (\mathrm{Sp}(\mathrm{Im}(\Delta)) \ltimes \mathrm{Im}(\Delta)) \times (\mathrm{Sp}(\mathrm{Im}(\Delta')) \ltimes \mathrm{Im}(\Delta')) \end{array}$$

projects V bijectively into the first coordinate $Im(\Delta)$ (and to 0 on the second coordinate $Im(\Delta'))$, though it continues to map $\mathrm{Sp}(V) \hookrightarrow \mathrm{Sp}(\mathrm{Im}(\Delta')) \ltimes \mathrm{Im}(\Delta')$ diagonally. Therefore, we obtain that as (projective) $\mathrm{Sp}(V) \ltimes V$-representations,

$$\omega_a \otimes \omega_b \cong \overline{\omega}_{ab(a+b)} \otimes \omega_{a+b}$$

(where now $\overline{\omega}_{ab(a+b)}$ is considered as the $\mathrm{Sp}(V) \ltimes V$-representation from letting V act trivially on the restriction of $\omega_{ab(a+b)}$ to $\mathrm{Sp}(V)$).

Since there is only one way to extend back to representations of $\mathrm{Sp}(V) \ltimes \mathbb{H}_N(\mathbb{F}_q)$, we can conclude (1.3.1). $\square$

We also note that

1.3.2. Lemma. *For all choices of $c, d \in \mathbb{F}_q^{\times}$, we have an isomorphism of $Sp(V_N)$-representations*

$$\omega_c[V_N] \otimes \omega_d[V_N] \cong \omega_{-c}[V_N] \otimes \omega_{-d}[V_N].$$

Proof. The claim is trivial if $q \equiv 1 \bmod 4$, since then -1 is a square and

$$\omega_c[V_N] \cong \omega_{-c}[V_N],\ \omega_d[V_N] \cong \omega_{-d}[V_N].$$

Suppose $q \equiv 3 \bmod 4$. Since $-1 \notin (\mathbb{F}_q^{\times})/(\mathbb{F}_q^{\times})^2$, we have either that

$$c \equiv d \in (\mathbb{F}_q^{\times})/(\mathbb{F}_q^{\times})^2 \text{ or } c \equiv -d \in (\mathbb{F}_q^{\times})/(\mathbb{F}_q^{\times})^2.$$

Again, the claim is clear if $c \equiv -d \in (\mathbb{F}_q^{\times})/(\mathbb{F}_q^{\times})^2$. Therefore, it remains to show that

$$\omega_c[V_N] \otimes \omega_c[V_N] \cong \omega_{-c}[V_N] \otimes \omega_{-c}[V_N]. \tag{1.3.2}$$

This fact follows from the fact that we may consider $\omega_c[V_N] \otimes \omega_c[V_N]$ as the restriction

$$\omega_c[V_N] \otimes \omega_c[V_N] \cong \mathrm{Res}^{\mathrm{Sp}(V_N \oplus V_N)}_{\mathrm{Sp}(V_N)}(\omega_c[V_N \oplus V_N])$$

using the diagonal inclusion

$$\mathrm{Sp}(V_N) \subseteq \mathrm{Sp}(V_N) \times \mathrm{Sp}(V_N) \subseteq \mathrm{Sp}(V_N \oplus V_N).$$

In this inclusion, we embed $V_N \subseteq V_N \oplus V_N$ diagonally, and tensor the form S_N by the diagonal identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is equivalent to tensor S_V by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which gives

$$\omega_{-c}[V_N] \otimes \omega_{-c}[V_N] \cong \mathrm{Res}^{\mathrm{Sp}(V_N \oplus V_N)}_{\mathrm{Sp}(V_N)}(\omega_c[V_N \oplus V_N]),$$

and therefore (1.3.2).

$\square$

Using this lemma, we observe that, in particular, $\omega_a[V_N] \otimes \omega_b[V_N]$ is always self-dual. We may therefore consider it as a degree 2 element, "inverse to itself" under the tensor product, up to copies of $\mathbb{C}V_N$. In fact, the tensor product acts on small tensor products of the oscillator representations of $\mathrm{Sp}(V_N)$ as the operation of an abelian group of order 4, depending on q mod 4:

Case 1: If $q \equiv 1 \mod 4$, then fixing $a, b \in \mathbb{F}_q^\times$ such that $b/a \notin (\mathbb{F}_q^\times)^2$, write

$$A = 1,\ B = \omega_a[V_N],\ C = \omega_b[V_N],\ D = \omega_a[V_N] \otimes \omega_b[V_N].$$

Here, we can see that A, B, C, D tensor corresponding for example to addition on elements

$$(0,0),\ (1,0),\ (0,1),\ (1,1) \in \mathbb{Z}/2 \times \mathbb{Z}/2$$

since $\overline{\omega}_a$ and $\overline{\omega}_b$ are self-dual, (with an additional copy of $\mathbb{C}V_N$ for every two that is cancelled in this addition).

Case 2: If $q \equiv 3 \mod 4$, then fixing an $a \in \mathbb{F}_q^\times$, write

$$A = 1,\ B = \omega_a[V_N],\ C = \omega_a[V_N] \otimes \omega_a[V_N],\ D = \omega_{-a}[V_N].$$

In this case, we see that the classes act with respect to the tensor product as a $\mathbb{Z}/4$-grading in a similar way, corresponding to $0, 1, 2, 3 \in \mathbb{Z}/4$, respectively, since, by Lemma 1.3.2,

$$\begin{gathered}\omega_a[V_N] \otimes \omega_a[V_N] \otimes \omega_a[V_N] = \\ (\omega_{-a}[V_N] \otimes \omega_{-a}[V_N]) \otimes \omega_a[V_N] = \\ \omega_{-a}[V_N] \otimes \mathbb{C}V_N.\end{gathered}$$

We then note that the tensor generation of $\mathrm{Sp}(V_N)$-representations by oscillator representations is *graded* by A, B, C, D, by the following

1.3.3. Proposition (following P. Deligne). *Suppose $n < N$. For all $a \in \mathbb{F}_q^\times$, we have*

$$Hom_{Sp(V_N)}(\omega_a[V_N], (\mathbb{C}V_N)^{\otimes n}) = 0. \tag{1.3.3}$$

For $a, b \in \mathbb{F}_q^\times$ such that $-b/a \notin (\mathbb{F}_q^\times)^2$, we have

$$Hom_{Sp(V_N)}(\omega_a[V_N] \otimes \omega_b[V_N], (\mathbb{C}V_N)^{\otimes n}) = 0. \tag{1.3.4}$$

Proof. Consider for an $n < N$

$$(\mathbb{C}V_N)^{\otimes n} = \mathbb{C}(V_N \oplus \cdots \oplus V_N). \tag{1.3.5}$$

Fix an $a \in \mathbb{F}_q^\times$. We begin by proving (1.3.4).

First note that for W a vector space with alternating form S_W, we may consider the subspace $\widehat{W}$ of linear embeddings in $\mathrm{Hom}_{\mathbb{C}}(W, V_N)$ preserving the forms on W and V_N. Then $\widehat{W}$ has a natural transitive action of $\mathrm{Sp}(V_N)$ by composition, giving it the structure of a representation of $\mathrm{Sp}(V_N)$. For an inclusion $\iota \in \widehat{W}$, we may consider the stabilizer subgroup

$$(\mathrm{Sp}(V_N))^\iota \subseteq \mathrm{Sp}(V_N)$$

fixing ι. By definition, we may also describe $\widehat{W}$ as the induction of the trivial representation of $(\mathrm{Sp}(V_N))^\iota$ to $\mathrm{Sp}(V_N)$

$$\widehat{W} \cong \mathrm{Ind}_{(\mathrm{Sp}(V_N))^\iota}^{\mathrm{Sp}(V_N)}(1). \tag{1.3.6}$$

In fact, these premutation representations, for all W, S_W of dimension $\leq n$, form a decomposition of (1.3.5)

$$(\mathbb{C}V_N)^{\otimes n} = \bigoplus_{(W,S_W),\, dim(W) \leq n} \widehat{W}. \tag{1.3.7}$$

Therefore, to prove the claimed statement, it suffices to prove that

$$\mathrm{Hom}_{\mathrm{Sp}(V_N)}(\omega_a[V_N], \widehat{W}) = 0$$

for every choice of W, S_W of dimensions $\leq n$. Now, by (1.3.6), we have

$$\mathrm{Hom}_{\mathrm{Sp}(V_N)}(\omega_a[V_N], \widehat{W}) = \mathrm{Hom}_{(\mathrm{Sp}(V_N))^\iota}(\omega_a[V_N], 1).$$

Now given a W, S_W with $dim(W) \leq N$, for a choice of form-preserving inclusion $\iota : W \hookrightarrow V$, there exists an orthogonal decomposition

$$V_N = V_N' \oplus V_N''$$

such that the image of ι is contained in V_N''. In particular, therefore, we have that the subgroup

$$\mathrm{Sp}(V_N') \subset \mathrm{Sp}(V_N') \times \mathrm{Sp}(V_N'') \subset \mathrm{Sp}(V_N)$$

contains the stabilizer subgroup $(\mathrm{Sp}(V_N))^\iota$. Thus, it would suffice to show that

$$\mathrm{Hom}_{\mathrm{Sp}(V_N')}(\mathrm{Res}_{\mathrm{Sp}(V_N')}^{\mathrm{Sp}(V_N)}(\omega_a[V_N]), 1) = 0.$$

This follows from the fact that the restriction

$$\mathrm{Res}^{\mathrm{Sp}(V_N)}_{\mathrm{Sp}(V'_N)}(\omega_a[V_N])$$

decomposes as a sum of copies of the oscillator representation $\omega_a[V'_N]$ for V_1, which does not contain any trivial representations.

Now fix $a, b \in \mathbb{F}_q^\times$ such that $-b/a \notin (\mathbb{F}_q^\times)^2$, i.e. such that $\overline{\omega}_a$ and $\overline{\omega}_b$ are not dual to each other. Consider

$$\begin{array}{c}\mathrm{Hom}_{\mathrm{Sp}(V_N)}(\omega_a[V_N] \otimes \omega_b[V_N], (\mathbb{C}V_N)^{\otimes n}) = \\ \mathrm{Hom}_{\mathrm{Sp}(V_N)}(\omega_a[V_N], \omega_{-b}[V_N] \otimes (\mathbb{C}V_N)^{\otimes n})\end{array}.$$

Again, using the decomposition (1.3.7) and (1.3.6), it suffices to prove, for every W, S_W of dimension $dim(W) \leq n$, for an $\iota \in \widehat{W}$,

$$\mathrm{Hom}_{\mathrm{Sp}(V_N)}(\omega_a[V_N], \omega_{-b}[V_N] \otimes \widehat{W}) =$$

$$\mathrm{Hom}_{\mathrm{Sp}(V_N)}(\omega_a[V_N], \omega_{-b}[V_N] \otimes \mathrm{Ind}^{\mathrm{Sp}(V_N)}_{(\mathrm{Sp}(V_N))^\iota}(1)) =$$

$$\mathrm{Hom}_{(\mathrm{Sp}(V_N))^\iota}(\mathrm{Res}^{\mathrm{Sp}(V_N)}_{(\mathrm{Sp}(V_N))^\iota}(\omega_a[V_N]), \mathrm{Res}^{\mathrm{Sp}(V_N)}_{(\mathrm{Sp}(V_N))^\iota}(\omega_{-b}[V_N])).$$

As before, since $dim(W) < N$, we can decompose $V_N = V'_N \oplus V''_N$ so that the image of ι is contained in V'_N, and therefore, it suffices to prove

$$\mathrm{Hom}_{\mathrm{Sp}(V'_N)}(\mathrm{Res}^{\mathrm{Sp}(V_N)}_{\mathrm{Sp}(V'_N)}(\omega_a[V_N]), \mathrm{Res}^{\mathrm{Sp}(V_N)}_{\mathrm{Sp}(V'_N)}(\omega_{-b}[V_N])) = 0,$$

which again follows since the restriction of the oscillator representation of $\mathrm{Sp}(V_N)$ corresponding to a, resp. $-b$, to $\mathrm{Sp}(V'_N)$ will consist of copies of the oscillator representation on V'_N corresponding to a, resp. $-b$, which are not isomorphic (and disjoint) since by assumption,

$$a \not\equiv -b \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2.$$

□

Therefore, the following definition is well-defined:

1.3.4. DEFINITION. *Say a simple object of $Rep(Sp(V_N))$ is in the* class *corresponding to an object A, B, C, D if it appears as a summand of that object, tensored with the nth tensor power* $(\mathbb{C}V_N)^{\otimes n}$ *of the permutation representation for some $n < N$.*

Now, tensoring A, B, C, D as objects of $\mathrm{Sp}(V_N)$, however, may give additional copies of $\mathbb{C}V_N$ upon cancellation - the number of which is non-trivial to calculate, and will vary depending again on q mod 4. We give an initial calculation of the number of these copies of $\mathbb{C}V_N$ for both cases of q mod 4, for the purpose of explicitly justifying that it does not depend on N.

We have, for a finite set S,

$$(A \oplus B \oplus C \oplus D)^{\otimes S} =$$
$$\bigoplus_{S_1 \amalg S_2 \amalg S_3 \amalg S_4 = S} A^{\otimes S_1} \otimes B^{\otimes S_2} \otimes C^{\otimes S_3} \otimes D^{\otimes S_4}$$

Since A is always trivial, it will contribute no new factors of $\mathbb{C}V_N$.

Case 1: $q \equiv 1$ mod 4. By the self-duality of B and C, and writing $D = B \otimes C$, we see that the term $A^{\otimes S_1} \otimes B^{\otimes S_2} \otimes C^{\otimes S_3} \otimes D^{\otimes S_4}$ will give

$$\lfloor \frac{|S_2| + |S_4|}{2} \rfloor + \lfloor \frac{|S_3| + |S_4|}{2} \rfloor$$

additional tensor factors of X. A copy of B, C, or D is present precisely when the first, second, or both terms are altered by the floor function.

Case 2: $q \equiv 3$ mod 4. We may re-write

$$A^{\otimes S_1} \otimes B^{\otimes S_2} \otimes C^{\otimes S_3} \otimes D^{\otimes S_4} = B^{\otimes S_2 \amalg 2 \cdot S_3} \otimes D^{\otimes S_4},$$

and B and D are dual, giving

$$min(|S_2| + 2 \cdot |S_3|, |S_4|) + \lfloor 2 \cdot \frac{||S_2| \amalg 2 \cdot |S_3| - |S_4||}{4} \rfloor$$

additional tensor factors of X, with a copy of B, C, or $D \otimes X$ present depending on $||S_2| \amalg 2 \cdot |S_3| - |S_4||$ mod 4.

Therefore, we may write

$$(A \oplus B \oplus C \oplus D)^{\otimes S} =$$
$$(A \otimes (\mathbb{C}V_N)^{\otimes S_A}) \oplus (B \otimes (\mathbb{C}V_N)^{\otimes S_B}) \oplus$$
$$(C \otimes (\mathbb{C}V_N)^{\otimes S_C}) \oplus (D \otimes (\mathbb{C}V_N)^{\otimes S_D}).$$

for finite sets S_A, S_B, S_C, S_D depending only on S and q.

1.4. The form of a Howe duality result over a finite field. Finally, in this subsection, we examine the idea of Howe duality for oscillator representations of finite symplectic groups. We recall the form of reductive dual pairs over a finite field, and in particular, the type I reductive dual pairs involving orthogonal groups.

1.4.1. DEFINITION. *A pair of subgroups* (G, H) *forms a* reductive dual pair *of subgroups in a symplectic group* $Sp(V_N)$ *if* G *and* H *are each others' commutants in* $Sp(V_N)$.

We now recall R. Howe's classification of reductive dual pairs in symplectic groups [**27**]:

(1) An irreducible reductive dual pair (G, H) in $\mathrm{Sp}(V)$ is of *type I* if the product $G \times H \subseteq \mathrm{Sp}(V)$ acts on V irreducible.

(2) An irreducible reductive dual pair (G, H) in $\mathrm{Sp}(V)$ is of *type II* otherwise, in which case there exists a maximal isotropic subspace $X \subseteq V$ which is invariant under both G and G'.

Type I reductive dual pairs in a symplectic group $\mathrm{Sp}(\mathbf{V}, \mathbf{S})$ are of the form

$$(\mathrm{Sp}(V, S), \mathrm{O}(W, B)),$$

such that $\mathbf{V} = V \otimes W$, $\mathbf{S} = S \otimes B$, and we embed the subgroups into $\mathrm{Sp}(\mathbf{V})$ according to the Kronecker tensor product

$$\mathrm{Sp}(V, S) \times \mathrm{O}(W, B) \hookrightarrow \mathrm{Sp}(\mathbf{V}, \mathbf{S}).$$

The problem of Howe duality considers the restriction of the oscillator representation

$$\omega[\mathbf{V}] = \omega[V \otimes W]$$

to $\mathrm{Sp}(V, S) \times \mathrm{O}(W, B)$ along this inclusion.

To see why the finite field context requires a more refined statement of Howe duality, consider a choice of eigenvalues $a_1, \dots, a_n$ of the matrix corresponding to B. Restricting the oscillator representation $\omega[V \otimes W]$ along the inclusion and further down to $\mathrm{Sp}(V)$

$$Res_{\mathrm{Sp}(V)}(\omega[V \otimes W]) \cong \omega_{a_1}[V] \otimes \cdots \otimes \omega_{a_n}[V].$$

Now recall that over a finite field, an orthogonal space W with symmetric bilinear form B must contain a large isotropic subspace. More specifically, the only orthogonal spaces with a fully anisotropic symmetric bilinear form are of dimensions $0, 1$, and 2. In particular, every symmetric bilinear form B is expressible

$$B = \bigoplus_{h_B} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus Z_B.$$

In particular, we may consider h_B copies of the tensor factors

$$\omega_1[V] \otimes \omega_{-1}[V] \cong \mathbb{C}V.$$

We note that, in particular, the $\mathrm{Sp}(V)$-fixed point appears with multiplicity two in the regular representation

$$1 \oplus 1 \subseteq \mathbb{C}V.$$

We also note a result with a similar proof as Proposition 1.3.3 (though the significance from the perspective of finite field Howe duality is different), over orthogonal groups instead of symplectic groups:

1.4.2. Lemma. *Conisder an orthogonal space and form (W, B), and consider the standard representation $\mathbb{C}W$ of the orthogonal group $O(W, B)$. Tensoring with the sign of determinant $\epsilon(det)$. For $N \leq h_W$, there are no $O(W, B)$-equivariant morphisms from this sign representation to a degree N tensor power of the standard representation*

$$Hom_{O(W,B)}(\epsilon(det), (\mathbb{C}W)^{\otimes N}) = 0.$$

Proof. As in the proof of Proposition 1.3.3, we may consider, for any orthogonal space (U, B_U), the $\mathrm{O}(W, B)$-representation $\widehat{U}$ consisting of the subspace of $\mathbb{C}$-linear linear maps from U to W preserving the forms on U and W. Then, again,

$$(\mathbb{C}W)^{\otimes N} = \bigoplus_{\substack{(U, B_U) \\ \dim(U) \leq N}} \widehat{U}.$$

Given a choice of inclusion $\iota : U \hookrightarrow W$, we may consider each $\widehat{U}$ as the induction of the trivial representation from the stabilizer subgroup $\mathrm{O}(W, B)^\iota$ fixing $\iota \in \widehat{U}$

$$\widehat{U} \cong \mathrm{Ind}_{\mathrm{O}(W,B)^\iota}^{\mathrm{O}(W,B)}(1).$$

It remains to show

$$\mathrm{Hom}_{\mathrm{O}(W,B)}(\epsilon(\mathrm{det}), \widehat{U}) = 0 \tag{1.4.1}$$

for every choice of (U, B_U) of dimension less than or equal to h_W. This follows since we may identify the left hand side of (1.4.1) with

$$\mathrm{Hom}_{\mathrm{O}(W,B)^\iota}(\epsilon(\mathrm{det}), 1) = 0. \tag{1.4.2}$$

Again, if $dim(U) \leq N \leq h_W$, then we may choose a decomposition

$$W = W' \oplus W''$$

into orthogonal spaces (denote the restrictions of B to W' and W'' by B' and B'') such that $\mathrm{O}(W, B)^\iota$ is contained in

$$\mathrm{O}(W', B') \subseteq \mathrm{O}(W', B') \times \mathrm{O}(W'', B'') \subseteq \mathrm{O}(W, B). \tag{1.4.3}$$

Then (1.4.2) is in particular contained in

$$\mathrm{Hom}_{\mathrm{O}(W',B')}(\epsilon(\mathrm{det}), 1)$$

(note that the restriction of $\epsilon(\mathrm{det})$ from $\mathrm{O}(W, B)$ to $\mathrm{O}(W', B')$ along (1.4.3) is still the sign of determinant representation of the smaller orthogonal group). This is then trivial, since both the sign representations $\sigma(det)$ and the trivial representation are irreducible.

□

CHAPTER 2

Stable endomorphism algebras and first decomposition statements

We are now ready to start exploring the endomorphism algebra method. As suggested by the Howe duality program, we shall look at the restriction of an oscillator representation of the form $\omega[V \otimes W]$ to the subgroup $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$ of $\mathrm{Sp}(V \otimes W)$. The first case to look at is when the dimension of one of the spaces V, W is large in comparison to the other. This is called the *stable range*. Let us say that, for example, the dimension of V is large in comparison with the dimension of W. Then we shall study in detail the endomorphism algebra of $\omega[V \otimes W]$ as a representation of $\mathrm{Sp}(V)$. Additively, the endomorphism algebra is isomorphic to the representation tensored with its dual, which turns out to be a permutation representation. Thus, a basis of the endomorphism algebra is given by orbits.

Surprisingly, however, it is possible to express the *composition* of endomorphisms by quite a simple formula on orbits. Examining these formulas, we see explicitly the action of $\mathrm{O}(W, B)$, notably *reflections* corresponding to elements of the projective space $\mathbb{P}(W)$ which are in the complement of the locus of the equation $B(w, w) = 0$. Elements which are in this locus, rather than reflections, give rise to idempotent endomorphisms, which, together, decompose the $\mathrm{Sp}(V)$-representation on $\omega[V \otimes W]$ into layers that can be precisely inductively decomposed into irreducible representations by counting dimensions!

This picture is less immediately visible when we reverse the roles of V and W, i.e. in the *orthogonal stable range*. This is because the symplectic group does not have generators as canonical as reflections in the case of an orthogonal group. However, ultimately, one can write satisfactory generators in the symplectic case also, and it is a nice demonstration our method to write the generators in orbit form and use this to prove a completely analogous decomposition in the orthogonal stable range as well.

Remark: The decomposition in the symplectic stable range was also proved by less explicit methods in [**47**].

In this chapter, we will explain the explicit construction of the decomposition using endomorphism algebras, and also a new analogous stable decomposition arising when we reverse the roles of V and W.

2.1. Statement of the stable decomposition theorem. The goal of this subsection is to make precise statements of what we are about to prove. First, we recall again from Subsection 1.4 our expectations for what a "Howe-duality type result" can look like when we work over $\mathbb{F}_q$: We hope for some relationship between the respective subsets of irreducible representations in $\widehat{\mathrm{Sp}(V)}$ and $\widehat{\mathrm{O}(W,B)}$ consisting of those appearing with non-zero multiplicity in

$$\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W]) \text{ and } \mathrm{Res}_{\mathrm{O}(W,B)}(\omega[V \otimes W])$$

where specifically we restrict $\omega[V \otimes W]$ along the Kronecker product inclusion

$$\mathrm{Sp}(V,S) \times \mathrm{O}(W,B) \hookrightarrow \mathrm{Sp}(V \otimes W, S \otimes B), \tag{2.1.1}$$

and then further along the inclusion of either the $\mathrm{Sp}(V)$ or $\mathrm{O}(W,B)$ factor of the product into the source of (2.1.1). (We again note the dependence of this restriction, not only on W and its dimension, but also on the equivalence class of B as an element of the Witt group of $\mathbb{F}_q$, which is hidden in the notation.)

However, since we cannot expect a true bijection between these subsets, we must make a careful choice of whether it will be more manageable to group terms into the form of distinct irreducible representations of $\mathrm{Sp}(V)$ tensored with certain (perhaps highly non-irreducible) multiplicity $\mathrm{O}(W,B)$-representations

$$\omega[V \otimes W] = \bigoplus_{\rho \in \widehat{\mathrm{Sp}(V)}} \rho \otimes \mathrm{Mult}(\rho), \tag{2.1.2}$$

or the other way around, as multiplicity $\mathrm{Sp}(V)$-representations tensored with distinct irreducible $\mathrm{O}(W,B)$-representations

$$\omega[V \otimes W] = \bigoplus_{\pi \in \widehat{\mathrm{O}(W,B)}} \mathrm{Mult}(\pi) \otimes \pi. \tag{2.1.3}$$

In cases when one of the members of the reductive dual pair is much larger than the other, however, this choice is "easy."

We define the following ranges of type I reductive dual pairs, where either the symplectic or orthogonal group factor is large enough compared to the other to make this choice easily:

2.1.1. DEFINITION. *Consider a symplectic $\mathbb{F}_q$-space V and an orthogonal $\mathbb{F}_q$-space W with a symmetric bilinear form B. We consider $(Sp(V), O(W,B))$ as a type I reductive dual pair in $Sp(V \otimes W)$.*

(1) *We say that $(Sp(V), O(W,B))$ lies in the* symplectic stable range *if the dimension of W is less than or equal to the dimension of a maximal isotropic subspace of V with respect to its symplectic form (which is always half the full dimension), meaning*

$$dim(W) \leq dim(V)/2. \tag{2.1.4}$$

(2) *We say that $(Sp(V), O(W,B))$ lies in the* orthogonal stable range *if the dimension of V is less than or equal to the dimension of a maximal isotropic subspace of W with respect to B, meaning*

$$dim(V) \leq h_B. \tag{2.1.5}$$

The precise cut-off of the conditions (2.1.4) and (2.1.5) (and the reasoning for why we call them stable ranges) is, perhaps, not so clear from our above heuristic perspective. However, it is illuminated when we consider the endomorphism algebra structures embodying decompositions of the form (2.1.2) and (2.1.3). We explain this (while unpacking how the stable range conditions allow us to make our easy choice between (2.1.2) or (2.1.3)) now:

Let us consider, on the one hand, a choice of reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ lying, say, in the symplectic stable range, so that $dim(W) \leq dim(V)/2$. On the level of the groups themselves, this condition does mean that in a rough sense $\mathrm{Sp}(V)$ forms a subgroup of much larger order in $\mathrm{Sp}(V \otimes W)$ than $\mathrm{O}(W,B)$ does. Heuristically, this suggests that the restriction of $\omega[V \otimes W]$ to the $\mathrm{Sp}(V)$-factor in the inclusion (2.1.1) preserves more of the oscillator representation's original structure, than its restriction to the small $\mathrm{O}(W,B)$-factor. This suggests, in turn, that in the symplectic stable range, (2.1.2) is a more reasonable way of grouping terms that (2.1.3). To study the multiplicity spaces of the irreducible $\mathrm{Sp}(V)$-representations in (2.1.2), we must consider the structure of the $\mathrm{Sp}(V)$-equivariant endomorphism algebra

$$\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W]). \tag{2.1.6}$$

On the other hand, in the orthogonal stable range, the orthogonal group is a "larger" subgroup than its symplectic group partner. Therefore, it intuitively seems preferable to consider a decomposition by grouping terms as in (2.1.3), and hence, in turn, we consider the

$\mathrm{O}(W, B)$-equivariant

$$\mathrm{End}_{\mathrm{O}(W,B)}(\omega[V \otimes W]). \tag{2.1.7}$$

The stable ranges of Definition 2.1.1 are precisely formed so that we have the following *stability effect* in the corresponding system of endomorphism algebras.

2.1.2. THEOREM. (1) *Fix a choice of orthogonal space (W, B). For any symplectic spaces V, V' such that $(Sp(V), O(W, B))$ and $(Sp(V'), O(W, B))$ lie in the symplectic stable range, there is an isomorphism of $\mathbb{C}$-algebras*

$$End_{Sp(V)}(\omega[V \otimes W]) \cong End_{Sp(V')}(\omega[V' \otimes W]).$$

(2) *Fix a choice of symplectic space V. For any symplectic spaces (W, B), (W', B') such that B and B' are equivalent in the Witt group $W(\mathbb{F}_q)$ and $(Sp(V), O(W, B))$ and $(Sp(V), O(W', B'))$ lie in the orthogonal stable range, there is an isomorphism of $\mathbb{C}$-algebras*

$$End_{O(W,B)}(\omega[V \otimes W]) \cong End_{O(W',B')}(\omega[V \otimes W']).$$

Note that if $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the symplectic stable range as a reductive dual pair in $\mathrm{Sp}(V \otimes W)$, then for every choice of $k = 0, \ldots, h_B$, the reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W[-k], B[-k]))$ in $\mathrm{Sp}(V \otimes W[-k])$ is also in the symplectic stable range. Similarly, if a pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the symplectic stable range as a reductive dual pair in $\mathrm{Sp}(V \otimes W)$, then for every choice of $k = 0, \ldots, dim(V)/2$, the reductive dual pair $(\mathrm{Sp}(V[-k]), \mathrm{O}(W, B))$ in $\mathrm{Sp}(V[-k] \otimes W)$ is also in the orthogonal stable range. Therefore, Theorem 2.1.2 can be summarized by the slogan that "once we have reached the stable range in one coordinate of the reductive dual pair, the corresponding endomorphism algebra is stable under adding more hyperbolics."

The stable range conditions (2.1.4) and (2.1.5) are also optimal, meaning that the endomorphism algebra structure of (2.1.6) (resp. (2.1.7)) immediately begins decaying if $dim(V)/2 < dim(W)$ (resp. $h_B < dim(V)$). We will see explicitly why this happens in Subsections 2.2 (resp. 2.3) below.

The utility of Theorem 2.1.2 is actually two-fold in determining the restriction of $\omega[V \otimes W]$, since it will allow us to construct the *interpolated category generated by oscillator representations* in Chapter 5. However, the goal of the present chapter is more concrete. In each stable range, we now apply the background of Chapter 1 to construct certain sets of generators (in both linear and algebraic senses) of the corresponding endomorphism algebra structure, in order to obtain explicit expressions of them. Specifically, we find the following

2.1.3. PROPOSITION. *Consider a reductive dual pair $(Sp(V), O(W,B))$ in $Sp(V \otimes W)$.*

(1) *If $(Sp(V), O(W,B))$ is in the symplectic stable range, as a $\mathbb{C}$-algebra, $End_{Sp(V)}(\omega[V \otimes W])$ decomposes as a product of matrix algebras of orthogonal group algebras*

$$\prod_{k=0}^{h_B} M_{|O(W,B)/P^k_{O(W,B)}|}\mathbb{C}O(W[-k], B[-k]). \tag{2.1.8}$$

(2) *If $(Sp(V), O(W,B))$ is in the orthogonal stable range, as a $\mathbb{C}$-algebra $End_{O(W,B)}(\omega[V \otimes W])$ decomposes as a product of matrix algebras of symplectic group algebras*

$$\prod_{k=0}^{dim(V)/2} M_{|Sp(V)/P^k_{Sp(V)}|}\mathbb{C}Sp(V[-k]). \tag{2.1.9}$$

(We interpret the orthogonal or symplectic group on a 0-dimensional space to be trivial groups with 1-dimensional group algebras.)

Finally, let us unpack Proposition 2.1.3 in terms of the concrete question of the decomposition of the restriction of as oscillator representation $\omega_\psi[V \otimes W]$ of $\mathrm{Sp}(V \otimes W)$

$$\mathrm{Res}_{\mathrm{Sp}(V)\times \mathrm{O}(W,B)}(\omega[V \otimes W])$$

into tensor products of irreducible $\mathrm{Sp}(V)$- and $\mathrm{O}(W,B)$-representations.

We now state the main result of this chapter.

2.1.4. THEOREM. *Consider a reductive dual pair $(Sp(V), O(W,B))$ in $Sp(V \otimes W)$.*

(1) *For $(Sp(V), O(W,B))$ in the symplectic stable range, there is a system of injections*

$$\eta_V^{W,B} : \widehat{O(W,B)} \hookrightarrow \widehat{Sp(V)} \tag{2.1.10}$$

(omitting the superscript when it is determined) where for a fixed choice of V, non-equivalent choices of (W,B) give injections with disjoint images, such that the restriction of $\omega[V \otimes W]$

to $Sp(V) \times O(W,B)$ decomposes as

$$\bigoplus_{k=0}^{h_B} \bigoplus_{\pi \in \widehat{O(W[-k],B[-k])}} \eta_V(\pi) \otimes Ind_{P_k^{O(W,B)}}(\pi^-) \tag{2.1.11}$$

where we consider π^- as the representation $\pi \otimes \epsilon(det)$ of the Levi factor $O(W[-k],B[-k]) \times GL_k(\mathbb{F}_q)$, which we inflate to the maximal parabolic $P_k^{O(W,B)}$.

(2) *For $(Sp(V), O(W,B))$ in the symplectic stable range, there is a system of injections*

$$\zeta^V_{W,B} : \widehat{Sp(V)} \hookrightarrow \widehat{O(W,B)} \tag{2.1.12}$$

(omitting the subscript when it is determined) where for a fixed choice of (W,B), non-equivalent choices of V give injections with disjoint images, such that the restriction of $\omega[V \otimes W]$ to $Sp(V) \times O(W,B)$ decomposes as

$$\bigoplus_{k=0}^{dim(V)/2} \bigoplus_{\rho \in \widehat{Sp(V[-k])}} Ind_{P^k_{Sp(V)}}(\rho^-) \otimes \zeta_{W,B}(\rho) \tag{2.1.13}$$

where we consider ρ^- as the representation $\rho \otimes \epsilon(det)$ of the Levi factor $Sp(V[-k]) \times GL_k(\mathbb{F}_q)$, which we inflate to the maximal parabolic $P^k_{Sp(V)}$.

Indeed, we can see that these endomorphism algebras are precisely compatible with the claimed decompositions (2.1.8) and (2.1.9). In fact, unwrapping Proposition 2.1.3 by properly interpreting the role of $\mathrm{O}(W,B)$- and $\mathrm{Sp}(V)$-equivariance, we can directly conclude Theorem 2.1.4 using an inductive argument.

Rewriting terms in the form of (2.1.2), we find that, for a choice of reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic stable range, for $\rho \in \widehat{\mathrm{Sp}(V)}$, its multiplicity is

$$\mathrm{Mult}(\rho) = \mathrm{Ind}_{P_k^{\mathrm{O}(W,B)}}(\pi^-)$$

if there exists some $k = 0, \dots, h_B$ and $\pi \in \widehat{\mathrm{O}(W[-k],B[-k])}$ such that $\eta_V^{W[-k],B[-k]}(\pi) = \rho$, and is

$$\mathrm{Mult}(\rho) = 0,$$

else. Similarly, rewriting terms in the form of (2.1.3), for a choice of reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the orthogonal stable range, for $\pi \in \widehat{\mathrm{O}(W,B)}$, its multiplicity is

$$\mathrm{Mult}(\pi) = \mathrm{Ind}_{P^k_{\mathrm{Sp}(V)}}(\rho^-)$$

if there exists some $k = 0, \dots, dim(V)/2$ and $\rho \in \widehat{\mathrm{Sp}(V[-k])}$ such that $\zeta^{V[-k]}_{W,B}(\rho) = \pi$, and is

$$\mathrm{Mult}(\pi) = 0,$$

else.

2.1.5. Remark. *After we have proved Theorem 2.1.4, we will be able to deduce the explicit computation of $\eta^V_{W,B}$ and $\zeta^{W,B}_V$ in terms of Lusztig's classification of irreducible representation of finite symplectic and orthogonal groups, which we do in Chapters 3 and 4 below. The present chapter, however, will solely be focused on proving the existence of the symplectic stable system of eta correspondences (2.1.10) and the orthogonal stable system of zeta correspondences (2.1.12), and will not involve the explicit description of irreducible representations of Sp(V) or O(W,B). Instead, we will prove Theorem 2.1.4 only using the structure of the endomorphism algebras of oscillator rerpesentations as we have outlined above, making the argument quite elementary. (In fact, arguing in this manner will eventually us to interpolate Theorem 2.1.4 in Chapter 6.)*

In Subsection 2.2, we focus on the symplectic stable range and begin considering its corresponding endomorphism algebra. We prove part 1 of Theorem 2.1.2 and begin setting up intermediate results for the proof of part 1 of Proposition 2.1.3. In Subsection 2.3, we prove similar results for the orthogonal stable range, proving part 2 of Theorem 2.1.2 and setting up for towards part 2 of Proposition 2.1.3. We also consider in what ways the story of the orthogonal stable range is fully symmetric to the symplectic stable range, and in which ways it differs. In Subsection 2.4, we prove a combinatorial lemma, which is the final key to the proof of Proposition 2.1.3. Finally, in Subsection 2.5, we conclude Proposition 2.1.3 and prove Theorem 2.1.4.

2.2. Symplectic endomorphism algebra invariants. In this subsection, we fix the context of reductive dual pairs $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic stable range. As we outlined in the introduction of this chapter, we intuitively suspect that it is best in this case to consider a decomposition graded by irreducible $\mathrm{Sp}(V)$-representations, meaning that we must examine $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$. There are several perspectives from which we can begin studying these endomorphisms: On the most basic level, we can ask about its structure as a $\mathbb{C}$-*vector space* and its dimension, which is what we begin with.

Of course, we also need information about its algebra structure if we hope to prove the stability effect, study the decomposition of $\omega[V \otimes W]$, or identify it with (2.1.8). To do this, we recall one of the models for the endomorphism algebra we described in Subsection 1.2 to deterime its *algebra* structure:

For any symplectic space $\mathbf{V}$ with form $\mathbf{S}$, the endomorphism algebra of the oscillator representation $\omega[\mathbf{V}]$ of $\mathrm{Sp}(\mathbf{V})$ can be identified with

$$(\mathrm{End}_{Vect}(\omega[\mathbf{V}]), \circ) \cong (\mathbb{C}\mathbf{V}, \star) \tag{2.2.1}$$

where the operation $\star$ is defined by taking, for $v_1, v_2 \in \mathbf{V}$,

$$(v_1) \star (v_2) = \psi(\frac{\mathbf{S}(v_1, v_2)}{2}) \cdot (u + v) \tag{2.2.2}$$

and extending linearly to an algebra operation on $\mathbb{C}V$. In particular, to find the G-equivariant endomorphism algebra of $\omega[\mathbf{V}]$ for some subgroup $G \subseteq \mathrm{Sp}(\mathbf{V})$, we can identify it with the subalgebra of (2.2.1) on the G-invariants of $\mathbb{C}\mathbf{V}$.

Considering the present situation, by putting $\mathbf{V} = V \otimes W$ and $G = \mathrm{Sp}(V) \subseteq \mathrm{Sp}(V \otimes W)$, we can identify the endomorphism algebra $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$ with

$$(\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W]), \circ) \cong ((\mathbb{C}V \otimes W)^{\mathrm{Sp}(V)}, \star), \tag{2.2.3}$$

where the operation $\star$ is calculated according to (2.2.2) (noting that we take $\mathbf{S} = S \otimes B$) so that for $v_1, v_2 \in V$ and $w_1, w_2 \in W$,

$$(v_1 \otimes w_1) \star (v_2 \otimes w_2) = \psi(\frac{1}{2} S(v_1, v_2) \cdot B(w_1, w_2)) \cdot (v_1 \otimes w_1 + v_2 \otimes w_2),$$

extending linearly to an operation on all vectors in $V \otimes W$ and then extending $\mathbb{C}$-linearly

Additionally, we will identify certain elements which form algebra generators, and observe their actions.

The vector space structure. We begin by recalling the form of $\mathrm{Sp}(V)$-orbits in $V \otimes W \cong V^{\oplus n}$. Specifically for an n-tuple of vectors $(v_1, \dots, v_n)$, its orbit is classified by the data of the linear dependence relations between the v_i's (which alone would classify the $GL(V)$-orbit of the n-tuple) and the values of the symplectic form $S(v_i, v_j)$ forming an antisymmetric $\mathbb{F}_q$-matrix A (giving an antisymmetric form on the span of $v_1, \dots, v_n$, obtainable as a restriction of S).

We note immediately that this data corresponding to basis elements of $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$ can be easily enumerated in the case when $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is symplectic stable.

2.2.1. Lemma. *For $(Sp(V), O(W,B))$ in the symplectic stable range, the dimension of the underlying $\mathbb{C}$-vector space structure of the $Sp(V)$-equivariant endomorphism algebra of $\omega[V \otimes W]$ is*

$$dim(End_{Sp(V)}(\omega[V \otimes W])) = 2(q+1)\dots(q^{dim(W)-1}+1). \tag{2.2.4}$$

In particular, note that since the formula (2.2.4) does not depend on the dimension of V, this Lemma implies the stability effect of endomorphisms stated in part 1 of Theorem 2.1.2, on the level of $\mathbb{C}$-vector spaces. The proof of Lemma 2.2.1 is fully elementary, proceeding analogously to the calculation of Grassmanian dimensions in order to compute general linear group fixed points on the regular representaitons. For completeness, we give this argument now.

Proof of Lemma 2.2.1. Write $\dim(W) = n$ and consider for the purpose of this proof $V \otimes W = V^{\oplus n}$. Let us consider the $\mathrm{Sp}(V)$-orbit of an n-tuple $(v_1, \dots, v_n)$. Write d for the dimension of the span of $v_1, \dots, v_n$ as a subspace of V. The linear dependence relations between the v_i's are enumerated by $d \times n$ $\mathbb{F}_q$-matrices in reduced row echelon form (attaining d-pivots), the number of which can be written as

$$\binom{n}{d}_q = \sum_{1 \le \ell_1 < \dots < \ell_d \le n} q^{d \cdot n - (\ell_1 + \dots + \ell_d) - \binom{d}{2}} \tag{2.2.5}$$

(taking ℓ_i to be the length of the ith row up to an including the ith pivot). Note that this is precisely the decomposition of the Grassmannian of d-dimensional subspaces in $W = \mathbb{F}_q^n$ into its Schubert cells.

Now we consider, for a choice of $(v_1, \dots, v_n)$ such that their span $\mathrm{Span}(v_1, \dots, v_n)$ is of dimension d, which choices of antisymmetric forms A are obtainable by restricting S to $Span(v_1, \dots, v_n)$ as a subspace of V. In general, the condition for A to be obtainable in this way is that

$$2N = dim(V) \geq d + dim(Ker(A)). \tag{2.2.6}$$

In particular, in the present situation by the symplectic stable range condition, we always have

$$dim(Ker(A)) \leq d \leq n \leq N.$$

Therefore, (2.2.6) always holds, for every choice of $d = 0, \dots, n$ and choice of antisymmetric $d \times d$-matrix A. There are $q^{\binom{d}{2}}$ choices of $d \times d$ antisymmetric $\mathbb{F}_q$-matrices. Combining with (2.2.5), we therefore conclude that the $dim(\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V\otimes W]))$, i.e. the number of $\mathrm{Sp}(V)$-orbits in $V^{\oplus n}$ is

$$\sum_{d=0}^{n} q^{\binom{d}{2}} \sum_{1\leq \ell_1<\cdots<\ell_d\leq n} q^{d\cdot n-(\ell_1+\cdots+\ell_d)-\binom{d}{2}},$$

which can be simplified as

$$\sum_{1\leq \ell_1<\cdots<\ell_d\leq n} \prod_{i=1}^{d} q^{n-\ell_i} = 2(q+1)\dots(q^{n-1}+1).$$

□

2.2.2. Example. *Consider the example where V is a symplectic space of dimension 2, and (W,B) is an orthogonal space of dimension 2. We may consider the reductive dual pair*

$$(SL_2(\mathbb{F}_q), O_2^{\pm}(\mathbb{F}_q))$$

in $Sp_4(\mathbb{F}_q)$. This pair is just barely outside of the symplectic stable range, with $dim(W)=1+dim(V)/2$. Nevertheless, the identification (2.2.3) still holds, and we may attempt to calculate its dimension by counting the $SL_2(\mathbb{F}_q)$-orbits on pairs of vectors in $\mathbb{F}_q^2$ using the same method as in the proof of Lemma 2.2.1. We separate orbits of pairs (v_1,v_2) according to $dim(Span(v_1,v_2))$:

- *The single orbit $(0,0)$ such that $dim(Span(v_1,v_2))=0$*
- *The $q+1$ orbits such that $dim(Span(v_1,v_2))=1$, corresponding to the number of one-dimensional subspaces in $\mathbb{F}_q^2$.*
- *The $q-1$ orbits such that $dim(Span(v_1,v_2))=2$, corresponding to the possible choices of skew-symmetric matrices*

This adds up to $2q+1$ total orbits, which is one less than the dimension of the endomorphism algebra (2.2.4) corresponding to a symplectic stable pair $(Sp(V), O_2^{\pm}(\mathbb{F}_q))$. More generally, this also applies to all reductive dual pairs $(Sp(V), O(W,B))$ which are just outside the sympelctic stable range meaning $dim(W)=1+dim(V)/2$, giving that the only missing orbit is corresponds to W as a $dim(W)$-dimensional subspace of itself and the 0 skew-symmetric matrix of size $dim(W)\times dim(W)$, in which case

$$dim(End_{Sp(V)}(\omega[V\otimes W])) = 2(q+1)\dots(q^{dim(W)-1}+1)-1.$$

The algebra structure. Now we examine the specific algebra stucture of the endomorphisms $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V\otimes W])$. Let us specifically consider, for each choice of $0 \neq w \in W$, the element

$$\sum_{v \in V} v \otimes w \in (\mathbb{C}V \otimes W)^{\mathrm{Sp}(V)}.$$

In the language we used above, these elements can be thought of as the sum of the orbit of $0 \in V \otimes W$ and the orbit consisting of n-tuples of V vectors whose span is of dimension 1, whose corresponding $1 \times n$ reduced row echelon form matrix is w as a row vector (divided by a scalar), and whose 1×1 antisymmetric matrix A is, of course, the 0 matrix. For purposes beyond enumeration, writing out $W = \mathbb{F}_q^n$ and considering $V \otimes W$ elements in coordinates $V^{\oplus n}$ is no longer useful.

2.2.3. Lemma. *As an algebra,*

$$End_{Sp(V)}(\omega[V \otimes W]) \cong ((\mathbb{C}V \otimes W)^{Sp(V)}, \star)$$

is generated by the elements

$$\sum_{v \in V} v \otimes w \tag{2.2.7}$$

corresponding to $0 \neq w \in W$.

Proof. We proceed by induction. Suppose for every choice of linearly independent $w_1, \dots, w_\ell \in W$ for $\ell < k$, and $\ell \times \ell$ antisymmetric matrices A with entries $a_{i,j}$, the element

$$\sum_{\substack{v_1, \dots, v_\ell \in V \\ S(v_i, v_j) = a_{i,j}}} v_1 \otimes w_1 + \dots + v_\ell \otimes w_\ell$$

is generated by order ℓ products of elements of the form (2.2.7).

□

However, different choices of $0 \neq w \in W$ can correspond to elements $\sum_{v \in V} v \otimes w$ that behave differently under the operation $\star$. We begin by considering "regular" choices generators, by which we mean those corresponding to $w \in W$ where $B(w, w) \neq 0$.

2.2.4. Lemma. *For each* $w_0 \in W$ *such that* $B(w_0, w_0) \neq 0$, *the element*

$$r_{w_0^\perp} := \frac{1}{q^N} \sum_{v \in V} v \otimes w_0 \tag{2.2.8}$$

acts bodily on $\omega[V \otimes W]$ *as the reflection in* W *about the hyperplane orthogonal to* w_0 *with respect to* B, *meaning that in the model of* $\omega[V \otimes$

$W]$ *as* $\mathbb{C}V_- \otimes W$*, it acts on* $x \otimes w \in V_- \otimes W$ *by*

$$r_{w_0^\perp}[x \otimes w] = [x \otimes (w - 2\frac{B(w,w_0)}{B(w_0,w_0)}w_0)] \tag{2.2.9}$$

Proof. Fix a vector $w_0 \in W$ such that $B(w_0, w_0) \neq 0$. Our goal is to consider the action of the proposed reflection $r^\perp_{w_0}$ on the oscillator representation $\omega[V \otimes W]$. We use the (dualized) Schrödinger model of the oscillator representation, with the underlying vector space expressed as the free $\mathbb{C}$-vector space on a maximal Lagrangian subspace. In this context, we take $(V \otimes W)_- = V_- \otimes W$, and it therefore suffices to consider the action of $r^\perp_{w_0}$ on a vector of the form $x \otimes w$ for $x \in V_-$ and a vector $w \in W$. We now compute $r^\perp_{w_0}[x \otimes w]$ by decomposing (2.2.8) and applying (1.2.4) to reduce the expression. By definition, we have

$$\begin{aligned} r^\perp_{w_0}[x \otimes w] &= \frac{1}{q^N}\sum_{v \in V}(v \otimes w_0)[x \otimes w] \\ &= \frac{1}{q^N}\sum_{v_\pm \in V_\pm}(v_+ \otimes w_0 + v_- \otimes w_0)[x \otimes w] \end{aligned} \tag{2.2.10}$$

Recalling (1.2.4), each term can be re-expressed as the following scalar multiple of a vector in $V_- \otimes W$:

$$\begin{gathered} (v_+ \otimes w_0 + v_- \otimes w_0)[x \otimes w] = \\ \psi(S(v_+, x)B(w_0, w) + \frac{S(v_+, v_-)B(w_0, w_0)}{2})[v_- \otimes w_0 + x \otimes w] \end{gathered}$$

Let us specifically consider the scalar coefficient. We have

$$\begin{aligned} &\psi(S(v_+, x)B(w_0, w) + \frac{S(v_+, v_-)B(w_0, w_0)}{2}) \\ &= \psi(S(v_+, B(w_0, w)x + \frac{B(w_0, w_0)}{2}v_-)) \end{aligned} \tag{2.2.11}$$

So far, therefore, we have reduced $r^\perp_{w_0}[x \otimes w]$ to

$$\frac{1}{q^N}\sum_{v_\pm \in V_\pm}\psi(S(v_+, B(w_0, w)x + \frac{B(w_0, w_0)}{2}v_-))[v_- \otimes w_0 + x \otimes w].$$

We note now that v_+ and v_- vary independently and only the scalar (and not the vector) of each term of the linear combination depends on v_+. In particular, for any choice of v_-, the coefficient of $[v_- \otimes w_0 + x \otimes w]$ is the sum over every $v_+ \in V$ of the characters (2.2.11). In particular, on the one hand, we notice that for every choice of $v_- \in V_-$ such that

$$B(w_0, w)x + \frac{B(w_0, w_0)}{2}v_- \neq 0,$$

this sum is zero. On the other hand, if

$$B(w_0,w)x+\frac{B(w_0,w_0)}{2}v_-=0, \tag{2.2.12}$$

every character (2.2.11) is equal to 1, adding up to a total coefficient of the order of V_+, which is q^N. The condition (2.2.12) precisely gives

$$v_-=-2\frac{B(w_0,w)}{B(w_0,w_0)}x.$$

The q^N coefficient for this term precisely cancels with the $1/q^N$ coefficient in the definition of $r^{\perp}_{w_0}$. Thus, we have reduced (2.2.10) to

$$\begin{aligned} r^{\perp}_{w_0}[x\otimes w] &= [x\otimes w-2\frac{B(w_0,w)}{B(w_0,w_0)}x\otimes w_0] \\ &= [x\otimes (w-2\frac{B(w_0,w)}{B(w_0,w_0)}w_0)] \end{aligned}$$

giving (2.2.9). □

On the other hand, the $\star$-generators $\sum_{v\in V} v\otimes w$ where w is isotropic in W (meaning $B(w,w)=0$) behave as *idempotents*. To see this clearly, we first also recall that we may express the restriction of the oscillator representation $\omega_b[V\otimes W]$ (associated to a character ψ_b corresponding to some $b\in\mathbb{F}_q^\times$) to a $\mathrm{Sp}(V)$-representation as a tensor product of oscillator representations: Let us write $a_1,\dots,a_n\in\mathbb{F}_q^\times$ for constants so that the symmetric bilinear form B is equivalent to the form corresponding to the diagonal matrix

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix} \tag{2.2.13}$$

Then

$$\mathrm{Res}_{\mathrm{Sp}(V)}(\omega_b[V\otimes W])=\omega_{a_1}[V]\otimes\cdots\otimes\omega_{a_n}[V], \tag{2.2.14}$$

(with $\mathrm{Sp}(V)$ acting diaongally on the right hand side). One can see that the (dualized) Schrödinger model and our presentation of the endomorphism algebra of oscillator representations is compatible with this facotrization.

Returning to the idempotancy of the “degenerate” generators of $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V\times W])$, we have the following

2.2.5. LEMMA. *Consider a choice $w_1, \dots, w_k \in W$ forming a basis of a k-dimensional isotropic subspace of W with respect to B. Then the linear combinations*

$$\frac{1}{q^{2N}} \sum_{v \in V} (v \otimes w_i) \in \mathbb{C}(V \otimes W) \tag{2.2.15}$$

form commuting idempotents whose composition

$$\frac{1}{q^{2kN}} \sum_{v_1 \in V} (v_1 \otimes w_1) \star \cdots \star \sum_{v_k \in V} (v_k \otimes w_k) \tag{2.2.16}$$

has image isomorphic to the $Sp(V)$-representation restriction

$$Res_{Sp(V)}(\omega[V \otimes W[-k]]).$$

Additionally, for two choices of $w_1, \dots, w_k$ which form bases of the same k-dimensional isotropic subspace in W, the corresponding idempotents (2.2.16) are the same.

PROOF. First, consider a choice of $w \in W$ such that $B(w, w) = 0$. We can calculate the composition of the endomorphism corresponding to $\sum_{v \in V}(v \otimes w)$ with itself

$$\sum_{v_1 \in V} (v_1 \otimes w) \star \sum_{v_2 \in V} (v_2 \otimes w) =$$
$$\sum_{v_1, v_2 \in V} \psi(S(v_1, v_2)B(w, w))(v_1 \otimes w + v_2 \otimes w).$$

The coefficient of each term is 1, since $B(w, w) = 0$. Changing variables, replacing v_2 by $v = v_1 + v_2$, this is re-expressed as

$$\sum_{v_1, v \in V} (v \otimes w) = q^{2N} \cdot \sum_{v \in V} (v \otimes w).$$

In particular, we find that the linear combinations (2.2.15) indeed form idempotents in the endomorphism algebra of $\omega[V \otimes W]$.

Next, we prove that they form commuting idempotents. For any choice of vectorss w_1, w_2 in W, the composition of $\sum_{v \in V}(v \otimes w_1)$ and $\sum_{v \in V}(v \otimes w_2)$ is

$$\sum_{v_1 \in V} (v_1 \otimes w_1) \star \sum_{v_2 \in V} (v_2 \otimes w_1) =$$
$$\sum_{v_1, v_2 \in V} \psi\left(\frac{B(w_1, w_2) \cdot S(v_1, v_2)}{2}\right) \cdot (v_1 \otimes w_1 + v_2 \otimes w_2).$$

In particular, if w_1 and w_2 both lie in an isotropic subspace of W with respect to B, the coefficients again are all 1 since $B(w_1, w_2) = 0$.

Therefore, the composition reduces to

$$\sum_{v_1,v_2\in V} (v_1 \otimes w_1 + v_2 \otimes w_2),$$

which is clearly symmetric in w_1 and w_2. Therefore, the idempotents (2.2.15) ommute, and in particular, their composition (2.2.16) is also an idempotent.

It remains to prove that the image of an idempotent of the form (2.2.16) is isomorphic to the restriction of $\omega[V \otimes W[-k]]$ to a $\mathrm{Sp}(V)$-representation. Fix $k \leq h_W$. For simplicity, without loss of generality, let us suppose B corresponds to a diagonal matrix (2.2.13) for some choice of $a_1, \dots, a_n \in \mathbb{F}_q^\times$. Further, let us suppose for $1 \leq i \leq 2k$, the diagonal entries are $a_i = (-1)^{i+1}$, so that we may write

$$B[-k] = \begin{pmatrix} a_{2k+1} & 0 & \dots & 0 \\ 0 & a_{2k+2} & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix} \tag{2.2.17}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\oplus k} \oplus B[-k].$$

Let us compare the restrictions of the oscillator representations $\omega[V \otimes W]$ and $\omega[V \otimes W[-k]]$ to $\mathrm{Sp}(V)$-representations with each other. By (2.2.14), we may express

$$\begin{aligned} &\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W]) \\ &= (\omega[V] \otimes \omega_{-1}[V])^{\otimes k} \otimes \mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-k]]) \\ &= (\mathbb{C}V)^{\otimes k} \otimes \mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-k]]). \end{aligned} \tag{2.2.18}$$

Let us consider an individual $\omega[V] \otimes \omega_{-1}[V]$ factor, by first treating the special case where the dimension of W is 2 and B is a fully split hyperbolic

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case, considering $W = \mathbb{F}_q^2$ and writing out coordinates, the vectors $w_\pm = (1, \pm 1)^T$ (and their scalar multiples) are the only isotropic vectors in W. Their corresponding idempotents (2.2.15) are the linear combinations

$$\frac{1}{q^{2N}} \sum_{v \in V} (v, \pm v). \tag{2.2.19}$$

Now the only term of each of these linear and hence the image of an idempotent (2.2.19) must be isomorphic to the trivial representation of $\mathrm{Sp}(V)$ (since the trivial representation is the only one-dimensional representation of a symplectic group).

Now let us return to the general case of n and B of the form (2.2.17). We begin by considering a specific example of a choice of linearly independent vectors $w_1, \dots, w_k$ in W whose span is isotropic with respect to B. Consider the case where (still putting $W = \mathbb{F}_q^n$ and writing vectors in coordinates) w_i is the n-tuple with all entries 0 except for the $(2i-1)$th and $2i$th which are put to be 1. We recall that in our model of the endomorphism of $\omega[V]$ as $\mathbb{C}V$, the zero vector $(0) \in \mathbb{C}V$ corresponds to the identity on $\omega[V]$. Then, as an endomorphism of (2.2.18), each sum $\sum_{v \in V}(v \otimes w_i)$ is a tensor product of $Id_{\omega[V]\otimes\omega_{-1}[V]}$ factors, except for the ith one, which is replaced by a factor

$$\sum_{v\in V}(v \otimes w_+) \in \mathrm{End}(\omega[V] \otimes \omega_{-1}[V]),$$

with $Id_{\omega[V\otimes W[-k]]}$. Therefore, the composition (2.2.16) of the idempotents corresponding to this choice of $w_1, \dots, w_k$ can be expressed as

$$(\frac{1}{q^{2N}} \sum_{v\in V}(v, \pm v))^{\otimes k} \otimes Id_{\omega[V\otimes W[-k]]}$$

on (2.2.18). This then has image

$$1^{\otimes k} \otimes \mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-k]]) = \mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-k]]).$$

It remains to conclude the claim for a general choice of $w_1, \dots, w_k$. This follows since any basis of any k-dimensional istoropic subspace can be transformed into any other using an orthogonal group element, by Witt's Theorem. Hence, the images of idempotents (2.2.16) for $w_1, \dots, w_k \in W$ generating a k-dimensional isotropic subspace are all isomorphic with each other, and our calculation of the image in the above particular case proves the claim in general.

$\square$

2.3. Orthogonal endomorphism algebra invariants. Similarly as in the previous subsection, we will now only consider orthogonal stable reductive dual pairs $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ and begin setting up the proof of part (2) of Proposition 2.1.3.

A certain level of symmetry with the symplectic stable range story can be seen. For example, we again can use the identification of the underlying $\mathbb{C}$-vector space of $\mathrm{End}_{Vect}(\omega[V \otimes W])$ with $\mathbb{C}V \otimes W$ to understand that the $\mathrm{O}(W,B)$-equivariant endomorphism subalgebra of $\omega[V\otimes W]$ can be identified with the $\mathrm{O}(W,B)$-invariant subalgebra

$$((\mathbb{C}V\otimes W)^{\mathrm{O}(W,B)},\star).$$

The vector space structure. As a vector space, the structure of the endomorphisms $\mathrm{End}_{\mathrm{O}(W,B)}(\omega[V\otimes W])$ can be considered in an entirely similar way as the vector space $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V\otimes W])$ in Subsection 2.2. For a $2N$-tuple $(w_1,\dots,w_{2N})$ of vectors $w_i\in W$, considered as an element of $V\otimes W\cong W^{\oplus 2N}$, its $\mathrm{O}(W,B)$-orbit is classified by the data of the linear dependence relations between the w_i's (which, again, would classify the $GL(W)$-orbit of the $2N$-tuple) and the values of $B(w_i,w_j)$ forming a symmetric $\mathbb{F}_q$-matrix A (giving a symmetric bilinear form on the span of $w_1,\dots,w_{2N}$, obtainable as a restriction of B).

2.3.1. LEMMA. *For $(Sp(V),O(W,B))$ in the orthogonal stable range, the dimension of the underlying vector space of the $O(W,B)$-equivariant endomorphism algebra of $\omega[V\otimes W]$ is*

$$dim(End_{O(W,B)}(\omega[V\otimes W]))=(q+1)(q^2+1)\dots(q^{dim(V)}+1) \tag{2.3.1}$$

PROOF. As in the symplectic case, we write $dim(V)=2N$ and consider $V\otimes W=W^{\oplus 2N}$. Consider a $2N$-tupe $(w_1,\dots,w_{2N})$ of vectors $w_i\in W$. Write d for the dimension of the span of $w_1,\dots,w_{2N}$. The linear dependence relations between the vectors are enumerated by $d\times 2N$ matrices over $\mathbb{F}_q$ in reduced row echelon form. Again, for the data of the symmetric bilinear form on the span of $w_1,\dots w_{2N}$, in general can by any symmetric form A such that

$$h_W\geq dim(Ker(A)), \tag{2.3.2}$$

which is ensured for any choice of symmetric matrix if we are given that the reductive dual pair $(\mathrm{Sp}(V),\mathrm{O}(W,B))$ is in the orthogonal stable range. The data of A is enumerated by a choice of $\binom{d+1}{2}=\binom{d}{2}+d$ scalars $a_{i\leq j}$ corresponding to each pair of indices $i\leq j\in\{1,\dots,d\}$. (We note that the cardinality of an orbit corresponding to a certain choice of $d\times 2N$ reduced row-echelon form matrix and $d\times d$ symmetric bilinear form, just like the range condition, depends on the form of B when n is even.)

Therefore, the number of $\mathrm{O}(W,B)$-orbits in

$$dim(\mathrm{End}_{\mathrm{O}(W,B)}(\omega[V\otimes W]))=\sum_{d=0}^{2N}\binom{2N}{d}_q\cdot q^{\binom{d}{2}+d},$$

which can be simplified using the naive expression for $\binom{2N}{d}_q$ to

$$dim(\mathrm{End}_{\mathrm{O}(W,B)}(\omega[V \otimes W])) = \sum_{1 \le \ell_1 < \cdots < \ell_d \le 2N} q^{d(2N+1)-(\ell_1+\cdots+\ell_d)} =$$

$$\sum_{1 \le \ell_1 < \cdots < \ell_d \le 2N} \prod q^{2N+1-\ell_i} = \prod_{j=1}^{2N} (q^j + 1),$$

as claimed.

□

The algebra structure. Now that we have fully discussed the structure of $\mathrm{End}_{\mathrm{O}(W,B)}(\omega[V \otimes W])$ as a $\mathbb{C}$-vector space, let us begin considering its algebra structure. At this point, however, the symmetry with the symplectic stable range argument begins to break. In Lemma 2.2.4, we chose the natural set of generators of an orthogonal group consisting of reflections which somewhat miraculously can be calculated to correspond exactly to generators of group algebras (or singular idempotents to lower oscillator representations). However, we may recall that unlike the orthogonal group, the symplectic group does not have such a simple set of generators.

We first treat the case of 2-dimensional symplectic sapce V and present the construction of the generators of $\mathrm{Sp}(V) = \mathrm{SL}_2(\mathbb{F}_q)$, embedding its group algebra in $\mathrm{End}_{\mathrm{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W])$.

2.3.2. Lemma. *Suppose $(SL_2(\mathbb{F}_q), O(W,B))$ is in the orthogonal stable range. Consider, for $a \in \mathbb{F}_q^\times$, the element*

$$g_a := \frac{1}{q^{n/2}} \sum_{w \in W} \psi(\frac{a}{2} B(w,w)) \cdot (w, aw) \tag{2.3.3}$$

in $(\mathbb{C}\mathbb{F}_q^2 \otimes W)^{O(W,B)}$. These elements generate a subalgebra of the endomorphism algebra $End_{O(W,B)}(\omega[\mathbb{F}_q^2 \otimes W])$ isomorphic to the group algebra $\mathbb{C}SL_2(\mathbb{F}_q)$ acting according to the action of $SL_2(\mathbb{F}_q)$ as a factor of $SL_2(\mathbb{F}_q) \times O(W,B)$ in $Sp(\mathbb{F}_q^2 \otimes W)$ on $\omega[\mathbb{F}_q^2 \otimes W]$.

Before proving this, we first note that it can be generalized to embed a symplectic group algebra into the orthogonally equivariant endomorphism algebra, for any chocie of reductive dual pair in the orthogonal stable range:

2.3.3. Corollary. *For any reductive dual pair in the orthogonal stable range, the group algebra $\mathbb{C}Sp(V)$ is a subalgebra of the algebra* *of $O(W,B)$-equivariant endomorphisms of $\omega[V\otimes W]$, generated by the images of elements under an inclusion*

$$End_{O(W,B)}(\omega[\mathbb{F}_q^2\otimes W])\hookrightarrow End_{O(W,B)}(\omega[V\otimes W]) \tag{2.3.4}$$

corresponding to a symplectic space inclusion $\mathbb{F}_q^2\hookrightarrow V$.

Proof. Applying Lemma 2.3.2, we in particular find that every transvection in $SL_2(\mathbb{F}_q)$ is generated by the elements (2.3.3) (in fact, we shall explicitly write out the linear combinations in $\mathbb{C}V\otimes W$ corresponding to the standard transvections in Lemma 2.3.2 below, and discuss thier generations by elements (2.3.3) at the end of this subsection).

□

Proof of Lemma 2.3.2. We use the same method as in the previous subsection, by finding elements of $\mathbb{C}(V\otimes W)^{\mathrm{O}(W,B)}$ and proving they act on $\omega[V\otimes W]$ according to the representation-theory action of corresponding generators of $\mathrm{Sp}(V)$.

Let us again recall how to apply an endomorphism in We will again use the Schrödinger model of $\omega[V\otimes W]$. Let us write $V=\Lambda^+\oplus\Lambda^-$ for V's decomposition into complementary Lagrangians with respect to S. According to (1.2.4), writing an element of the algebra $(\mathbb{C}(V\otimes W),\star)$ as $(v_1^++v_1^-,\dots,v_n^++v_n^-)$ for $v_i^\pm\in\Lambda^\pm$, and writing an element of $\omega[V\otimes W]=\mathbb{C}W\otimes\Lambda^-$ as $(x_1,\dots,x_n)$ for $x_i\in\Lambda^-$, we have

$$\begin{gathered}(v_1^++v_1^-,\dots,v_n^++v_n^-)\cdot(x_1,\dots,x_n)=\\ \psi(\sum_{j=1}^n a_i\cdot(S(v_i^+,x_i)+\frac{S(v_i^+,v_i^-)}{2}))\cdot(v_1^-+x_1,\dots,v_n^-+x_n)\end{gathered} \tag{2.3.5}$$

(where ψ denotes the non-trivial additive character corresponding to our choice of oscillator representation $\boldsymbol{\omega}$).

To write this in terms of the symmetric bilinear form B and vectors $u\in W$, let us fix bases of the Lagrangians Λ^+, Λ^- such that, with respect to the basis $e_1^+,\dots,e_N^+,e_1^-,\dots,e_N^-$ of V, the symplectic form S is

$$\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}.$$

Then, alternatively, writing an element of $W\otimes V$ as $(z_1^+,z_1^-,\dots,z_N^+,z_N^-)$ for $z_i^\pm\in W\otimes\mathbb{F}_q\{e_i^\pm\}$, and an element of $W\otimes\Lambda^-$ as $(u_1,\dots,u_N)$ for

$u_i \in W \otimes \mathbb{F}_q\{e_i^-\}$, we have

$$(2.3.6)\qquad \begin{gathered}(z_1^+, z_1^-, \ldots, z_N^+, z_N^-) \cdot (u_1, \ldots, u_N) = \\ \psi(\sum_{i=1}^{N} B(z_i^+, u_i) + \frac{B(z_i^+, z_i^-)}{2}) \cdot (u_1 + z_1^-, \ldots, u_N + z_N^-)\end{gathered}$$

Now we can pick elements of $\mathbb{C}(V \otimes W)^{\mathrm{O}(W,B)}$ designed to act as generators of $\mathrm{Sp}(V)$.

Let us first consider the case when $dim(V) = 2$. In this case, we may reduce (2.3.6) to

$$(z^+, z^-) \cdot (u) = \psi(B(z^+, u) + \frac{B(z^+, z^-)}{2}) \cdot (u + z^-).$$

From this perspective, the action of the matrices in $SL_2(\mathbb{F}_q)$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

on the oscillator representation should correspond to transformations

$$(2.3.7)\quad (u) \mapsto \frac{1}{(-q)^{n/2}} \sum_{w \in W} \psi(B(-u, w)) \cdot (w), \quad (u) \mapsto \psi(\frac{tB(u,u)}{2}) \cdot (u)$$

respectively. The matrices

$$\begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix}$$

act by $(u) \mapsto \epsilon(s) \cdot (s \cdot u)$, for $s \in \mathbb{F}_q^\times$. Now consider, for example, the operators given by the action of

$$g_t = \frac{\epsilon(t)}{(-q)^{n/2}} \cdot \sum_{z \in W} \psi(\frac{tB(z,z)}{2}) \cdot (z, tz) \in \mathbb{C}(W \otimes V)^{\mathrm{O}(W)}$$

for $t \in \mathbb{F}_q^\times$. Applied to $u \in W$, these endomorphisms give

$$g_t \cdot (u) = \frac{\epsilon(t)}{(-q)^{n/2}} \cdot \sum_{z \in W} \psi(B(z, u) + B(z, tz)) \cdot (u + tz),$$

which, replacing $w = u + tz$, can be simplified to

$$\begin{gathered}\frac{\epsilon(t)}{(-q)^{n/2}} \cdot \sum_{w \in W} \psi(B(\frac{w-u}{t}, w)) \cdot (w) = \\ \frac{\epsilon(t)}{(-q)^{n/2}} \cdot \sum_{w \in W} \psi(\frac{B(w,w)}{t}) \cdot \psi(B(\frac{-u}{t}, w)) \cdot (v).\end{gathered}$$

Therefore, each g_t corresponds to the group action of the composition of matrices

$$\begin{pmatrix} 1 & 0 \\ 2/t & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & t \\ -1/t & 2 \end{pmatrix} \tag{2.3.8}$$

on $\omega[W \otimes V]$ (note that $\epsilon(t) = \epsilon(1/t)$). These matrices generate $\mathrm{SL}_2(\mathbb{F}_q)$.

Now, for general V, $\dim(V) = 2N$, we may find these generators for all choices of 1-dimensional subspaces in a Lagrangian (and its dual). This system of group algebras over $\mathrm{SL}_2(\mathbb{F}_q)$ corresponding to choices of isotropic 1-dimensional subspace of V therefore generate $\mathrm{Sp}(V)$. Hence, we get

$$\mathbb{C}\mathrm{Sp}(V) \subseteq \mathrm{End}_{\mathrm{O}(W)}(\omega[W \otimes V]). \tag{2.3.9}$$

Additionally, since these endomorphisms encode the geometric action of $\mathrm{Sp}(V) \subseteq \mathrm{Sp}(W \otimes V)$ on $\omega[W \otimes V]$ and are, in particular, bijective, they are inexpressible as compositions factoring through a lower degree tensor power of $\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]) \cong \mathbb{C}W^-$.

□

The matrices (2.3.8) are not, of course, the usual choice of generating elements of $\mathrm{SL}_2(\mathbb{F}_q)$. A more usual choice of generators would be

$$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

These generators are more complicated to express in terms of linear combinations in and involve Gaussian sum coefficients.

Let us write $q = p^\ell$ for p an odd prime, and $\ell \in \mathbb{N}$. To avoid confusion, we denote the quadratic multiplicative characters of $\mathbb{F}_q$ and $\mathbb{F}_p$ by

$$\epsilon_p : \mathbb{F}_p^\times \to \{\pm 1\}, \qquad \epsilon_q : \mathbb{F}_q^\times \to \{\pm 1\},$$

respectively. As usual, we extend these to 0 by $\epsilon_p(0) = \epsilon_q(0) = 0$. Denoting the norm of the field extension by $N_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q^\times \to \mathbb{F}_p^\times$, we have

$$\epsilon_p \circ N_{\mathbb{F}_q/\mathbb{F}_p} = \epsilon_q. \tag{2.3.10}$$

We may re-write the classical quadratic Gauss sum as

$$\sum_{n \in \mathbb{F}_p} e^{\frac{2\pi i}{p} n^2} = \sum_{m \in \mathbb{F}_p} (1 + \epsilon_p(m)) \cdot e^{\frac{2\pi i}{p} n^2} = \sum_{m \in \mathbb{F}_p} \epsilon_p(m) \cdot e^{\frac{2\pi i}{p} n^2} \tag{2.3.11}$$

(since the linear sum of characters is 0, and for each $m \in \mathbb{F}_p$, there are exactly $1 + \epsilon_p(m)$ elements in $\mathbb{F}_p$ whose square is m), which is well-known to equal

$$\sum_{n \in \mathbb{F}_p} e^{\frac{2\pi i}{p} a \cdot n^2} = \epsilon_p(a) \cdot \sqrt{\epsilon_p(-1) \cdot p}. \tag{2.3.12}$$

Now the same argument as (2.3.11) can be applied to give

$$\sum_{x \in \mathbb{F}_q} \psi(x^2) = \sum_{x \in \mathbb{F}_q} e^{\frac{2\pi i}{p} Tr_{\mathbb{F}_q/\mathbb{F}_p}(x^2)} =$$

$$\sum_{y \in \mathbb{F}_q} \epsilon_q(y) \cdot e^{\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(y)}.$$

Applying the Hasse-Davenport relation for Gauss sums to (2.3.12), we get that

$$\sum_{y \in \mathbb{F}_q} \epsilon_q(y) \cdot e^{\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(y)} = (-1)^{\ell+1} \cdot \left(\sqrt{\epsilon_p(-1) \cdot p}\right)^{\ell},$$

which simplifies to give

$$\sum_{x \in \mathbb{F}_q} \psi(x^2) = (-1)^{\ell+1} \sqrt{\epsilon_q(-1) \cdot q} \tag{2.3.13}$$

Now, for $c \in \mathbb{F}_q^{\times}$, again since a linear sum of characters vanishes, we have

$$\sum_{x \in \mathbb{F}_q} \psi(c \cdot x^2) = \epsilon_q(c) \cdot \sum_{x \in \mathbb{F}_q} \psi(x^2).$$

Combining this with (2.3.13), we find that

$$\sum_{u \in W} \psi(c \cdot B(u,u)) = \sum_{u_1, \ldots, u_n \in \mathbb{F}_q} \psi(\sum_{i=1}^{n} c \cdot a_i \cdot u_i^2) =$$

$$\prod_{i=1}^{n} \sum_{u_i \in W} \psi(c \cdot a_i \cdot u_i^2) = \epsilon_q(c^n \cdot a_1 \ldots a_n) \cdot (\sum_{x \in \mathbb{F}_q} \psi(x^2))^n =$$

$$(-1)^{n(\ell+1)} \cdot \mathrm{disc}(B) \cdot q^{n/2} \cdot \epsilon_q(c)^n \cdot \epsilon_q(-1)^{n/2},$$

where $\mathrm{disc}(B)$ denotes is discriminant, i.e. $\epsilon_q(\det(B))$. For notational brevity, we denote these coefficients by

$$\begin{gathered} K(c) := \sum_{u \in W} \psi(c \cdot B(u,u)) = \\ (-1)^{n(\ell+1)} \cdot \mathrm{disc}(B) \cdot q^{n/2} \cdot \epsilon_q(c)^n \cdot \epsilon_q(-1)^{n/2}. \end{gathered} \tag{2.3.14}$$

2.3.4. Proposition. *Suppose we are given an orthogonal space (W, B) such that $(SL_2(\mathbb{F}_q), O(W, B))$ forms a reductive dual pair in the orthogonal stable range.*

(1) *For $a \neq 1 \in \mathbb{F}_q^\times$, the matrix*

$$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \in SL_2(\mathbb{F}_q)$$

as an element of $End_{O(W,B)}(\omega[\mathbb{F}_q^2 \otimes W])$ corresponds to the element

$$\alpha_a := \frac{\epsilon(a)}{q^n} \cdot \sum_{y^+, y^- \in W} \psi(-\frac{a+1}{2(a-1)} \cdot B(y^+, y^-)) \cdot (y^+, y^-)$$

of $(\mathbb{C}V \otimes W)^{O(W,B)}$.

(2) *The matrix*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{F}_q)$$

as an element of $End_{O(W,B)}(\omega[\mathbb{F}_q^2 \otimes W])$ corresponds to the element

$$\beta := \frac{1}{K(1) \cdot (-q)^{n/2}} \sum_{y^+, y^- \in W} \psi\left(\frac{1}{4}(B(y^+, y^+) + B(y^-, y^-))\right) \cdot (y^+, y^-)$$

of $(\mathbb{C}V \otimes W)^{O(W,B)}$.

(3) *For $b \in \mathbb{F}_q^\times$, the advection matrix*

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in SL_2(\mathbb{F}_q)$$

as an element of $End_{O(W,B)}(\omega[\mathbb{F}_q^2 \otimes W])$ corresponds to the element

$$\gamma_b := \frac{1}{K(\frac{-1}{2b})} \sum_{z \in W} \psi(-\frac{1}{2b} B(z, z)) \cdot (z, 0)$$

of $(\mathbb{C}V \otimes W)^{O(W,B)}$.

2.3.5. Remark. *While on a surface level, these formulas may seem too complicated to work with, especially in comparison with the reflection elements seen in Subsection 2.2, examining them more closely actually reveals that they are quite natural. The coefficients $(a+1)/(a-1)$ appearing in α_a can be interpreted as Cayley transforms, and the Gaussian coefficient appearing, for example, in β is exactly the coefficient of a Fourier transform, interpreted in the finite finite field context.*

Proof of Proposition 2.3.4. We begin with the proof of (1): For an element $u \in W \cong W \otimes \Lambda^-$, for $a \in \mathbb{F}_q^\times$, we can calculate the following:

$$\alpha_a(u) =$$

$$\frac{\epsilon(a)}{q^n} \cdot \sum_{y^+, y^- \in W} \psi\left((\frac{1}{2} - \frac{a+1}{2(a-1)}) \cdot B(y^+, y^-) + B(y^+, u) \right) \cdot (y^- + u) \tag{2.3.15}$$

$$= \frac{\epsilon(a)}{q^n} \cdot \sum_{y^+, y^- \in W} \psi(B(y^+, -\frac{y^-}{a-1} + u)) \cdot (y^- + u).$$

The sum runs over arbitrary choices of y^+, meaning that for fixed $u \in W$ and chosen $y^- \in W$, the coefficient sum

$$\sum_{y^+ \in W} \psi(B(y^+, -\frac{y^-}{a-1} + u)) \tag{2.3.16}$$

of the vector $(y^- + u)$ is a linear sum of characters, and is therefore 0, unless $u = y^-/(a-1)$, in which case (2.3.16) is q^n. Hence, the only contributing choice of y^- is $y^- = (a-1) \cdot u$. Therefore, (2.3.15) simplifies as

$$\alpha_a(u) = \epsilon(a) \cdot \frac{q^n}{q^n} \cdot ((a-1) \cdot u + u) = (a \cdot u),$$

agreeing with the action of the proposed matrix on the oscillator representation $\mathbb{C}W$.

Now we prove (2): For $u \in W$, at each choice of $y^+, y^- \in W$, applying the corresponding term of the sum in β (disregarding the coefficient, for now) gives

$$\psi\left(\frac{1}{4}(B(y^+, y^+) + B(y^-, y^-)) \right) \cdot (y^+, y^-)(u) =$$

$$\psi\left(\frac{1}{4}(B(y^+, y^+) + 2B(y^+, y^-) + B(y^-, y^-)) + B(y^+, u) \right) \cdot (y^- + u) =$$

$$\psi(B(\frac{y^+ + y^-}{2}, \frac{y^+ + y^-}{2}) + B(y^+, u)) \cdot (y^- + u)$$

Therefore, we have

$$\beta(u) =$$

$$\frac{1}{K(1) \cdot (-q)^{n/2}} \sum_{y^+, y^- \in W} \psi(B(\frac{y^+ + y^-}{2}, \frac{y^+ + y^-}{2}) + B(y^+, u)) \cdot (y^- + u). \tag{2.3.17}$$

Renaming variables using $z = y^- + u$, we may rewrite this as

$$\frac{1}{K(1)\cdot(-q)^{n/2}}\sum_{y^+,z\in W}\psi(B(\frac{y^++z-u}{2},\frac{y^++z-u}{2})+B(y^+,u))\cdot(z). \tag{2.3.18}$$

Now we may also notice that

$$B(\frac{y^++z-u}{2},\frac{y^++z-u}{2})+B(y^+,u)=$$

$$B(\frac{y^++z+u}{2},\frac{y^++z+u}{2})-B(z,u),$$

allowing us to rewrite (2.3.18) as

$$\frac{1}{K(1)\cdot(-q)^{n/2}}\sum_{z,y^+\in W}\psi(B(\frac{y^++z+u}{2},\frac{y^++z+u}{2})-B(z,u))\cdot(z).$$

Renaming variables using $w = (y^+ + z + u)/2$ gives

$$\frac{1}{K(1)\cdot(-q)^{n/2}}\sum_{z,w\in W}\psi(B(w,w))\psi(-B(z,u))\cdot(z),$$

which, applying (2.3.14), reduces to

$$\beta(u)=\frac{1}{(-q)^{n/2}}\sum_{z\in W}\psi(-B(z,u))\cdot(z),$$

which is precisely the action (2.3.7).

Finally, we prove (3): For $u \in W$,

$$\gamma_b(u) =$$

$$\frac{1}{K(\frac{-1}{2b})}\sum_{z\in W}\psi(-\frac{1}{2b}B(z,z))\cdot(z,0)(u)= \tag{2.3.19}$$

$$\frac{1}{K(\frac{-1}{2b})}\sum_{z\in W}\psi(-\frac{1}{2b}B(z,z)+B(z,u))\cdot(u).$$

Now we may notice that

$$B(z,u)-\frac{1}{2b}B(z,z)=-\frac{1}{2b}(-2b\cdot B(z,u)+B(z,z))=$$

$$-\frac{1}{2b}\left(B(bu-z,bu-z)-b^2\cdot B(u,u)\right)=$$

$$-\frac{1}{2b}\cdot B(bu-z,bu-z)+\frac{b}{2}\cdot B(u,u).$$

Therefore, substituting $w = bu - z$, we can rewrite (2.3.19) as

$$\gamma_b(u) = \frac{1}{K(\frac{-1}{2b})} \sum_{w\in W} \psi(-\frac{1}{2b} \cdot B(w,w)) \cdot \psi(\frac{b}{2} \cdot B(u,u)) \cdot (u). \tag{2.3.20}$$

Since, by definition,

$$\sum_{w\in W} \psi(-\frac{1}{2b} \cdot B(w,w)) = K(\frac{-1}{2b}),$$

(2.3.20) then reduces to

$$\gamma_b(u) = \psi(\frac{b}{2} \cdot B(u,u)) \cdot (u),$$

agreeing precisely with (2.3.7).

□

Again, it is instructive to see the lower "degenerate" idempotents, whose images restrictions of lower oscillator representations, involving smaller symplectic groups. Analogously to Lemma 2.2.5, we have the following

2.3.6. Lemma. *Consider a choice of $v_1, \dots, v_k \in V$ forming a basis of a k-dimensional isotropic subspace of V. Then the linear combinations*

$$\frac{1}{q^n} \sum_{w\in W} (v_i \otimes w) \in \mathbb{C}(V \otimes W) \tag{2.3.21}$$

form commuting idempotents whose composition

$$\frac{1}{q^{kn}} \sum_{w_1\in W} (v_1 \otimes w_1) \star \dots \star \sum_{w_k\in W} (v_k \otimes w_k) \in \mathbb{C}(V \otimes W) \tag{2.3.22}$$

has image isomorphic to the $O(W,B)$-representation restriction

$$Res_{O(W,B)}(\omega[V[-k] \otimes W]).$$

Additionally, for two choices of $v_1, \dots, v_k$ which form abses of the same k-dimensional isotropic subspace in V, the corresponding idempotents (2.3.22) are the same.

Proof. Just as before, we first find that for any $v \in V$, the composition of the endomorphism corresponding to a sum $\sum_{w\in W}(v \otimes w)$

with itself can be calculated as

$$\sum_{w_1\in W}(v\otimes w_1)\star\sum_{w_2\in W}(v\otimes w_2)=$$
$$\sum_{w_1,w_2\in W}\psi(S(v,v)B(w_1,w_2))(v\otimes w_1+v\otimes w_2)=$$
$$\sum_{w_1,w_2\in W}(v\otimes(w_1+w_2)),$$

which, after changing variables (for example, replacing w_2 by w_1+w_2), can be reduced to q^n times $\sum_{w\in W}(v\otimes w)$. Hence, the linear combinations (2.3.21) indeed form idempotents in the endomorphism algebra of the oscilaltor representation.

For any choice of vectors v_1, v_2 in V, the composition of $\sum_{w\in W}(v_1\otimes w)$ and $\sum_{w\in W}(v_2\otimes w)$ is

$$\sum_{w_1\in W}(v_1\otimes w_1)\star\sum_{w_2\in W}(v_2\otimes w_2)=$$
$$\sum_{w_1,w_2\in W}\psi(\frac{S(v_1,v_2)B(w_1,w_2)}{2})(v_1\otimes w_1+v_2\otimes w_2)$$

and hence, if v_1 and v_2 both lie in an isotropic subspace of V (and in particular $S(v_1,v_2)=0$), the coefficients are all 1, reducing the composition to the sum over $w_1,w_2\in W$ of vectors $(v_1\otimes w_1+v_2\otimes w_2)$, which is symmetric in v_1 and v_2. We can conclude that the idempotents (2.3.21) commute and their composition (2.3.22) is also an idempotent.

It remains to compute the image of (2.3.22). First note that we may factor the inclusion along which we restrict $\omega[V\otimes W]$ through the inclusion

$$(\mathrm{SL}_2(\mathbb{F}_q)\times \mathrm{O}(W,B))^N\hookrightarrow \mathrm{Sp}_{2N}(\mathbb{F}_q)\times\mathrm{O}(W,B)$$

according to direct sum of matrices, if we consider the symplectic form defining $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ as a sum of 2-dimensional hyperbolics

$$S=\begin{pmatrix}0&1\\-1&0\end{pmatrix}^{\oplus N}. \tag{2.3.23}$$

We may consider

$$\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[V\otimes W])\cong(\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[\mathbb{F}_q^2\otimes W]))^{\otimes N}. \tag{2.3.24}$$

In the case of $N=1$, we find that for any vector $v\in V=\mathbb{F}_q^2$, the image of the corresponding idempotent (2.3.21) is 1-dimensional since the only term of the sum (2.3.21) with non-zero trace is $1/q^n$ times the zero vector, and

$$tr((0))=dim(\omega[\mathbb{F}_q^2\otimes W])=q^n.$$

Therefore, the image of the idempotent corresponding to any vector in $\mathbb{F}_q^2$ is isomorphic to the trivial representation.

Now we consider general N. Using (2.3.24), we can compare the restrictions of $\omega[V \otimes W]$ and $\omega[V[-k] \otimes W]$ to find

$$\begin{aligned} &\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[V \otimes W]) \cong \\ &(\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]))^{\otimes k} \otimes \mathrm{Res}_{\mathrm{O}(W,B)}(\omega[V[-k] \otimes W]). \end{aligned} \tag{2.3.25}$$

Again, we first prove the claim for a specific choice of $v_1, \ldots, v_k$. Say (assuming (2.3.23)) we choose vectors $v_1, \ldots, v_k$ by putting v_i to have all 0 coordinates except for the $2i$-th, which is 1. Each v_i can be considered as endomorphisms of (2.3.25), as tensor products of identity endomorphism on $k-1$ of the $\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W])$ tensor factor, with the ith tensor factor consisting of an idempotent with the trivial representation as its image, tensored with the identity on $\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[V[-k] \otimes W])$. In particular, the composition is a degree k tensor product of idempotents with trivial representation images, tensored with the identity on $\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[V[-k] \otimes W])$, and thus its image is isomorphic to $\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[V[-k] \otimes W])$.

Finally, Witt's Theorem can be applied to prove that for general choices of linearly independent vectors $v_1, \ldots, v_k$ whose span forms an isotropic subspace of V, the images of the compositions (2.3.22) are all isomorphic.

□

An explicit computation. In this paragraph, we return to the context of 2-dimensional symplectic spaces, and verify a relation between the endomorphisms of the oscillator representation implied by their correspondence with elements of the group algebra $\mathbb{C}\mathrm{SL}_2(\mathbb{F}_q)$. Now that we have calculated the elements of $(\mathbb{C}\mathbb{F}_q^2 \otimes W)^{\mathrm{O}(W,B)}$ corresponding to the usual generators of $SL_2(\mathbb{F}_q)$, they must. Fix a scalar $a \in \mathbb{F}_q^\times$. We have the following relation of matrices

$$\begin{pmatrix} 0 & a \\ -1/a & 2 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2/a & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in $SL_2(\mathbb{F}_q)$. The purpose of this paragraph is to complete the calculation that

$$g_a \star \alpha_a = \gamma_{2/a} \star \beta. \tag{2.3.26}$$

The composition $g_a \star \alpha_a$ is $\frac{\epsilon(a)^2}{q^n(-q)^{n/2}} = \frac{1}{q^n(-q)^{n/2}}$ times the sum over all choices of $y^+, y^-, z \in W$ of terms

$$\psi(\frac{a}{2}B(z,z) - \frac{a+1}{2(a-1)}B(y^+,y^-)) \cdot (z,az) \star (y^+,y^-). \tag{2.3.27}$$

Writing out

$$(z,az) \star (y^+,y^-) = \psi(\frac{1}{2}(B(z,y^-) - a \cdot B(z,y^+))),$$

each term (2.3.27) can be simplified to the pair of vector $(y^+ + z, y^- + tz)$ multiplied by the coefficient

$$\psi(\frac{a}{2}B(z,z) - \frac{a+1}{2(a-1)}B(y^+,y^-) + \frac{1}{2}B(z,y^-) - \frac{a}{2}B(z,y^+)).$$

By considering

$$-\frac{a+1}{2(a-1)} = \frac{1}{2} - \frac{a}{a-1}, \qquad -\frac{a}{2} = \frac{a}{2} - a,$$

this can be rewritten as

$$\psi(\frac{1}{2}B(y^+ + z, y^- + tz) - \frac{a}{a-1}B(y^+, y^- + (a-1)z)).$$

Substituting $u = y^+ + z$, $v = y^- + tz$ gives

$$\psi(\frac{1}{2}B(u,v) - \frac{a}{a-1}B(u-z, v-z)).$$

Therefore, we have reduced $g_a \star \alpha_a$ to

$$\frac{1}{q^n(-q)^{n/2}} \sum_{z,u,v \in W} \psi(\frac{1}{2}B(u,v) - \frac{a}{a-1}B(u-z,v-z)) \cdot (u,v). \tag{2.3.28}$$

Writing

$$B(u-z, v-z) = B(u,v) - B(u+v,z) + B(z,z),$$

we may "complete the square" by noticing that

$$-B(u+v,z) + B(z,z) =$$
$$B(z - \frac{u+v}{2}, z - \frac{u+v}{2}) - B(\frac{u+v}{2}, \frac{u+v}{2}).$$

Substituting variables using $w = z - (u+v)/2$, putting the terms together, we get

$$B(u-z,v-z) = B(u,v) + B(w,w) - B(\frac{u+v}{2}, \frac{u+v}{2}) =$$

$$B(w,w) - B(\frac{u-v}{2}, \frac{u-v}{2}).$$

Therefore, (2.3.28) reduces to

$$g_a \star \alpha_a =$$

$$\frac{1}{q^n(-q)^{n/2}} \sum_{w,u,v\in W} \psi(\frac{1}{2}B(u,v) - \frac{a}{a-1}(B(w,w) - B(\frac{u-v}{2}, \frac{u-v}{2}))) \cdot (u,v) =$$

(2.3.29)

$$\frac{K(-a/(a-1))}{q^n(-q)^{n/2}} \sum_{u,v\in W} \psi(\frac{1}{2}B(u,v) + \frac{a}{a-1}B(\frac{u-v}{2}, \frac{u-v}{2})) \cdot (u,v).$$

Now let us consider the other side of (2.3.26). The composition $\gamma_{2/a} \star \beta$ is $\frac{1}{(-q)^{n/2}K(-a/4)K(1)}$ times the sum over all choices of $y^+, y^-, z \in W$ of terms

$$\psi(-\frac{a}{4}B(z,z) + \frac{1}{4}(B(y^+,y^+) + B(y^-,y^-))) \cdot (z,0) \star (y^+,y^-)$$

Writing out

$$(z,0) \star (y^+,y^-) = \psi(\frac{1}{2}B(z,y^-)) \cdot (z+y^+, y^-),$$

this term is the pair of vectors $(z+y^+, y^-)$ multiplied by the coefficient

$$\psi(-\frac{a}{4}B(z,z) + \frac{1}{4}(B(y^+,y^+) + B(y^-,y^-)) + \frac{1}{2}B(z,y^-)) =$$

$$\psi(-a \cdot B(\frac{z}{2}, \frac{z}{2}) + B(\frac{y^+ - y^-}{2}, \frac{y^+ - y^-}{2}) + \frac{1}{2}B(z+y^+, y^-))$$

Substituting variables $u = z + y^+$, $v = y^-$, $w = z/2$ we reduce $\gamma_{2/a} \star \beta$ to the coefficient $\frac{1}{(-q)^{n/2}K(-a/4)K(1)}$ times

$$\sum_{u,v,w\in W} \psi(-aB(w,w) + B(\frac{u-v}{2} - w, \frac{u-v}{2} - w) + \frac{1}{2}B(u,v)) \cdot (u,v).$$

Writing

$$B(\frac{u-v}{2} - w, \frac{u-v}{2} - w) =$$

$$B(\frac{u-v}{2}, \frac{u-v}{2}) - 2B(\frac{u-v}{2}, w) + B(w,w),$$

we have

$$-aB(w,w) + B(\frac{u-v}{2} - w, \frac{u-v}{2} - w) =$$

$$-(a-1) \cdot B(w,w) - 2 \cdot B(\frac{u-v}{2}, w) + B(\frac{u-v}{2}, \frac{u-v}{2}).$$

Completing the square gives

$$B(w,w)+\frac{2}{a-1}B(\frac{u-v}{2},w)=$$

$$B(w+\frac{u-v}{2(a-1)},w+\frac{u-v}{2(a-1)})-\frac{1}{(a-1)^2}B(\frac{u-v}{2},\frac{u-v}{2}).$$

Replacing variables $x=w+\frac{1}{2(a-1)}(u-v)$ gives

$$-aB(w,w)+B(\frac{u-v}{2}-w,\frac{u-v}{2}-w)+\frac{1}{2}B(u,v)=$$

$$-(a-1)B(x,x)+(1+\frac{a-1}{(a-1)^2})B(\frac{u-v}{2},\frac{u-v}{2})+\frac{1}{2}B(u,v)=$$

$$-(a-1)B(x,x)+\frac{a}{a-1}B(\frac{u-v}{2},\frac{u-v}{2})+\frac{1}{2}B(u,v)$$

Thus, $\gamma_{2/a}\star\beta$ is the factor $\frac{1}{(-q)^{n/2}K(-a/4)K(1)}$ times

$$\sum_{u,v,x\in W}\psi(-(a-1)B(x,x)+\frac{a}{a-1}B(\frac{u-v}{2},\frac{u-v}{2})+\frac{1}{2}B(u,v))\cdot(u,v)=$$

$$K(-(a-1))\cdot\sum_{u,v\in W}\psi(\frac{a}{a-1}B(\frac{u-v}{2},\frac{u-v}{2})+\frac{1}{2}B(u,v))\cdot(u,v).$$

This agrees with our above calculation of $g_a\star\alpha_a$ in (2.3.29), up to a constant. It remains to check that the constants precisely agree, i.e.

$$\frac{K(-(a-1))}{(-q)^{n/2}K(-a/4)K(1)}=\frac{K(-a/(a-1))}{q^n(-q)^{n/2}}. \tag{2.3.30}$$

Recalling (2.3.14), first note that since

$$K(c)=(-1)^{n(\ell+1)}\cdot\operatorname{disc}(B)\cdot q^{n/2}\cdot\epsilon_q(c)^n\cdot\epsilon_q(-1)^{n/2}$$

only depends on $\epsilon_q(c)$, we have $K(-a/4)=K(-a)$. We can therefore simplify (2.3.30) to

$$q^n\cdot K(-(a-1))=K(-a/(a-1))\cdot K(-a)\cdot K(1).$$

Next, the signs, i.e. the factors $(-1)^{n(\ell+1)}\operatorname{disc}(B)$ in each K factor will cancel, since both the left and right hand side have and odd number of K factors. Further, collecting factors, both sides have a factor of $q^n(-q)^{n/2}$, which we may factor out. This reduces the claim to

$$\epsilon_q(-(a-1))^n\cdot\epsilon(-1)^{n/2}=\epsilon_q(-a/(a-1))^n\epsilon_q(-a)^n\epsilon_q(-1)^{3n/2}.$$

Dividing both sides by $\epsilon_q(-1)^{n/2}$ and collecting terms gives

$$\epsilon_q(-(a-1))^n = \epsilon_q(\frac{-a}{a-1}\cdot(-a)\cdot(-1))^n,$$

which holds, since $\epsilon_q(-(a-1)) = \epsilon_q(-1/(a-1))$.

2.4. Combinatorics of the edomorphism algebra dimensions. The purpose of this subsection is to prove that the dimensions of the underlying $\mathbb{C}$-vectors spaces of the algebras (2.1.8) and (2.1.9) are equal to the dimensions of $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V\otimes W])$ and $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V\otimes W])$ calculated in Lemmas 2.2.1 and 2.3.1, respectively.

We denote the Gaussian binomial coefficients by

$$\binom{a}{b}_q := \frac{(q^a-1)\cdot(q^{a-1}-1)\dots(q^{a-b+1}-1)}{(q^b-1)\cdot(q^{b-1}-1)\dots(q-1)}. \tag{2.4.1}$$

The key lemma of this subsection is the following:

2.4.1. Lemma. *For every $p\in\mathbb{N}_0$, for every $r>b\in\mathbb{N}_0$,*

$$Q_{r+p}\cdot Q_{r-1+p}\dots Q_{b+p} = \sum_{a=0}^{p} q^{a(b+a-1)}\cdot\binom{r-b+1}{a}_q\cdot\prod_{i=p-a+1}^{p}(q^i-1)\cdot\prod_{j=b+a}^{r}Q_j \tag{2.4.2}$$

Applying Lemma 2.2.1, we can conclude

2.4.2. Corollary. *For a reductive dual pair $(Sp(V), O(W,B))$ in the symplectic stable range the dimension*

$$dim(Hom_{Sp(V)}(\omega[V\otimes W[-\ell]],\omega[V\otimes W])) \tag{2.4.3}$$

is equal to the sum over $k=0,\dots,h_B-\ell$ of terms

$$\begin{gathered}|O(W[-\ell],B[-\ell])/P^k_{O(W[-\ell],B[-\ell])}|\cdot\\ |O(W,B)/P^{k+\ell}_{O(W,B)}|\cdot|O(W[-k-\ell],B[-k-\ell])|.\end{gathered} \tag{2.4.4}$$

In particular, for $\ell=0$, this gives

$$dim(End_{Sp(V)}(\omega[V\otimes W])) = \sum_{k=0}^{h_B}|O(W,B)/P^k_{O(W,B)}|^2\cdot|O(W[-k],B[-k])|. \tag{2.4.5}$$

Similarly, we apply Lemma 2.3.1 to conclude the following

2.4.3. Corollary. *For a reductive dual pair $(Sp(V), O(W,B))$ in the orthogonal stable range the dimension*

$$dim(Hom_{O(W,B)}(\omega[V[-\ell]\otimes W],\omega[V\otimes W])) \tag{2.4.6}$$

is equal to the sum over $k = 0, \dots, dim(V)/2 - \ell$ of terms

$$|Sp(V[-\ell])/P^{\ell}_{Sp(V[-\ell])}| \cdot |Sp(V)/P^{k+\ell}_{Sp(V)}| \cdot |Sp(V[-k-\ell])|. \tag{2.4.7}$$

In particular, for $\ell = 0$, this gives

$$dim(End_{O(W,B)}(\omega[V \otimes W])) = \sum_{k=0}^{dim(V)/2} |Sp(V)/P^{k}_{Sp(V)}|^2 \cdot |Sp(V[-k])|. \tag{2.4.8}$$

(In fact, we will find that the formulas in Corollary 2.4.3 are precisely scaled versions of the formulas in Corollary 2.4.2.)

Assuming Lemma 2.4.1, let us begin with the

Proof of Corollary 2.4.2. Fix a type I reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the symplectic stable range and a choice of $\ell = 0, \dots, h_B$. By definition B is obtained by adding ℓ copies of hyperbolic plane forms to $B[-\ell]$

$$B = B[-\ell] \oplus \bigoplus_{\ell} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In paricular, this gives an isomorphism of $\mathrm{Sp}(V)$-representations between the restriction $\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$ and the tensor product

$$\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]]) \otimes (\omega_1[V] \otimes \omega_{-1}[V])^{\otimes \ell} \cong \mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]]) \otimes (\mathbb{C}V)^{\otimes \ell}.$$

Hence, we obtain an identification

$$\mathrm{Hom}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]], \omega[V \otimes W]) \cong \mathrm{Hom}_{\mathrm{Sp}(V)}(1, \mathbb{C}(V^{\oplus(n-\ell)})),$$

and therefore, by applying Lemma 2.2.1, we find that the dimension of the Hom-space $\mathrm{Hom}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]], \omega[V \otimes W])$ is

$$2(q+1)\dots(q^{n-\ell-1}+1), \tag{2.4.9}$$

which we denote in this subsection as $2Q_1 \dots Q_{n-\ell-1}$. It remains the reduce the sum of terms (2.4.7) to this product.

First, we write out the explicit formulae for the factors involved in (2.4.7). To begin, we recall the classical formulae for the order of a parabolic quotient $|\mathrm{O}(W,B)/P_\ell^{O(W,B)}|$:

$$\begin{aligned}
&\binom{m}{\ell}_q \prod_{j=m-\ell+1}^{m}(q^j+1) && \text{if } n=2m+1\\
&\binom{m}{\ell}_q \prod_{j=m-\ell}^{m-1}(q^j+1) && \text{if } n=2m,\ dim(Z)=0\\
&\binom{m-1}{\ell}_q \prod_{j=m-\ell+1}^{m}(q^j+1) && \text{if } n=2m,\ dim(Z)=2
\end{aligned} \tag{2.4.10}$$

Since these formulae depend on the parity of the dimension of W and, in the case when $dim(W)$ is even, whether B is fully split or not, we now split the proof into corresponding cases.

Case 1: The dimension of W is odd, and we write $n = 2m+1$. Note that m is the number of hyperbolics h_Q in Q. In this case, for $k = 0, \dots m-\ell$, by (2.4.10) we find

$$|\mathrm{O}(W,B)/P_{k+\ell}^{\mathrm{O}(W,B)}| = \binom{m}{k+\ell}_q \prod_{i=m-k-\ell+1}^{m} Q_i.$$

We have a similar formula for $|\mathrm{O}(W[-\ell],B[-\ell])/P_k^{O(W[-\ell],B[-\ell])}|$, which we write out in terms of factors

$$\frac{(q^{2(m-\ell)}-1)\dots(q^{2(m-k-\ell+1)}-1)}{(q^k-1)\dots(q-1)} = \frac{\displaystyle\prod_{j=m-k-\ell+1}^{m-\ell}(q^{2j}-1)}{(q^k-1)\dots(q-1)}.$$

Finally, we also recall that the order formular for odd finite orthogonal groups is equal to

$$|\mathrm{O}(W[-k-\ell],B[-k-\ell])| = 2\cdot q^{(m-k-\ell)^2}\prod_{j=1}^{m-k-\ell}(q^{2j}-1)$$

Hence, considering again that the dimension (2.4.3) can be re-expressed as the product (2.4.9) the claim of Corollary 2.4.2 reduces to the statement

$$2Q_1\dots Q_{2m-\ell} = \sum_{k=0}^{m-\ell} 2q^{(m-k-\ell)^2}\binom{m}{k+\ell}_q \prod_{i=m-k-\ell+1}^{m} Q_i \frac{\displaystyle\prod_{j=1}^{m-\ell}(q^{2j}-1)}{(q^k-1)\dots(q-1)} \tag{2.4.11}$$

We may divide both sides of (2.4.11) by $2Q_1 \dots Q_{m-\ell}$ (expanding the factors $q^{2j}-1 = Q_j \cdot (q^j - 1)$, giving

$$Q_{m-\ell+1} \dots Q_{2m-\ell} = \sum_{k=0}^{m-\ell} q^{(m-k-\ell)^2} \binom{m}{k+\ell}_q \prod_{i=m-k-\ell+1}^{m} Q_i \prod_{j=k+1}^{m-\ell} (q^j - 1). \tag{2.4.12}$$

Replacing

$$\binom{m}{k+\ell}_q = \binom{m}{m-k-\ell}_q, \tag{2.4.13}$$

this follows exactly from Lemma 2.4.1 by putting $r = m,\ p = m-\ell,\ b = 1$ and substituting $a = m-k-\ell$.

Case 2: The dimension of W is even, and we write $n = 2m$. The case of whether B is fully split or not still separates the formula for the order of parabolic quotients of $\mathrm{O}(W, B)$.

Case 2A: First we treat the fully split case, meaning we suppose that the symmetric bilinear form B decomposes completely into hyperbolics, i.e. $h_B = m$. Applying (2.4.10), we now find

$$|\mathrm{O}(W,B)/P_{k+\ell}^{\mathrm{O}(W,B)}| = \binom{m}{k+\ell}_q \cdot \prod_{i=m-k-\ell}^{m-1} Q_i.$$

Again, we have a similar formula for $|\mathrm{O}(W[-\ell], B[-\ell])/P_k^{\mathrm{O}(W[-\ell],B[-\ell])}|$

$$\frac{\prod_{j=m-k-\ell}^{m-\ell-1} (q^{j+1}-1) Q_j}{(q^k-1)\dots(q-1)}.$$

Finally, we recall the formula for the order of split even orthogonal groups $\mathrm{O}(W[-k-\ell], B[-k-\ell]) = \mathrm{O}^+_{2(m-k-\ell)}(\mathbb{F}_q)$:

$$2q^{(m-k-\ell)(m-k-\ell-1)} \cdot (q^{m-k-\ell} - 1) \prod_{j=1}^{m-k-\ell-1} (q^{2j} - 1)$$

Again, by expressing the dimension (2.4.3) as the product (2.4.9), we find that the statement of the corollary can be rewritten as
(2.4.14)

$$2Q_1 \dots Q_{2m-\ell-1} =$$

$$\sum_{k=0}^{m-\ell} q^{(m-k-\ell)(m-k-\ell-1)} \binom{m}{k+\ell}_q \prod_{i=m-k-\ell}^{m-1} Q_i \frac{(q^{m-\ell}-1)\prod_{j=1}^{m-\ell-1}(q^j-1)Q_j}{(q^k-1)\dots(q-1)}$$

Again, we may divide both sides of (2.4.16) by $2Q_1 \dots Q_{m-\ell-1}$, giving
(2.4.15)

$$Q_{m-\ell} \dots Q_{2m-\ell-1} =$$

$$\sum_{k=0}^{m-\ell} q^{(m-k-\ell)(m-k-\ell-1)} \cdot \binom{m}{k+\ell}_q \cdot \prod_{i=m-k-\ell}^{m-1} Q_i \cdot \prod_{j=k+1}^{m-\ell} (q^j-1).$$

Again, rewriting (2.4.13), this statement follows directly from applying Lemma 2.4.1 to $r = m-1$, $p = m-k$, $b = 0$ and substituting $a = m-k-\ell$.

Case 2B: Now we treat the case where the symmetric bilinear form B is not fully split, meaning that it has a 2-dimensional anisotropic part and $h_B = m-1$. In this case, applying (2.4.10) gives

$$|\mathrm{O}(W,B)/P_{k+\ell}^{\mathrm{O}(W,B)}| = \binom{m-1}{k+\ell}_q \cdot \prod_{i=m-k-\ell+1}^{m} Q_i.$$

Similarly, we have a formula for the order

$$|\mathrm{O}(W[-k],B[-k])/P_\ell^{\mathrm{O}(W[-k],B[-k])}|$$

of the quotient of the lower orthogonal group $\mathrm{O}(W[-k],B[-k])$ by its maximal parabolic, which we expand into factors

$$\frac{\prod_{i=m-k-\ell+1}^{m-\ell} (q^{i-1}-1)Q_i}{(q^k-1)\dots(q-1)}.$$

Finally, we also recall the order of the non-split even orthogonal groups to write the formula for $\mathrm{O}(W[-k-\ell],B[-k-\ell]) = \mathrm{O}^-_{2(m-k-\ell)}(\mathbb{F}_q)$ as

$$2q^{(m-k-\ell)(m-k-\ell-1)} \cdot (q^{m-k-\ell}+1) \prod_{j=1}^{m-k-\ell-1} (q^{2j}-1)$$

Again, by using (2.4.9) and the above formulae, we can reduce the claim of the corollary to the statement

(2.4.16)
$$2Q_1 \dots Q_{2m-\ell-1} =$$
$$\sum_{k=0}^{m-\ell-1} q^{(m-k-\ell)(m-k-\ell-1)} \cdot \binom{m-1}{k+\ell}_q \prod_{i=m-k-\ell+1}^{m} Q_i \frac{Q_{m-\ell} \prod_{j=1}^{m-\ell-1} (q^j-1) \cdot Q_j}{(q^k-1)\dots(q-1)}$$

Similarly as in the previous cases, we may divide both sides of (2.4.16) by $2Q_1 \dots Q_{m-\ell}$, giving

(2.4.17)
$$Q_{m-\ell+1} \dots Q_{2m-\ell-1} =$$
$$\sum_{k=0}^{m-\ell-1} q^{(m-k-\ell)(m-k-\ell-1)} \cdot \binom{m-1}{k+\ell}_q \cdot \prod_{i=m-k-\ell+1}^{m} Q_i \cdot \prod_{j=k+1}^{m-\ell-1} (q^j-1).$$

Rewriting $\binom{m-1}{k+\ell}_q = \binom{m-1}{m-k-\ell-1}_q$ a final time, we obtain the claimed equality by applying Lemma 2.4.1 to $r = m,\ p = m-k-1,\ b = 2$ and substituting $a = m-k-\ell-1$.

□

It remains to prove Lemma 2.4.1. We proceed by induction. We note that the statement does not reduce well if we replace q by 1, and therefore it cannot be considered a direct "quantization" of a classical formula.

Proof of Lemma 2.4.1. To prove (2.4.2), we begin by rewriting

$$Q_{r+p} \dots Q_{b+p} =$$
$$Q_r \dots Q_b + \sum_{k=b}^{r} (Q_{p+k} - Q_k) \cdot \prod_{j=k+1}^{r} Q_j \cdot \prod_{j'=b}^{k-1} Q_{j'+p}. \tag{2.4.18}$$

Now, for each $b \le k \le r$, we have

$$Q_{p+k} - Q_k = q^k \cdot (q^p - 1),$$

so we may rewrite (2.4.18) as

$$Q_{r+p} \dots Q_{b+p} =$$
$$Q_r \dots Q_b + \sum_{k=b}^{r} q^k (q^p - 1) \prod_{j=k+1}^{r} Q_j \cdot \prod_{j'=b}^{k-1} Q_{j'+p}. \tag{2.4.19}$$

Our goal is now to process each term of the right hand side of (2.4.19) to convert, step by step, the highest appearing $Q_{j'+p}$ factor into the next smallest Q_j not yet appearing. In the statement of Lemma 2.4.1, all terms consist of multiples of products of the form

$$Q_r \cdot Q_{r-1} \dots Q_{b+a+1} \cdot Q_{b+a},$$

so we cannot skip any Q_j's in the process. We make the following

2.4.4. Claim. *For $b \leq k \leq r$, we have*

(2.4.20)

$$q^k(q^p - 1) \prod_{j=k+1}^{r} Q_j \cdot \prod_{j'=b}^{k-1} Q_{j'+p} =$$

$$\sum_{a=1}^{k-b+1} \left(\sum_{a+b-1 \leq \ell_1 \leq \cdots \leq \ell_a = k} q^{\ell_1 + \cdots + \ell_a} \right) \cdot \prod_{i=p-a+1}^{p} (q^i - 1) \cdot \prod_{j=b+a}^{r} Q_j$$

Proof of Claim 2.4.4. We will proceed inductively, step by step, converting each factor $Q_{j'+p}$ in a term of the previous step's reduction of (2.4.20), starting with the largest appearing j', into a sum of the next lower Q_j not yet appearing, with the appropriate error term of q^j multiplied by a factor $(q^{j'+p-j} - 1)$. This process will terminate in $k - b$ steps (we are already done when $k = b$).

The induction hypothesis is that after n steps, we will have reduced (2.4.20) to the sum, over $a = 1, \dots, n+1$ of terms

$$\left(\sum_{a-n+k-1 \leq \ell_1 \leq \cdots \leq \ell_a = k} q^{\ell_1 + \cdots + \ell_a} \right) \prod_{i=p-a+1}^{p} (q^i - 1) \prod_{j=k-n+a}^{r} Q_j \prod_{j'=b}^{k-n-1} Q_{j'+p} \tag{2.4.21}$$

Let us describe the first step of this process for (2.4.20). The largest appearing j' is $j' = k - 1$. The next lower Q_j factor not yet appearing is for $j = k$. Therefore, this step uses the replacement

$$Q_{k-1+p} = Q_k + q^k(q^{p-1} - 1).$$

This gives

$$q^k(q^p-1)\prod_{j=k}^{r}Q_j\cdot\prod_{j'=b}^{k-2}Q_{j'+p}+$$

$$q^{k+k}(q^p-1)(q^{p-1}-1)\prod_{j=k+1}^{r}Q_j\cdot\prod_{j'=b}^{k-2}Q_{j'+p}$$

proving (2.4.21) at $n=1$.

Suppose (2.4.21) holds at Step n. We now need to perform Step $(n+1)$. For $1\le a\le n+1$, consider the term

$$\left(\sum_{a-n+k-1\le\ell_1\le\cdots\le\ell_a=k}q^{\ell_1+\cdots+\ell_a}\right)\cdot\prod_{i=p-a+1}^{p}(q^i-1)\cdot\prod_{j=k-n+a}^{r}Q_j\cdot\prod_{j'=b}^{k-n-1}Q_{j'+p},\tag{2.4.22}$$

of (2.4.21).

The highest occuring $Q_{j'+p}$ is at $j'=k-n-1$. The next lower Q_j factor not yet appearing is for $j=k-(n+1)+a$. Therefore, in this term, we must use the replacement

$$Q_{k-n-1+p}=Q_{k-(n+1)+a}+q^{k-(n+1)+a}\cdot(q^{p-a}-1).$$

This reduces (2.4.22) to

$$\begin{aligned}&\left(\sum_{a-n+k-1\le\ell_1\le\cdots\le\ell_a=k}q^{\ell_1+\cdots+\ell_a}\right)\cdot\prod_{i=p-a+1}^{p}(q^i-1)\cdot\prod_{j=k-(n+1)+a}^{r}Q_j\cdot\prod_{j'=b}^{k-n-2}Q_{j'+p}+\\&+\left(\sum_{a-n+k-1=\ell_0\le\ell_1\le\cdots\le\ell_a=k}q^{\ell_0+\ell_1+\cdots+\ell_a}\right)\cdot\prod_{i=p-a}^{p}(q^i-1)\cdot\prod_{j=k-n+a}^{r}Q_j\cdot\prod_{j'=b}^{k-n-2}Q_{j'+p}\end{aligned}\tag{2.4.23}$$

These terms appear in the $(n+1)$th inductive step; the first one occurs in the expression (2.4.21) with n replaced by $n+1$ with no reindexing of a or ℓ_i, and the second one occurs after replacing a by $a+1$ and shifting $\ell_0\le\cdots\le\ell_a$ to $\ell_1\le\cdots\le\ell_{a+1}$.

Therefore, we may proceed inductively, and at Step $n=k-b$, we obtain the reduction (2.4.20).

□

Recombining the terms (2.4.20) according to (2.4.19), we get

$$Q_{r+p}\dots Q_{b+p} =$$

$$\sum_{a=0}^{p}\left(\sum_{a+b-1\leq\ell_1\leq\cdots\leq\ell_a\leq r} q^{\ell_1+\cdots+\ell_a}\right)\cdot\prod_{i=p-a+1}^{p}(q^k-1)\cdot\prod_{j=b+a}^{r}Q_j \tag{2.4.24}$$

(note that the $a=0$ term arises from the single term $Q_r\dots Q_b$ in (2.4.19)).

Finally, we compute

$$\sum_{a+b-1\leq\ell_1\leq\cdots\leq\ell_a\leq r} q^{\ell_1+\cdots+\ell_a} =$$

$$q^{a(a+b-1)}\cdot\sum_{0\leq\ell_1\leq\cdots\leq\ell_a\leq r-a-b+1} q^{\ell_1+\cdots+\ell_q} =$$

$$q^{a(a+b-1)}\cdot\binom{r-b+1}{a}_q,$$

by the Gaussian binomial coefficient theorem. Plugging this into (2.4.24) gives (2.4.2).

□

2.5. Induction and the proof of the stable decomposition theorem. The proofs of Theorem 2.1.4 and Propostion 2.1.3 proceed alongside each other.

Proof of Proposition 2.1.3 and Theorem 2.1.4. We begin with part (1), considering reductive dual pairs $(\mathrm{Sp}(V),\mathrm{O}(W,B))$ in the symplectic stable range. We will proceed by fixing a choice of symplectic space V and then perform induction on possible orthogonal spaces (W,B). We note that this induction has finitely many steps, since for a fixed V, there are precisely $dim(V)$ possible non-equivalent choices of such (W,B). Our induction hypothesis is that we suppose the statement of the theorem holds for every choice of (W,B) such that the maximal dimension of an isotropic subspace h_B is strictly less than m. To proceed, we must prove that the statement holds for reductive dual pairs $(\mathrm{Sp}(V),\mathrm{O}(W,B))$ where the maximal dimension of an isotropic subspace in (W,B) is $h_B=m$. Consider such a (W,B).

We know that for every $1\leq\ell\leq m$, the lower orthogonal spaces $(W[-\ell],B[-\ell])$ have maximal isotropic dimensions equal to $h_B-\ell$, and therefore, the induction hypothesis holds for reductive dual pairs

$$(\mathrm{Sp}(V), \mathrm{O}(W[-\ell], B[-\ell])).$$

In particular, the induction hypothesis gives a system of injections

$$\eta_V^{W[-\ell],B[-\ell]} : \widehat{\mathrm{O}(W[-\ell], B[-\ell])} \hookrightarrow \widehat{\mathrm{Sp}(V)}$$

with disjoint images. Let us pick out the top parts of the restrictions

$$\omega[V \otimes W[-\ell]]^{top} = \bigoplus_{\pi \in \widehat{\mathrm{O}(W[-\ell], B[-\ell])}} \eta^V_{W[-\ell],B[-\ell]}(\pi) \otimes \pi,$$

forming a sub-$\mathrm{Sp}(V) \times \mathrm{O}(W[-\ell], B[-\ell])$-representation of the restriction

$$\mathrm{Res}_{\mathrm{Sp}(V)\times \mathrm{O}(W[-\ell],B[-\ell])}(\omega[V \otimes W[-\ell]]).$$

From the perspective of endomorphism algebras, we have

$$\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]]^{top}) \cong \mathbb{C}\mathrm{O}(W[-\ell], B[-\ell]). \tag{2.5.1}$$

Additionally, restricting the decomposition obtained from the induction hypothesis further down to $\mathrm{Sp}(V)$ gives a decomposition

$$\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]]) = \bigoplus_{k=0}^{h_B-\ell} \frac{|\mathrm{O}(W[-\ell], B[-\ell])|}{|P_k^{\mathrm{O}(W[-\ell],B[-\ell])}|} \mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-k-\ell]]^{top}), \tag{2.5.2}$$

Our first step is to prove that a sum of multiplicity $|\mathrm{O}(W, B)/P_\ell^{\mathrm{O}(W,B)}|$ of copies of $\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]]^{top})$ appears as a sub-$\mathrm{Sp}(V)$-representation of $\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$. We check this by considering the Hom-spaces

$$\mathrm{Hom}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]]^{top}, \omega[V \otimes W]). \tag{2.5.3}$$

Applying (2.5.2), we can decompose the Hom-space

$$\mathrm{Hom}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-\ell]], \omega[V \otimes W]) \tag{2.5.4}$$

as the sum, indexed by $k = 0, 1, \ldots, h_B - \ell$, of copies of theses Hom-spaces

$$\mathrm{Hom}_{\mathrm{Sp}(V)}(\omega[V \otimes W[-k-\ell]]^{top}, \omega[V \otimes W]) \tag{2.5.5}$$

with multiplicity $|\mathrm{O}(W[-\ell], B[-\ell])/P_k^{\mathrm{O}(W[-\ell],B[-\ell])}|$ each. In particular, the dimension of (2.5.4) is equal to the sum over $k = 0, 1, \ldots, h_B - \ell$ of the dimension of (2.5.5) multiplied by coefficients equal to the quotient orders $|\mathrm{O}(W[-\ell], B[-\ell])/P_k^{\mathrm{O}(W[-\ell],B[-\ell])}|$. We have already calculated, however, explicit formulas for the left hand sides of these equalities (the dimensions of (2.5.4)) in Subsection 2.4. By Corollary 2.4.2, we already know that these equalities hold if we replace (2.5.5) by the products

$$\frac{|\mathrm{O}(W,B)|}{|P_{k+\ell}^{\mathrm{O}(W,B)}|}|\mathrm{O}(W[-k-\ell],B[-k-\ell])|.$$

Therefore considering the relationship between the dimensions of (2.5.4) and (2.5.5) as a linear system on (2.5.5), we can conclude that

$$dim(\mathrm{Hom}_{\mathrm{Sp}(V)}(\omega[V\otimes W[-\ell]]^{top},\omega[V\otimes W])) = \\ \frac{|\mathrm{O}(W,B)|}{|P_{\ell}^{\mathrm{O}(W,B)}|}|\mathrm{O}(W[-\ell],B[-\ell])|$$

Next, let us consider the $\mathrm{Sp}(V)\times\mathrm{O}(W[-\ell],B[-\ell])$-equivariant Hom-spaces

$$\mathrm{Hom}_{\mathrm{Sp}(V)\times\mathrm{O}(W[-\ell],B[-\ell])}(\omega[V\otimes W[-\ell]]^{top},\omega[V\otimes W]) \tag{2.5.6}$$

which are endowed with the structure of modules over the group algebras (2.5.1). The restriction of $\omega[V\otimes W]$ to $\mathrm{Sp}(V)\times\mathrm{O}(W[-\ell],B[-\ell])$ is isomorphic to

$$\mathrm{Res}_{\mathrm{Sp}(V)\times\mathrm{O}(W[-\ell],B[-\ell])}(\omega[V\otimes W[-\ell]])\otimes(\mathbb{C}V)^{\otimes\ell}, \tag{2.5.7}$$

where $\mathrm{O}(W[-\ell],B[-\ell])$ acts trivially on the $\mathbb{C}V$ tensor factors. We find that $\omega[V\otimes W[-\ell]]^{top}$ appears as a summand of (2.5.7) of multiplicity $|\mathrm{O}(W,B)/P_{\ell}^{\mathrm{O}(W,B)}|$. Now, since we know that

$$\mathrm{Res}_{GL(V_-\otimes W)}(\omega[V\otimes W])=\mathbb{C}(V_-\otimes W)\otimes\epsilon(det),$$

we obtain, adjunction, that the sum over $1\leq\ell\leq h_B$ of the lower layers

$$\bigoplus_{\pi\in\widehat{\mathrm{O}(W[-\ell],B[-\ell])}}\eta_V^{W[-\ell],B[-\ell]}(\pi)\otimes\mathrm{Ind}_{P^{\ell}_{\mathrm{O}(W,B)}}(\pi\otimes\epsilon(det))$$

as in the claim. From the prespective of endomorphism algebras, these summands contribute endomorphism algebras

$$\prod_{\ell=1}^{h_B}M_{|\mathrm{O}(W,B)/P_{\ell}^{\mathrm{O}(W,B)}|}(\mathbb{C}\mathrm{O}(W[-\ell],B[-\ell])),$$

all occuring independently from the top part of $\omega[V\otimes W]$ as a $\mathrm{Sp}(V)\times\mathrm{O}(W,B)$-representation. We know by the arguments of Subsection 2.2 that the endomorphism algebra of the top part is $\mathbb{C}\mathrm{O}(W,B)$, so that we find

$$\prod_{\ell=0}^{h_B}M_{|\mathrm{O}(W,B)/P_{\ell}^{O(W,B)}|}(\mathbb{C}\mathrm{O}(W[-\ell],B[-\ell]))$$

as a subalgebra of $\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W])$. This must be an equality, since the dimensions of these algebras match by Corollary 2.4.2.

The proof for (2) of Proposition 2.1.3 and Theorem 2.1.4 follows entirely symmetrically, swapping the roles of the symplectic and orthogonal groups, by inductively applying Lemma 2.3.6, Corollary 2.3.3 to prove that (2.1.9) is a subalgebra of $\mathrm{End}_{\mathrm{O}(W,B)}(\omega[V \otimes W])$ and then applying Corollary 2.4.3 to see that this inclusion must be an equality. $\square$

CHAPTER 3

Classifying irreducible representation of a finite symplectic or orthogonal group

In Chapter 2, we used the endomorphism algebra structure of oscillator representations to prove the existence of a certain system of one-to-one correspondences

$$\eta^V_{W,B} : \widehat{\mathrm{O}(W,B)} \hookrightarrow \widehat{\mathrm{Sp}(V)}$$

for $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic stable range and

$$\zeta^{W,B}_V : \widehat{\mathrm{Sp}(V)} \hookrightarrow \widehat{\mathrm{O}(W,B)}$$

for $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the orthogonal stable range such that, in the corresponding case, the restriciton of an oscillator representation $\omega[V \otimes W]$ can be decomposed only in terms of the layers of the appropriate correspondence and (signed) parabolic inductions.

From the perspective of representation theory, we would like to describe explicitly the effect of the eta and zeta correspondences on irreducible representations. To do so, we prepare in this chapter by recalling the classification of irreducible representations obtianed from G. Lusztig's parametrization of irreducible characters [**45**], specifically in the case of symplectic and orthogonal groups $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, $\mathrm{O}_{2m+1}(\mathbb{F}_q)$, and $\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)$.

First, in Subsection 3.1, we present a brief overview of the classification we will use and the general theory underlying. In Subsection 3.2, we discuss the story of split and non-split tori in reductive finite groups G of rank 1, including $\mathrm{SL}_2(\mathbb{F}_q)$, $\mathrm{SO}_3(\mathbb{F}_q)$, and $\mathrm{SO}^{\pm}_2(\mathbb{F}_q)$. In Subsections 3.3, 3.4, and 3.5, we describe the semisimple, unipotent, and central data classifying the irreducible representations for symplectic groups and special orthogonal groups. In Subsection 3.5, we also describe the rules for passing from special orthogonal group representations to those of full orthogonal groups.

3.1. Overview. Consider a finite reductive group $G(\mathbb{F}_q)$. The classification of irreducible $G(\mathbb{F}_q)$-representations we discuss in this chapter can be considered as approaching the question with the philosophy of "automorphic forms."

In a classical setting, we expect (in broad terms) for there to be a certain range of cuspidal irreducible representations such that every general irreducible representation can be constructed in the parabolic induction of an inflation of a cuspidal representation of a Levi subgroup. In a finite field setting, an anologue of this story can be produced, but a certain technical complication arises from the existence of *non-split tori*, which are not contained in any maximal parabolic subgroup of $G(\mathbb{F}_q)$ (though they can be considered within the fixed points of a parabolic in $G(\overline{\mathbb{F}_q})$ under a twisted Frobenius). These additional tori lead to the existence of a large set of irreducible $G(\mathbb{F}_q)$-representations which are not in the true principle series of $G(\mathbb{F}_q)$, but in some ways behave as though they were.

For example, consider the case of $G(\mathbb{F}_q) = SL_2(\mathbb{F}_q)$. The only maximal parabolic subgroup of $SL_2(\mathbb{F}_q)$ is the Borel subgroup, consisting say of upper-triangular matrices

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

for $a \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$. Its Levi subgroup consists of diagonal matrices where $b = 0$ and is isomorphic to the split torus $GL_1(\mathbb{F}_q) = \mathbb{F}_q^\times$. Considering characters $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$, inflating to Borel representations (letting the unipotent factor act trivially) and inducing to $SL_2(\mathbb{F}_q)$ yields representations of dimension $q+1$ (with the representation constructed from χ being isomorphic to the one from χ^{-1}). For $\chi \neq 1, \epsilon$, this defines a range of $(q-3)/2$ irreducible representations of dimension $(q+1)/2$, (while for $\chi = \epsilon$, this induction decomposes into two non-isomorphic $(q+1)/2$-dimensional irreducible representation, and for $\chi = 1$, it decomposes into the trivial representation and the q-dimensional Steinberg representation). Theses are the genuine "principal series" representations of $SL_2(\mathbb{F}_q)$. This only gives about half of the irreducible representations with this construction. There are $(q-1)/2$ other irreducible representations, of dimension $q+1$ each and another non-isomorphic pair of irreducible representations of dimension $(q+1)/2$ each. Intuitively, the structure of these representations seems to be obtained by replacing the role of the split torus $\mathbb{F}_q^\times$ by the the non-split torus isomorphic to the cyclic group $\mu_{q+1} \cong \mathbb{F}_{q^2}^\times / \mathbb{F}_q^\times$. We recall that this non-split torus can be considered, for example, to consist of

$$\begin{pmatrix} a & \alpha \cdot b \\ b & a \end{pmatrix}$$

for a fixed non-square $\alpha \in \mathbb{F}_q^\times$, for choices of $a, b \in \mathbb{F}_q$ with $a^2 - \alpha \cdot b^2 = 1$ (i.e. for $a + \sqrt{\alpha} b \in \mathbb{F}_{q^2}^\times$ with norm 1).

To accomodate the non-split tori T in a group $G(\mathbb{F}_q)$ (and the representations that appear to correspond to their characters), we replace parabolic induction by a generalization called *Deligne-Lusztig induction* [**9**] which we will denote by R_T^G. We will treat Deligne-Lusztig induction as black box for our purposes, but for a proper treatment, we refer the reader to [**4, 9, 13, 45**].

For now, let us consider a connected group G with connected center (for the cases of symplectic and orthogonal groups, we will discuss what must be changed in the following story separately). Let us write G^* for the dual finite reductive group whose simple roots are the simple coroots of G (see Definition 5.12 of [**9**]). Conjugacy classes (s) of semisimple elements s in $G^*(\mathbb{F}_q)$ then can be considered to each correspond to a set of pairs (T, θ) of maximal tori $T \subseteq G(\mathbb{F}_q)$ and characters $\theta : T \to \mathbb{C}^\times$. (We note that this assumes a fixed bijection between characters on a cyclic group and its elements.) Then for such each (s), we may define the (s)-*Lusztig series* $\mathscr{E}(G, (s))$ of irreducible characters χ on G such that there exists a choice of (T, θ) corresponding to (s) such that

$$\langle \chi, R_T^G(\theta) \rangle \neq 0 \in \mathbb{Z}$$

(we note that $R_T^G(\theta)$ may produce a virtual representation). These Lusztig series can be found to partition the set of irreducible representations of $G(\mathbb{F}_q)$ into disjoint subsets indexed by the semisimple conjugacy classes $(s) \in G^*(\mathbb{F}_q)$.

We note that $(s) = (1) \in G^*(\mathbb{F}_q)$ corresponds to any pair $(T, 1)$ consisting of a torus $T \subseteq G(\mathbb{F}_q)$ and its trivial character. The irreducible $G(\mathbb{F}_q)$-representations appearing in the (1)-Lusztig series $\mathscr{E}(G, (1))$ are called the irreducible *unipotent representations* of $G(\mathbb{F}_q)$. We will write $\widehat{G(\mathbb{F}_q)}_u$ instead of $\mathscr{E}(G, (1))$ for conveniene in this case. (For example, in the case of $SL_2(\mathbb{F}_q)$, the irreducible unipotent representations are simply the trivial representation and the Steinberg representation.) We also note that there is a bijection

$$\begin{gathered} \mathscr{E}(G, (1)) \to \mathscr{E}(G^*, (1)) \\ u \mapsto \widetilde{u} \end{gathered} \tag{3.1.1}$$

Now fix a choice of a conjugacy class (s) of a semisimple element in $G^*(\mathbb{F}_q)$. There is now a bijection

$$\mathscr{E}(G, (s)) \to \widehat{Z_{G^*}(s)}_u \tag{3.1.2}$$

identifying the irreducible representations in the (s)-Lusztig series of $G(\mathbb{F}_q)$ with the set of irreducible unipotent representations of the centralizer of s in $G^*(\mathbb{F}_q)$. We will denote the $G(\mathbb{F}_q)$-representation corresponding to a semisimple conjugacy class (s) in $G^*(\mathbb{F}_q)$ and a unipotent irreducible representation $u \in \widehat{Z_{G^*}(s)}_u$ by

$$r^G[(s), u]. \tag{3.1.3}$$

The dimension of this representation is precisely

$$\dim(r^G[(s), u]) = \frac{|G(\mathbb{F}_q)|_{q'}}{|Z_{G^*(\mathbb{F}_q)}(s)|_{q'}} \dim(u)$$

(see, for example, [**5**], Theorem 8.4.8). For connected groups G with connected center, the representations $r^G[(s), u]$ are precisely the irreducible representations of $G(\mathbb{F}_q)$.

Next let us consider a connected G with disconnected center. (Specifically, so that we can treat the case of symplectic groups $G = \mathrm{Sp}_{2N}$ with center $\mathbb{Z}/2$.) It turns out that we can instead partition $\widehat{G(\mathbb{F}_q)}$ into disjoint sets indexed by semisimple conjugacy classes $(s) \in G^*(\mathbb{F}_q)$ which are in bijective correspondence with the sets of unipotent irreducible representations of the centralizer of s (see Theorem 13.23 of [**13**]). For a non-connected group $H(\mathbb{F}_q)$, which $Z_{G^*(\mathbb{F}_q)}(s)$ may be in this case, we define the unipotent irreducible representations to consist of all irreducible $H(\mathbb{F}_q)$-representations contained in an induction

$$\mathrm{Ind}_{H^\circ}^H(u) \tag{3.1.4}$$

for an irreducible unipotent representation u of $H's$ identity component. We still denote the set of irreducible unipotent representations of such an H by $\widehat{H}_u$.

In the case of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, for $(s) \in \mathrm{SO}_{2N+1}(\mathbb{F}_q) = \mathrm{Sp}_{2N}^*(\mathbb{F}_q)$, there are two choices of classes which are distinct over $\mathbb{F}_q$ but geometrically conjungate precisely when s has -1 eigenvalues, in which case there are two choices distinguished by whether the restriction of the symmetric bilinear form defining $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$ to the -1-eigenspace of s is split or not (see Definition 3.2.2). These are the cases of (s) whose centralizers are disconnected (specifically, they have an even orthogonal group factor $\mathrm{O}_{2\ell}^{\pm}(\mathbb{F}_q)$ is -1 is an eigenvalue of s of multiplicity 2ℓ). In this case, the induction (3.1.4) decomposes into two non-isomorphic irreducible unipotent representations unless the $\mathrm{O}_{2\ell}^{\pm}(\mathbb{F}_q)$-factor corresponds to a degenerate symbol (see Subsection 3.4 below). When the splitting occurs, we will denote the splitting by

$$\mathrm{Ind}_{Z_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(s)^\circ}^{Z_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(s)}(u) = u^+ \oplus u^-. \tag{3.1.5}$$

We shall denote the representations corresponding to such an (s), $u \in \widehat{Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(s)^\circ}_u$, and the choice of sign corresponding to a summand of (3.1.5), by

$$r^{\mathrm{SO}_{2N+1}}[(s), u, +1],\ r^{\mathrm{SO}_{2N+1}}[(s), u, -1].$$

To summarize, we say that irreducible representations of a connected group $G(\mathbb{F}_q)$ are classified by a certain collection of data:

3.1.1. Definition. *Define the* $G(\mathbb{F}_q)$-classification data *corresponding to an irreducible representation* ρ *of* G *to consist of*

(1) *The choice of conjugacy class* (s) *of a semisimple element* $s \in G^*(\mathbb{F}_q)$ *such that* ρ *is in a (rational) Lusztig series corresponding to* (s)*. We call* (s) *the* semisimple data *corresponding to* ρ*.*
(2) *The choice of a unipotent irreducible representation* u *of the identity component* $\widehat{Z_{G^*}(s)^\circ}_u$ *of* s*'s centralizer bijectively corresponding to* ρ*. We call this the* unipotent data *corresponding to* ρ*.*
(3) *When* $Z(G)$ *is disconnected, we also consider the choice of data specifying an irreducible summand of* $\mathrm{Ind}_{Z_{G^*}(s)^\circ}^{Z_{G^*}(s)}(u)$ *For cases of* $Z(G) = \mathbb{Z}/2$*, this data can be captured, when needed, as a* central sign*, and otherwise it is trivial.*

For classification data $[(s), u, \pm 1]$*, we write*

$$\rho = r^G[(s), u, \pm 1] \tag{3.1.6}$$

for the corresponding $G(\mathbb{F}_q)$*-representation. We omit* G *from the notation when it is fixed.*

This can be applied to cases including $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ and the special orthogonal groups $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$, $\mathrm{SO}^{\pm}_{2m}(\mathbb{F}_q)$. We treat the process of inducing to the full orthogonal groups as a separate step. In this case, since we are extending only by $\mathbb{Z}/2$, we can deduce the classification data for the full orthogonal groups by hand. For a treatment of the classification of irreducible representation in the general case of a disconnected reductive finite groups, we refer the reader to [**14**]. In the reminder of this chapter, we discuss each aspect of the classification data in more detail.

3.2. Tori and introducing the semisimple elements. The purpose of this subsection is to consider the form of maximal tori in $\mathrm{Sp}_{2r}(\mathbb{F}_q)$, $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$, or $\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$. Let us write $G(\mathbb{F}_q)$ for one of these groups. We recall that the superscript sign for the even special orthogonal groups is $+$ when the underlying symmetric bilinear form is fully split and $-$ when it is not fully split (i.e. when the maximal isotropic subspace of $\mathbb{F}_q^{2r}$ with respect to the symmetric bilinear form is of dimension r and when it is of dimension $r-1$, respectively). A maximal torus in such a group consists of a product of special orthogonal groups

$$\mathrm{SO}^{\pm}_2(\mathbb{F}_{q^{r_1}}) \times \cdots \times \mathrm{SO}^{\pm}_2(\mathbb{F}_{q^{r_k}}) \tag{3.2.1}$$

of maximal rank $r = r_1 + \cdots + r_k$ over $\mathbb{F}_q$, where, in the case of G an even special orthogonal group $\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$, the sign in the superscript is equal to the product of the signs appearing in (3.2.1) (for tori in odd special orthogonal groups and symplectic groups, there is no condition on the appearing signs). Note that we embed each $\mathrm{SO}^{\pm}_2(\mathbb{F}_{q^{r_i}})$ into $\mathrm{SO}^{\pm}_{2r_i}(\mathbb{F}_q)$ to consider its elements as matrices of the correct size. We describe this in more detail later. Recall that (using the standard choices of symmetric bilinear forms)

$$\mathrm{SO}^{+}_2(\mathbb{F}_q) = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid x \in \mathbb{F}_q^{\times}\} \cong \mu_{q-1} \tag{3.2.2}$$

$$\begin{aligned} \mathrm{SO}^{-}_2(\mathbb{F}_q) = \{\begin{pmatrix} a & \alpha\cdot b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{F}_q\ a^2 - \alpha b^2 = 1\} \cong \\ \mathbb{F}_{q^2}^{\times}/\mathbb{F}_q^{\times} \cong \mu_{q+1}, \end{aligned} \tag{3.2.3}$$

where in (3.2.3), $\alpha \in \mathbb{F}_q^{\times}$ is fixed to be an element which is not a square, and we consider $\mathbb{F}_{q^2} = \mathbb{F}_q[\sqrt{\alpha}]$, whose norm 1 elements $a + \sqrt{\alpha}b$ form a subgroup of $\mathbb{F}_{q^2}^{\times}$ which is isomorphic to μ_{q+1}. Note that the eigenvalues of the elements (3.2.3) are precisely $a \pm \sqrt{\alpha}b$.

Every maximal torus in the group $G(\mathbb{F}_q)$ is $G(\mathbb{F}_q)$-conjugate to a torus of the form (3.2.1). In particular, every semisimple element $(s) \in G$ is conjugate to an element of (3.2.1), which is classified by the conjugacy classes of the list of its eigenvalues, under the action of the Weyl group of G.

Let us change coordinates so that we can write the symmetric bilinear form B defining an orthogonal group $\mathrm{O}(W, B)$ as

$$B = \bigoplus_{h_B} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus Z_B, \tag{3.2.4}$$

for anisotropic part Z_B of dimension $0, 1, 2$, and so that we can write the symplectic form S defining a symplectic group $\mathrm{Sp}(V, S)$ as

$$S = \bigoplus_{dim(V)/2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.2.5}$$

we can in fact consider the tori (3.2.1) as embedded in G directly by taking a direct sum of matrices. We note that in the case of G an odd special orthogonal group $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$, to embed a torus of the form (3.2.1) into G, we need to insert a "forced" diagonal entry 1 to obtain a matrix of size $2r+1$. In other words, every semisimple element of $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$ has a single extra 1 eigenvalue in addition to the eigenvalues detected by its conjugacy class in (3.2.1), so that the total number of 1 eigenvalues can be odd. The placement of this entry is according to whether the product (3.2.1) is a subgroup of $\mathrm{SO}^+_{2r}(\mathbb{F}_q)$ or $\mathrm{SO}^-_{2r}(\mathbb{F}_q)$, which is governed by the product signs in the superscripts.

Therefore, we can see that the data of a list of elements

$$(\lambda_1, \dots, \lambda_k) \in \prod_{i=1}^{k} \mu_{q^{r_i}\pm 1} \tag{3.2.6}$$

for $\lambda_i \in \mu_{q^{r_i}\pm 1}$ determines a semisimple element of $\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$ obtained by taking a direct sum of matrices in $\mathrm{SO}^{\pm}_{2r_i}(\mathbb{F}_q)$ corresponding to each λ_i, so that in $\mathrm{SO}^{\pm}_{2r_i}(\mathbb{F}_{q^{r_i}})$, each is conjugate

$$\bigoplus_{j=0}^{r_i-1} \begin{pmatrix} \lambda_i^{q^j} & 0 \\ 0 & \lambda_i^{-q^j} \end{pmatrix}. \tag{3.2.7}$$

To make this form easier to discuss, let us introduce a notation for these blocks: Write A_{λ_i} for the element of $\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$ conjugate to (3.2.7). We note that replacing λ_i by λ_i^q produces the same semisimple conjugacy class.

This is enough information about semisimple elements for our purposes. We note that in the case of symplectic groups, we may embed each factor $\mathrm{SO}^{\pm}_2(\mathbb{F}_{q^{r_i}})$ of the torus into $\mathrm{SL}_2(\mathbb{F}_{q^{r_i}})$, so the conjugacy class corresponding to (3.2.6) further only depends on the equivalence class

$$(\lambda_1, \dots, \lambda_k) \in \prod_{i=1}^{k} \mu_{q^{r_i}\pm 1}/(\lambda \sim \lambda^{-1}). \tag{3.2.8}$$

In the case of odd special orthogonal groups $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$, there is one more technical detail. As we mentioned, the semisimple elements of $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$ correspond to semisimple elements of $\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$, embedded into $\mathrm{SO}_{2r}^{\pm}(\mathbb{F}_q)$ by adding a single diagonal 1 entry where it is "forced to be." For $2r$ by $2r$ matrices which can be considered in groups (3.2.1) for more than one choice of signs (meaning some factors are equal to the identity matrix I or $-I$), then we must consider whether these two choices of where to insert the final "forced" diagonal entry 1 give different conjugacy classes in $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$ or not. The only cases of eigenvalues which can correspond to elements of either choice of torus splitness are 1 and -1 eigenvalues. If only 1 eigenvalues are present, it is indistinguishable where we add the extra 1 diagonal entry. So we find that these two choices give different conjugacy classes if and only if the $2r$ by $2r$ element considered in (3.2.1) has any -1 eigenvalues, in which case the resulting two choices of elements will turn out to have different centralizers and cannot be conjugate in $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$. We also note that odd special orthogonal groups, like symplectic groups, do have a large enough Weyl group to consider eigenvalue data only up equivalence class as in (3.2.8).

To summarize, we may consider any semisimple element of a group $G(\mathbb{F}_q)$ as conjugate to a sum of blocks

$$s \sim A_{\lambda_1} \oplus \cdots \oplus A_{\lambda_k},$$

with an additional 1 inserted in the case of $G = \mathrm{SO}_{2r+1}(\mathbb{F}_q)$, considering the extra data of a choice of where precisely if one of the λ_i is equal to -1. We write may write the superscript $(A_{\lambda_1} \oplus \cdots \oplus A_{\lambda_k})^{\pm}$ in the case of $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$ to emphasize whether the element is considered in a split or non-split torus (i.e. a torus in $\mathrm{SO}_{2r}^{\pm}(\mathbb{F}_q)$). Further, if we must refer to a maximal torus containing s, let us also always choose to minimize the field extensions $\mathbb{F}_{q^{r_i}}$ needed in (3.2.1) to contain the eigenvalues of s, so that there are no $r_i' < r_i$ such that $\lambda_i \in \mathbb{F}_{q^{r_i'}} \subset \mathbb{F}_{q^{r_i}}$.

To see how our description of semisimple elements works in practice, let us discuss the examples of rank 1 groups G. The even cases of $\mathrm{SO}_2^{\pm}(\mathbb{F}_q) = \mu_{q\mp 1}$ are not large enough to show any interesting effects.

3.2.1. Example ($G(\mathbb{F}_q) = \mathrm{SL}_2(\mathbb{F}_q)$ and $\mathrm{SO}_3(\mathbb{F}_q)$). *For either case of $G(\mathbb{F}_q) = SL_2(\mathbb{F}_q)$ or $SO_3(\mathbb{F}_q)$, the only maximal tori are isomorphic to the special orthogonal groups $SO_2^{\pm}(\mathbb{F}_q) \sim \mu_{q\mp 1}$.*

On the one hand, in the case of $G(\mathbb{F}_q) = SL_2(\mathbb{F}_q)$, our theory gives that there are

(1) *two central elements*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which can be considered in both $SO_2^{\pm}(\mathbb{F}_q)$.

(2) $(q-3)/2$ *central elements conjugate to*

$$A_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \sim \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \tag{3.2.9}$$

for $\lambda \in \mathbb{F}_q^\times \smallsetminus \{\pm 1\} = \mu_{q-1} \smallsetminus \{\pm 1\}$

(3) $(q-1)/2$ *central elements* $A_\mu \in SO_2^-(\mathbb{F}_q)$ *which are conjugate to*

$$\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \sim \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix} \tag{3.2.10}$$

in $SL_2(\mathbb{F}_{q^2})$*, for any norm* 1 *elements* $\mu \in \mathbb{F}_{q^2} \smallsetminus \{\pm 1\}$*, i.e. elements of* $\mu_{q+1} \smallsetminus \{\pm 1\}$

We obtain a total of $2q$ *semisimple conjugacy classes in* $SL_2(\mathbb{F}_q)$*.*

On the other hand, in $G(\mathbb{F}_q) = SO_3(\mathbb{F}_q)$*, we must consider more carefully how the blocks* A_λ*,* A_μ *can be embedded as* 3×3 *matrices in* $G(\mathbb{F}_q)$*. For this, let us suppose the symmetric bilinear form* B *defining* $G(\mathbb{F}_q) = SO(\mathbb{F}_q^3, B)$ *is of the form (3.2.4) with* -1 *discriminant (the case of* 1 *discriminant is entirely similar, with a reversed choice of where the forced* 1*'s are placed). For a choice of* $\lambda \in \mathbb{F}_q^\times$*, corresponding to a block* $A_\lambda \in SO_2^+(\mathbb{F}_q)$*, we can embed it as the element*

$$A_\lambda^+ := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(due to the ambiguity in the case when $\lambda = -1$*, it is meaningful in this case to keep track of which sign is in the superscript of the choice of* $SO_2^\pm(\mathbb{F}_q)$ *we start with). To see why* A_λ^+ *and* $A_{\lambda^{-1}}^+$ *are conjugate, we have*

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, for choices of $\mu = a + \sqrt{\alpha} b$ *of norm* 1 *in* $\mathbb{F}_{q^2}$*, we can embed* $A_\mu \in SO_2^-(\mathbb{F}_q)$ *as a* 3×3 *matrix in* $SO_3(\mathbb{F}_q)$*. A similar argument gives that* A_μ^- *and* $A_{\mu^{-1}}^-$ *are conjugate in* $SO_3(\mathbb{F}_q)$*.*

Now consider the elements A_{-1}^+ *and* A_{-1}^-*. In* $SO_3(\mathbb{F}_q)$*, there are two singular conjugacy classes of type* $(0,1)$*: the conjugacy classes of*

$$\sigma_1^+ := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1^- := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which have centralizers $O_2^+(\mathbb{F}_q)$, $O_2^-(\mathbb{F}_q)$ *in* $SO_3(\mathbb{F}_q)$*, respectively (since here the centralizer is the orthogonal group on the two coordinates corresponding to the two* -1 *entries in* $\sigma_1^{\pm}$*). Note that in* $SL_2(\mathbb{F}_q)$*, this effect does not occur, and conjugacy classes of semisimple elements are in fact in all cases determined. We therefore find that there are a total of* $2q+1$ *semisimple conjugacy classes in* $SO_3(\mathbb{F}_q)$*.*

In $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$, elements $\sigma_n^{\pm}$ which play a role generalizing $\sigma_1^{\pm}$ exist and play a very important role in considering oscillator representations. We define them now:

3.2.2. Definition. *Let us consider an odd special orthogonal group* $SO_{2r+1}(\mathbb{F}_q)$*. Consider the element consisting of a sum of* r *copies of* A_{-1} *lying in either a split or non-split choice of torus in* $SO_{2r}^{\pm}(\mathbb{F}_q) \subset SO_{2r+1}(\mathbb{F}_q)$*. We write*

$$\sigma_r^+ := (\bigoplus_r A_{-1})^+$$
$$\sigma_r^- := (\bigoplus_r A_{-1})^-$$

for the $(2r+1)\times(2r+1)$ *matrices in* $SO_{2r+1}(\mathbb{F}_q)$ *obtained by adding the* 1 *forced by considering the sum of* A_{-1}*'s as an element of the split and non-split special orthogonal group, respectively.*

We find that the centralizers of these semisimple elements are

$$Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(\sigma_r^+) \cong \mathrm{O}_{2r}^+(\mathbb{F}_q)$$
$$Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(\sigma_r^-) \cong \mathrm{O}_{2r}^-(\mathbb{F}_q).$$

In particular σ_r^+ and σ_r^- cannot be conjugate in $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$. We note that they are geometrically conjugate in $\mathrm{SO}_{2r+1}(\overline{\mathbb{F}_q})$.

3.3. Semisimple data. In the previous subsection, we discussed how the data of a conjugacy class of a semisimple element in a symplectic or special orthogonal group is equivalent to the data of its eigenvalues under the action of the Weyl group (with the possible additional data of a sign encoding split/non-splitness when -1 is an eigenvalue in an odd special orthogonal group). In this subsection, we begin considering semisimple conjugacy classes from the perspective of their role as a component of classification data.

First, as described in Subsection 3.1, the semisimple component of the classification data for irreducible representations of $G(\mathbb{F}_q)$ consists of a semisimple conjugacy class in the dual group $G^*(\mathbb{F}_q)$. Since for general linear, unitary, and even special orthogonal groups, all roots have the same length, these groups are all self-dual

$$\begin{gathered}\mathrm{GL}_r^*(\mathbb{F}_q)=\mathrm{GL}_r(\mathbb{F}_q),\ (\mathrm{U}_r^-)^*(\mathbb{F}_q)=\mathrm{U}_r^-(\mathbb{F}_q),\\ (\mathrm{SO}_{2r}^{\pm})^*(\mathbb{F}_q)=\mathrm{SO}_{2r}^{\pm}(\mathbb{F}_q),\end{gathered}\tag{3.3.1}$$

while for the symplectic and odd special orthogonal groups,

$$\mathrm{Sp}_{2r}^*(\mathbb{F}_q)=\mathrm{SO}_{2r+1}(\mathbb{F}_q),\ \mathrm{SO}_{2r+1}^*(\mathbb{F}_q)=\mathrm{Sp}_{2r}(\mathbb{F}_q).\tag{3.3.2}$$

As we discussed above, any symplectic or special orthogonal group, every maximal torus is isomorphic to a product of the form (3.2.1) of $\mathrm{SO}_2^{\pm}$ factors, possibly on field extensions of $\mathbb{F}_q$, such that the sum of the degrees of these field extensions adds up to the total rank, and in the case of tori in even special orthogonal groups, the sign $\pm$ denoting whether or not the full group is equal to the product of signs appearing in each $\mathrm{SO}_2^{\pm}$ factor.

Fix a semisimple conjugacy class $(s)\in G^*(\mathbb{F}_q)$ as the semisimple component of a choice of classification data for an irreducible representation of $G(\mathbb{F}_q)$. A compatible choice of unipotent data consists of an irreducible unipotent representation of the centralizer $Z_{G^*(\mathbb{F}_q)}(s)$. For a complete rigorous description of the form of centralizers of semisimple elements for any finite groups of Lie type, see [**4**].

To compute the centralizer of $(s)\in G^*(\mathbb{F}_q)$, consider s as an element of some maximal torus T isomorphic to (3.2.1), minimizing the field extension degrees r_i where possible. Say that (s) is obtained from a sum of blocks

$$A_{\lambda_1}\oplus\cdots\oplus A_{\lambda_k}\in\mathrm{SO}_2^{\pm}(\mathbb{F}_{q^{r_1}})\times\cdots\times\mathrm{SO}_2^{\pm}(\mathbb{F}_{q^{r_k}})$$

for $\lambda_i\in\mu_{q^{r_i}\mp 1}$.

For choices of $\lambda_i\neq\pm1$ occuring with multiplicity, we obtain centralizers of type A or ${}^2A$, depending on the sign in the superscript of $\lambda_i\in\mathrm{SO}_2^{\pm}(\mathbb{F}_{q^{r_i}})$ (this sign is fixed, since $\lambda_i\neq\pm1$). To be specific, say there are j blocks $A_{\lambda_{i_1}},\dots,A_{\lambda_{i_j}}$ such that $\lambda_{i_1}=\cdots=\lambda_{i_j}=a$ for some fixed $a\in\mu_{q^n\mp1}$ (and not in any $\mu_{q^{n'}\mp1}$ for $n'<n$). These blocks then contribute a centralizer factor $\mathrm{U}_j^{\pm}(\mathbb{F}_{q^n})$ again recalling that we use the notation $\mathrm{U}_j^+=\mathrm{GL}_j$. Let us change notation so that $\lambda_1,\dots,\lambda_t$ are all the distinct choices of eigenvalues not equal to ±1 appearing in s, with $\lambda_i\in\mu_{q^{r_i}\mp1}$ (with minimal chosen r_i) of multiplicity j_i, each. Say each A_{λ_i} contributes a factor in $\mathrm{SO}_2^{\pm}(\mathbb{F}_{q^{r_i}})$. The form of the centralizer factors contributed by blocks consisting of eigenvalues ±1 depends on the specific choice of G^* and G. Let us consider the specific cases now.

For a semisimple conjugacy class (s) in $G^*(\mathbb{F}_q)=\mathrm{Sp}_{2r}(\mathbb{F}_q)$ (to describe representations of $G(\mathbb{F}_q)=\mathrm{SO}_{2r+1}(\mathbb{F}_q)$), a semisimple conjugacy class $(s) \in \mathrm{Sp}_{2r}(\mathbb{F}_q)$ has centralizer

$$Z_{\mathrm{Sp}_{2r}(\mathbb{F}_q)}(s) = \prod_{i=1}^{t} \mathrm{U}^{\pm}_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{Sp}_{2p}(\mathbb{F}_q) \times \mathrm{Sp}_{2\ell}(\mathbb{F}_q) \tag{3.3.3}$$

where s has 1 as an eigenvalue of multiplicity $2p$ and -1 as an eigenvalue of multiplicity 2ℓ. To match rank, we must have

$$r = \sum_{i=1}^{t} j_i r_i + p + \ell. \tag{3.3.4}$$

Next, say we must consider a semisimple conjugacy class (s) in $G^*(\mathbb{F}_q) = \mathrm{SO}_{2r+1}(\mathbb{F}_q)$ (to describe representations of $G(\mathbb{F}_q) = \mathrm{Sp}_{2r}(\mathbb{F}_q)$), a semisimple element $(s) \in \mathrm{SO}_{2r+1}(\mathbb{F}_q)$ has centralizer

$$Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s) = \prod_{i=1}^{t} \mathrm{U}^{\pm}_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}_{2p+1}(\mathbb{F}_q) \times \mathrm{O}^{\pm}_{2\ell}(\mathbb{F}_q) \tag{3.3.5}$$

with identity component

$$Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)^\circ = \prod_{i=1}^{t} \mathrm{U}^{\pm}_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}_{2p+1}(\mathbb{F}_q) \times \mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q),$$

if s has 1 as an eigenvalue of multiplicity $2p+1$ and -1 as an eigenvalue of multiplicity 2ℓ. We recall that a single 1 eigenvalue is automatic in this case, since it must be added to embed a torus T in $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$. Again, the sign of the final factor $\mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q)$ corresponds to whether we pick $T \subset \mathrm{SO}^{+}_{2r}(\mathbb{F}_q)$ or $\mathrm{SO}^{-}_{2r}(\mathbb{F}_q)$, depending on whether (s) is conjugate to a product of blocks involving σ^{+}_{ℓ} or σ^{-}_{ℓ} (recalling Definition 3.2.2). As in the case of the symplectic group, the rank must satisfy (3.3.4).

Finally, in the case of semisimple conjugacy classes in $G^*(\mathbb{F}_q) = \mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$ (to describe representations of $G(\mathbb{F}_q) = G^*(\mathbb{F}_q) = \mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$), a semisimple $(s) \in \mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$ has centralizer

$$Z_{\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)}(s) = \prod_{i=1}^{t} \mathrm{U}^{\pm}_{j_i}(\mathbb{F}_{q^{r_i}}) \times (\mathrm{O}^{\pm}_{2p}(\mathbb{F}_q) \times \mathrm{O}^{\pm}_{2\ell}(\mathbb{F}_q))^{\det=1} \tag{3.3.6}$$

where s has 1 as an eigenvalue of multiplicity $2p$ and -1 as an eigenvalue of multiplicity 2ℓ (and where $(\mathrm{O}^{\pm}_{2p}(\mathbb{F}_q) \times \mathrm{O}^{\pm}_{2\ell}(\mathbb{F}_q))^{\det=1}$ denotes the group of consisting of pairs of matrices in $\mathrm{O}^{\pm}_{2p}(\mathbb{F}_q) \times \mathrm{O}^{\pm}_{2\ell}(\mathbb{F}_q)$ such that the product of their determinants is 1). The identity component of this centralizer is

$$Z_{\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)}(s)^\circ = \prod_{i=1}^{t} \mathrm{U}^{\pm}_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}^{\pm}_{2p}(\mathbb{F}_q) \times \mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q).$$

Again, the signs in the superscripts are restricted by the total sign in the superscript of $G(\mathbb{F}_q) = \mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$, and the ranks must satisfy (3.3.4).

3.4. Unipotent data. Now we consider the unipotent component of classification data, given a choice of conjugacy class of a semisimple element (s) in the dual $G^*(\mathbb{F}_q)$ for $G(\mathbb{F}_q)$ for example a symplectic or special orthogonal group. This data consists of an irreducible unipotent representation u of s's centralizer, which we have computed as (3.3.3), (3.3.5), or (3.3.6) depending on the choice of G. We note that these unipotent representations are precisely those obtained in the induction of an irreducible unipotent representation of the identity component of s's centralizer.

We may consider such a u as a tensor product of irreducible unipotent representations of the factors appearing in the centralizer (3.3.3), (3.3.5), or (3.3.6) of (s), which may consist of some general linear or non-split unitary groups (corresponding to blocks of eigenvalues not equal to ± 1 in s) and a pair of groups which are symplectic or special orthogonal with type determined by the case of $G^*(\mathbb{F}_q)$ (corresponding to blocks of 1 and -1 eigenvalues).

The unipotent representations of a group of the form

$$\mathrm{U}^{\pm}_{r}(\mathbb{F}_q),\ \mathrm{Sp}_{2r}(\mathbb{F}_q),\ \mathrm{SO}_{2r+1}(\mathbb{F}_q),\ \mathrm{O}^{\pm}_{2r}(\mathbb{F}_q), \tag{3.4.1}$$

i.e. the connected groups of types A or ${}^2A$, B, C, and D or ${}^2D$ over $\mathbb{F}_q$, are governed by the combinatorial data of *symbols*, which we use [**44, 45**] as references for. In general, the symbols of a finite group of Lie type are related to the classification of the irreducible representations of its corresponding Weyl group, in combination with the data of the *unipotent cuspidal representations*. We will return to some properties of the Lusztig symbols in Subsection 7.1. Here, we specifically focus on the concrete classifications of unipotent irreducible representations in each specific case of (3.4.1), and their dimensions. In fact, it will turn out that for our purposes of decomposing the restricted oscillator representation, the most relevant factors of (3.3.3), (3.3.5), and (3.3.6) are precisely those corresponding to eigenvalues equal to ± 1 and it is actually enough to describe the symbols for symplectic and orthogonal groups. However, for completeness, we also describe the symbols for the general linear and non-split unitary groups as a warm-up case, to give some sense of the relationship with the Weyl group.

The classification of irreducible unipotent representations of the groups $\mathrm{U}^{+}_{r}(\mathbb{F}_q) = \mathrm{GL}_r(\mathbb{F}_q)$ is by the *symbols of A-type* and rank r, which are defined to be increasing sequences

$$(\lambda_1, \dots, \lambda_a) \tag{3.4.2}$$

of integers $\lambda_i \geq 0$ such that

$$\sum_{i=1}^{a} \lambda_i = r + \binom{a}{2}. \tag{3.4.3}$$

Its dimension is

$$(q-1)\frac{\prod\limits_{1\leq i<j\leq a} (q^{\lambda_j} - q^{\lambda_i})}{q^{c(a)} \prod\limits_{i=1}^{a} \prod\limits_{j=1}^{\lambda_i} (q^j - 1)} |\mathrm{GL}_r(\mathbb{F}_q)|_{q'} \tag{3.4.4}$$

where we write $c(a) = \sum_{i=1}^{a} \binom{a-i}{2}$. The same classification applies to the irreducible representations of $\mathrm{U}_r^-(\mathbb{F}_q)$, with *symbols of* ${}^2A$*-type* and rank r also defined as increasing sequences of the form (3.4.2). The dimension of the corresponding representation

$$(q+1)\frac{\prod\limits_{1\leq i<j\leq a} (q^{\lambda_j} + (-1)^{\lambda_i+\lambda_j} q^{\lambda_i})}{q^{c(a)} \prod\limits_{i=1}^{a} \prod\limits_{j=1}^{\lambda_i} (q^j - (-1)^j)} |\mathrm{U}_r^-(\mathbb{F}_q)|_{q'}, \tag{3.4.5}$$

(also obtained by replacing q by $-q$ and taking the absolute value in (3.4.4)), with the same choice of $c(a)$.

The classification of irreducible unipotent representations of the groups $\mathrm{Sp}_{2r}(\mathbb{F}_q)$ and $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$ are the same, since they are dual groups and have the same Weyl groups (recall our notation (3.1.1)). A *symbol of* B*- or* C*-type* and rank r is defined to be a pair of increasing sequences

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

for $\lambda_i, \mu_i \in \mathbb{Z}_{\geq 0}$ such that $(\lambda_1, \mu_1) \neq (0,0)$, $a - b$ is odd (the "defect condition"), and

$$\sum_{i=1}^{a} \lambda_i + \sum_{i=1}^{b} \mu_i = r + \frac{(a+b-1)^2}{4} \tag{3.4.6}$$

(the "rank condition"). We take switching rows to give the same symbol. The irreducible unipotents are in bijective correspondence with this combinatorial data (and we denote the representation corresponding to a symbol by the symbol itself).

Similarly, for the case of $\mathrm{SO}^+_{2r}(\mathbb{F}_q)$ (resp. $\mathrm{SO}^-_{2r}(\mathbb{F}_q)$), irreducible unipotent representations correspond to symbols of D- (resp. ${}^2D$-type) and rank r, which are defined to consist of pairs of increasing sequences $\binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ for $\lambda_i, \mu_i \in \mathbb{Z}_{\geq 0}$ such that $(\lambda_1, \mu_1) \neq (0,0)$, the "defect condition" $a-b \equiv 0 \bmod 4$ (resp. $\equiv 2 \bmod 4$, for $\mathrm{SO}^-_{2r}(\mathbb{F}_q)$) and the "rank condition"

$$\sum_{i=1}^{a} \lambda_i + \sum_{i=1}^{b} \mu_i = r + \frac{(a+b)(a+b-2)}{4} \tag{3.4.7}$$

(the same rank condition is used for $\mathrm{SO}^-_{2r}(\mathbb{F}_q)$). Again, we denote a unipotent representation the same as its corresponding symbol. In the case of $\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$, choosing $a=b$, $\lambda_i = \mu_i$ gives a *degenerate symbol*

$$\begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \lambda_1 < \dots < \lambda_a \end{pmatrix}. \tag{3.4.8}$$

This symbol corresponds to a unipotent $\mathrm{SO}^+_{2r}(\mathbb{F}_q)$-representation which splits into two irreducible equi-dimensional pieces, which we call its *degenerations*.

We can additionally calculate the dimension of a unipotent representation $\binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ of a symplectic or special orthogonal group G can be calculated as the following formula

$$\frac{\prod\limits_{1\leq i<j\leq a} (q^{\lambda_j} - q^{\lambda_i}) \prod\limits_{1\leq i<j\leq b} (q^{\mu_j} - q^{\mu_i}) \prod\limits_{\substack{1\leq i\leq a \\ 1\leq j\leq b}} (q^{\lambda_i} + q^{\mu_j})}{2^{d(a,b)} q^{c(a,b)} \prod\limits_{1\leq i\leq a} \prod\limits_{j=1}^{\lambda_i} (q^{2j}-1) \prod\limits_{1\leq i\leq b} \prod\limits_{j=1}^{\mu_i} (q^{2j}-1)} |G|_{q'} \tag{3.4.9}$$

where we write

$$d(a,b) = \lfloor \frac{a+b-1}{2} \rfloor$$

and

$$c(a,b) = \sum_{i=1}^{\lfloor (a+b)/2 \rfloor} \binom{a+b-2i}{2}.$$

The induction of an $\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$-irreducible unipotent representation to $\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)$ splits into two non-isomorphic equidimensional irreducible unipotent representations

$$\begin{gathered}\operatorname{Ind}_{\mathrm{SO}_{2r}^{\pm}(\mathbb{F}_q)}^{\mathrm{O}_{2r}^{\pm}(\mathbb{F}_q)}(\begin{pmatrix}\lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b\end{pmatrix}) = \\ \begin{pmatrix}\lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b\end{pmatrix}^{+} \oplus \begin{pmatrix}\lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b\end{pmatrix}^{-},\end{gathered} \tag{3.4.10}$$

with an exception in the split case, since for both irreducible $\mathrm{SO}_{2r}^{+}(\mathbb{F}_q)$-unipotent representations corresponding to summands of a degenerate symbol $\binom{\lambda_1<\cdots<\lambda_a}{\lambda_1<\cdots<\lambda_a}$, their inductions to an $\mathrm{O}_{2r}^{+}(\mathbb{F}_q)$ are irreducible and isomorphic. We simply this single irreducible unipotent representation by the symbol $\binom{\lambda_1<\cdots<\lambda_a}{\lambda_1<\cdots<\lambda_a}$. In particular, the dimension of each $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}^{\pm}$ and each single irreducible unipotent representation corresponding to each degenerate symbol $\binom{\lambda_1<\cdots<\lambda_a}{\lambda_1<\cdots<\lambda_1}$ can be computed by (3.4.9) with G taken to be the special orthogonal group.

Since the symbols come from the irreducible representations of Weyl groups, they also turn out to obey a "Pieri rule" for induction, which we describe in Subsection 7.2. However, the symbols for symplectic and (special) orthogonal groups do not exactly correspond to the combinatorial data classifying irreducible representations of the associated Weyl group. Rather, the symbols correspond to choices of irreducible representations of a lower Weyl group, combined with a certain "shift," determining the defect of the symbols. (We describe this more concretely in Subsection 7.1.) This effect is caused by the existence of *unipotent cuspidal representations* for symplectic and special orthogonal groups. From the perspective of using Deligne-Lusztig induction to adapt the classical "Harish-Chandra" philosophy to the finite field context, these unipotent cuspidal representations can be thought of as playing the role of the genuine cusp forms. These unipotent cuspidal representations correspond to symbols of the form

$$\begin{pmatrix}0 < 1 < 2 < \cdots < k \\ \emptyset\end{pmatrix}, \tag{3.4.11}$$

giving a representation of a symplectic group or odd special orthogonal group if k is even, a representation of a split even special orthogonal group if k is 3 mod 4, and a representation of a non-split even special orthogonal group if k is 1 mod 4. For k even, writing $k = 2d$, the rank condition (3.4.6) specifies that (3.4.11) gives a unipotent cuspidal representation in

$$\mathrm{Sp}_{2(d^2+d)}(\mathbb{F}_q) \text{ and } \mathrm{SO}_{2(d^2+d)+1}(\mathbb{F}_q).$$

For k odd, writing $k = 2d - 1$, the rank condition (3.4.7) specifies that (3.4.11) gives a unipotent cuspidal representation of

$$\mathrm{SO}^{\pm}_{2d^2}(\mathbb{F}_q)$$

with the superscript sign determined by k mod 4 as described above.

These unipotent cuspidal representations are quite mysterious. Our full generality Howe duality statement, however, may serve to illuminate some aspects of their structure, at least giving machinery to construct them "concretely" and calculate their characters which in some cases may be easier to compute than the approach through character sheaves and cells of symbols (which we do not discuss here but refer the reader to [**45**] and [**11**] for more details on this other approach). We discuss this in Subsection 7.5. Additionally, beyond the key role they play in the representation theory of the finite algebraic groups, the unipotent cuspidal representations are also, by deisgn, representations without Whittaker models; can be interpreted as a "source" of counterexamples to the naive generalized Ramanujan conjecture [**31**].

We also note that the non-split unitary groups also have unipotent cuspidal representations. These unipotent cuspidal representations correspond to increasing sequences of odd integers

$$(1, 3, 5, \dots, 2d - 1),$$

which the rank condition (3.4.3) specifies correspond to irreducible unipotent representations of the non-split unitary groups of rank $\binom{d+1}{2}$.

3.5. Central data and extension data. We have almost completely determined the classification data for symplectic and special orthogonal groups, having discussed the semisimple and unipotent compotents in enough detail to use for the purpose of decomposing the restricted oscillator representations. The semisimple and unipotent data is enough to describe the irreducible representations odd special orthogonal groups, since they have trivial center and are connected. This is not enough for symplectic groups or even special orthogonal groups, since they have non-trivial centers, and hence, certain classification data also require us to specify *central sign data*.

Additionally, recalling our intended application to describing the eta and zeta correspondences, we need to describe the irreducible representations of the full orthogonal groups. This is simple enough for the odd orthogonal groups, since the center of $\mathrm{O}_{2r+1}(\mathbb{F}_q)$ splits off to give a product

$$\mathrm{O}_{2r+1}(\mathbb{F}_q) = \mathrm{SO}_{2r+1}(\mathbb{F}_q) \times \mathbb{Z}/2. \tag{3.5.1}$$

However, this no longer works for even orthogonal groups, and a more careful treatment is needed. In any case, we will parametrize the irreducible representations of these full, disconnected orthogonal groups according to *extension data*.

The goal of this subsection is to describe the classification data indexing the irreducible representations of symplectic and orthogonal groups in each case.

First we consider $\mathrm{Sp}_{2r}(\mathbb{F}_q)$. Its center is $\mathbb{Z}/2$. There are two distinct irreducible representations $r[(s), u, \pm 1]$ corresponding to semisimple data (s) and unipotent data u if and only if s has any -1 eigenvalues and the factor of u corresponding to the factor of s's centralizer associated to the -1 eigenvalues corresponds to a non-degenerate symbol. This is because it is for precisely these s's where there are two choices of rational conjugacy classes in $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$ (including (s) itself) which are geometrically conjugate to s, and therefore $Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)$ is disconnected. For an irreducible unipotent representation u of $Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)^\circ$, its induction can be decomposed into pieces (3.1.5) (see (3.4.10)) unless u corresponds to a degenerate symbol. For u, u' which are the same except for the -1-egienvalue factors which correspond to distinct summands of a degenerate symbol, they give the same irreducible $\mathrm{Sp}_{2r}(\mathbb{F}_q)$-representation since

$$\mathrm{Ind}_{Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)^\circ}^{Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)}(u) = \mathrm{Ind}_{Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)^\circ}^{Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)}(u').$$

In the case where there are these two representations, we consider the sum

$$r[(s), u] = r[(s), u, +1] \oplus r[(s), u, -1]$$

We then have

$$\dim(r[(s), u, \pm 1]) = \frac{1}{2}\dim(r[(s), u]) = \frac{|\mathrm{Sp}_{2r}(\mathbb{F}_q)|_{q'}}{2|Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)|_{q'}}\dim(u)$$

Otherwise, $r[(s), u]$ is already irreducible of dimension equal to $\dim(u)$ times $|\mathrm{Sp}_{2r}(\mathbb{F}_q)|_{q'}/|Z_{\mathrm{SO}_{2r+1}(\mathbb{F}_q)}(s)|_{q'}$.

Now let us consider how to pass from the representatios of special orthogonal groups to those of full orthogonal groups. In both even and odd cases, we can see that every irreducible representation of the full orthogonal group is contained in the induction of a representation corresponding to certain semisimple and unipotent data of the special orthogonal group. We also note that, particularly in the even case, considering central sign data on the special orthogonal group level is somewhat counterproductive, since we must induce again up to the orthogonal group, which may destroy the irreducibility.

For odd orthogonal groups $O_{2r+1}(\mathbb{F}_q)$, since we have (3.5.1), we find that the irreducible representations of $O_{2r+1}(\mathbb{F}_q)$ are precisely the irreducible representations of $SO_{2r+1}(\mathbb{F}_q)$ tensored with either the trivial or sign representation of the center $\mathbb{Z}/2 = Z(O_{2r+1}(\mathbb{F}_q))$. Let us write

$$r^{O_{2r+1}(\mathbb{F}_q)}[(s), u]^{\pm 1} := r^{SO_{2r+1}(\mathbb{F}_q)}[(s), u] \otimes (\pm 1), \tag{3.5.2}$$

indexing $O_{2r+1}(\mathbb{F}_q)$-representations by the "extended" classification data of

(1) The *semisimple data* of the conjugacy class of a semisimple element $s \in SO^*_{2r+1}(\mathbb{F}_q) = Sp_{2r}(\mathbb{F}_q)$.
(2) The *unipotent data* of an irreducible unipotent representation of
$$Z_{Sp_{2r}(\mathbb{F}_q)}(s).$$
(3) A *extension sign* ± 1, indicating whether the central $\mathbb{Z}/2$ factor of (3.5.1) acts by the trivial or sign representation.

We note that for both choices of extension sign,

$$\dim(r^{O_{2r+1}(\mathbb{F}_q)}[(s), u]^{\pm 1}) = \dim(r^{SO_{2r+1}(\mathbb{F}_q)}[(s), u]).$$

Let us now consider $O^{\pm}_{2r}(\mathbb{F}_q)$-representations. Unlike the odd orthogonal groups, $O^{\pm}_{2r}(\mathbb{F}_q)$, the group $SO^{\pm}_{2r}(\mathbb{F}_q)$ retains non-trivial center $\mathbb{Z}/2$, and $O^{\pm}_{2r}(\mathbb{F}_q)$ can only be considered as a semidirect product of the special orthogonal group with a (separate) copy of $\mathbb{Z}/2$. Attempting to pass from $SO^{\pm}_{2r}(\mathbb{F}_q)$ to $O^{\pm}_{2r}(\mathbb{F}_q)$ leads to several distinct effects.

We examine, for choices of semisimple data (s) and unipotent data u for $SO^{\pm}_{2r}(\mathbb{F}_q)$, the induction

$$\mathrm{Ind}^{O^{\pm}_{2r}(\mathbb{F}_q)}_{SO^{\pm}_{2r}(\mathbb{F}_q)}(r^{SO^{\pm}_{2r}(\mathbb{F}_q)}[(s), u]) \tag{3.5.3}$$

where, say, $r^{SO^{\pm}_{2r}(\mathbb{F}_q)}[(s), u]$ denotes the sum of all irreducible $SO^{\pm}_{2r}(\mathbb{F}_q)$-representations corresponding to the data of (s) a semisimple conjugacy class in $SO^{\pm}_{2r}(\mathbb{F}_q)$, any irreducible unipotent summand of the induction of an irreducible unipotent representation u of the identity component $Z_{SO^{\pm}_{2r}(\mathbb{F}_q)}(s)^{\circ}$, and any other potential central data.

First, we notice that as long as (s) has some eigenvalues not equal to ± 1, there are choices of semisimple data (s') for $SO^{\pm}_{2r}(\mathbb{F}_q)$ such that s and s' are not conjugate in the special orthogonal group, but they are conjugate in $O^{\pm}_{2r}(\mathbb{F}_q)$ (precisely since replacing A_λ by $A_{\lambda^{-1}}$ factors give conjugate elements in $O^{\pm}_{2r}(\mathbb{F}_q)$, as in (3.2.9) and (3.2.10)). For such cases, we find

$$\mathrm{Ind}^{O^{\pm}_{2r}(\mathbb{F}_q)}_{SO^{\pm}_{2r}(\mathbb{F}_q)}(r^{SO^{\pm}_{2r}(\mathbb{F}_q)}[(s), u]) \cong \mathrm{Ind}^{O^{\pm}_{2r}(\mathbb{F}_q)}_{SO^{\pm}_{2r}(\mathbb{F}_q)}(r^{SO^{\pm}_{2r}(\mathbb{F}_q)}[(s'), u]).$$

Consider now an irreducible unipotent representation u of the identity component $Z_{\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)}(s)^{\circ}$. We also see another effect since we are now inducing unipotent representations all the way up to the full centralizer $Z_{\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)}(s)$ of s, meaning we must consider symbol splittings (and the coincidence of the degenerate symbols) for the inductions of even special orthogonal group symbols corresponding to the two possible factors of $Z_{\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)}(s)$ obtained from 1 and -1 eigenvalues in s. Ultimately, we find that (3.5.3) splits into $2^{\alpha(s)+\beta(s)}$ irreducible equidimensional $\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)$-representations, where we put

$$\begin{aligned} \alpha(s) &= \begin{cases} 1, & \text{if } 1 \text{ is an eigenvalue of } s \\ 0, & \text{else} \end{cases} \\ \beta(s) &= \begin{cases} 1, & \text{if } -1 \text{ is an eigenvalue of } s \\ 0, & \text{else.} \end{cases} \end{aligned} \tag{3.5.4}$$

In conclusion, we may enumerate irreducible representations of the groups $\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)$ according to the "extended" classification data

(1) The *semisimple data* of the conjugacy class of a semisimple element $s \in \mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)$, under conjugacy by elements in the full orthogonal group $\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)$.
(2) The *unipotent data* u consisting of a unipotent representation of $Z_{\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)}(s)^{\circ}$, which is irreducible on all A, ${}^2A$-factors and which corresponds to a symbol on the factors obtained from the ± 1 eigenvalues of s. We allow possible degenerate on these factors symbols (that we do not decompose in order to avoid over-counting).
(3) The *extension sign data* γ, consisiting of $\alpha(s)+\beta(s)$ independent choices of sign. If s has both $+1$ and -1 eigenvalues, we write $\gamma = (\pm 1, \pm 1)$, listing the sign associated to the presence of 1 eigenvalues first. When only one of $\alpha(s)$ and $\beta(s)$ is non-zero, we write γ as the single choice of sign itself. When both $\alpha(s)$ and $\beta(s)$ are 0. We write γ_{α}, resp. γ_{β} for the signs corresponding to the 1 eigenvalues, resp. -1 eigenvalues, if present.

We write

$$r^{\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)}[(s), u]^{\gamma} \tag{3.5.5}$$

for the irreducible $\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)$-representation corresponding to a choice of extended classification data. Its dimension is

$$\begin{gathered}\dim(r^{\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)}[(s),u]^{\gamma}) = \frac{2}{2^{\alpha(s)+\beta(s)}} \cdot \dim(r^{\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)}[(s),u]) = \\ \frac{|\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)|_{q'}}{2^{\alpha(s)+\beta(s)}|Z_{\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)}(s)|_{q'}} \cdot \dim(\mathrm{Ind}^{Z_{\mathrm{O}^{\pm}_{2r}(\mathbb{F}_q)}(s)}_{Z_{\mathrm{SO}^{\pm}_{2r}(\mathbb{F}_q)}(s)^{\circ}}(u))\end{gathered} \tag{3.5.6}$$

3.5.1. Remark. *We note that our description of the irreducible representations of the orthogonal groups, passing through the special orthogonal groups, is somewhat ad hoc. Specifically, our notation of the extension data is non-canonical. Alternatively, we may consider the theory of classification data for disconnected groups directly, for example as devlopped in* [**14**]. *We also refer to S.-Y. Pan* [**50, 51**] *for a description of the classification of irreducible representations of symplectic and orthogonal groups. We reconcile this with the present notation in Subsection 7.4.*

CHAPTER 4

The explicit stable computation

We are now ready to make the Howe duality decomposition in the stable range explicit. In view of what was said in Chapters 2 and 3, we have reduced the problem to the following question: For reductive dual pairs $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ lying in the symplectic stable range, given a choice of $\pi \in \widehat{\mathrm{O}(W,B)}$, what is the classification data of its image under the eta correspondence $\eta^V_{W,B}(\pi) \in \widehat{\mathrm{Sp}(V)}$? Similarly for reductive dual pairs $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ lying in the orthogonal stable range, given a choice of $\rho \in \widehat{\mathrm{Sp}(V)}$, what is the classification data of its image under the zeta correspondence $\zeta^{W,B}_V(\rho) \in \widehat{\mathrm{O}(W,B)}$? Answering these questions fully solves the problem of finite field Howe duality in the stable ranges, and is the purpose of this chapter.

We begin by stating the constructions in broad terms as Theorem 4.1.1 in Subsection 4.1. Then in Subsection 4.2, we describe the explicit definitions of our candidate correspondences $\phi^V_{W,B}$ and $\psi^{W,B}_V$ for the eta and zeta correspondences. In Subsection 4.3, we set up the combinatorics of the dimension of the top part of the oscillator representation with respect to stable reductive dual pairs. We also discuss the effect our proposed correspondences have on the dimension of irreducible representations and conclude the key set-up result Proposition 4.1.2. In Subsection 4.5, we deduce Theorem 4.1.1 from Proposition 4.1.2

4.1. The explicit stable classification theorem. We will begin by introducing two natural candidates for the eta and zeta correspondences, which we denote by

$$\phi^V_{W,B} : \widehat{\mathrm{O}(W,B)} \hookrightarrow \widehat{\mathrm{Sp}(V)} \tag{4.1.1}$$

for $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic stable range (meant to give $\eta^V_{W,B}$) and by

$$\psi^{W,B}_V : \widehat{\mathrm{Sp}(V)} \hookrightarrow \widehat{\mathrm{O}(W,B)} \tag{4.1.2}$$

for $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the orthogonal stable range (meant to give $\zeta^V_{W,B}$). The precise description of our candidates must depend on the choice of (W, B), since the classification data for irreducible representations of odd and even (split or non-split) orthogonal groups differs in structure because of type. However, every case of $\phi^V_{W,B}$ and $\psi^{W,B}_V$ can be described according to the following rough recipe. Suppose our input irreducible representation's classification data has semisimple component (s) and unipotent component u. We must alter this data to obtain new classification data of a group with different rank and type:

(1) Alter the semisimple data to produce appropriate semisimple data of the right rank and type for the output representation by adding ± 1 eigenvalues. If W is even dimensional, only add 1 eigenvalues to make rank. If W is odd dimensional, only add -1 eigenvalues to make rank and add (for $\phi^V_{W,B}$) or remove (for $\psi^{W,B}_V$) a single 1 eigenvalue forced to make type.

(2) Consider the factor of u forming a unipotent irreducible representation of the factor of corresponding to 1 eigenvalues if W is even-dimensional and corresponding to -1 eigenvalues if W is odd dimensional. This corresponds to a symbol which must have a certain forced type and a range of possible ranks. We alter u by preserving all of its other factors and changing this symbol by adding a single new entry on the end of one of its rows. This switches the parity of its defect, and the new entry is precisely determined to make rank.

In both of these steps, depending on whether we are treating $\phi^V_{W,B}$ or $\psi^{W,B}_V$ and on the case of (W, B), there may be more than one choice of how to alter either part of the data in a way compatible with the above description. The cases when this occurs precisely correspond to cases when we have additional extension data, which can be used to resolve any ambiguities. The choice of output extension data (when it is needed) is also determined from the extension data of the input data and the discriminant of the form B in the case when W is odd-dimensional. We find that we can make these constructions to form systems of one-to-one correspondences of the form (4.1.1) and (4.1.2). In Subsection 4.2, we provide an explicit discussion of the different cases and concretely construct the two candidate correspondences.

The goal of this chapter is to prove that this guess is, in fact correct:

4.1.1. Theorem. *Consider a symplectic $\mathbb{F}_q$-space V and an orthogonal $\mathbb{F}_q$-space W with a symmetric bilinear form B.*

(1) *Suppose we have a reductive dual pair* $(Sp(V), O(W, B))$ *in* $Sp(V \otimes W)$ *in the symplectic stable range. Then*

$$\eta^V_{W,B} = \phi^V_{W,B}.$$

(2) *Suppose we have a reductive dual pair* $(Sp(V), O(W, B))$ *in* $Sp(V \otimes W)$ *in the orthogonal stable range. Then*

$$\zeta^{W,B}_V = \psi^{W,B}_V.$$

The main tool we use to prove Theorem 4.1.1 is actually dimension. In Chapter 3, we kept a certain focus on the formulas needed to compute the dimension of any irreducible representation of a symplectic or orthogonal group from its classification data. Exploiting certain aspects of the induction multiplier and the form of the dimension of a symbol will be key.

First we consider the "very stable" cases, where the dimension of V is assumed to be very large compared to the dimension of W or vice versa. More specifically, we will first prove that the dimensions of the $\mathrm{Sp}(V)$-representations $\eta^V_{W,B}(\rho)$ and $\phi^V_{W,B}(\rho)$ (resp. the $\mathrm{O}(W, B)$-representations $\zeta^{W,B}_V(\rho)$ and $\psi^{W,B}_V(\rho)$) always match for $\rho \in \widehat{\mathrm{O}(W, B)}$ (resp. $\rho \in \widehat{\mathrm{Sp}(V)}$) for $N >> n$ (resp. $n >> N$).

We define, for ρ an irreducible representation of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, its N-*rank* to be

$$rk_N(\rho) = \lceil \frac{deg_q(dim(\rho))}{N} \rceil.$$

Similarly, for ρ an irreducible representation of $\mathrm{O}(W, B)$ with $dim(W) = n$, define its n-*rank* to be

$$rk_n(\rho) = \lceil \frac{deg_q(dim(\rho))}{n} \rceil.$$

(We note that this is not the same concept as either of the rank concepts defined by Howe and Gurevich in [**24, 25**].)

4.1.2. Proposition. *Consider a symplectic* $\mathbb{F}_q$*-space* V *and an orthogonal* $\mathbb{F}_q$*-space* W *with a symmetric bilinear form* B*, and write* $dim(V) = 2N$ *and* $dim(W) = n$.

(1) *Consider* $N >> n$*. Then the disjoint union of the images of the eta correspondences*

$$\eta^V_{W,B} : \widehat{O(W, B)} \hookrightarrow \widehat{Sp(V)}$$

for the symplectic space V *of dimension* $2N$ *and the two choices of orthogonal space* (W, B) *of dimension* n *is precisely the set of irreducible representations of* $Sp(V)$ *of* N*-rank* n*.*

(2) *Consider $n >> N$. Then the signed image of the zeta correspondences*

$$\zeta^V_{W,B} : \widehat{Sp(V)} \hookrightarrow \widehat{O(W,B)}$$

consisting of irreducible representations obtained by switching the signs in the extension data of $\zeta^V_{W,B}(\rho)^\pm$ for an irreducible representation ρ of $Sp(V)$ for the symplectic space V of dimension $2N$ and an orthogonal space (W,B) of dimension n is precisely the set of irreducible representations of $O(W,B)$ of N-rank n.

We can then exploit the fact that we know that the system of eta correspondences $\eta^V_{W,B}$ all have disjoint images (and similarly for $\zeta^{W,B}_V$) to use an inductive argument to show that we can just barely reach the right dimension if $\eta^V_{W,B}$ matches $\phi^V_{W,B}$ exactly (and $\zeta^{W,B}_V$ matches $\psi^{W,B}_V$). Therefore, we conclude Theorem 4.1.1 in the very stable ranges.

Once we have proved Theorem 4.1.1 in these warm-up situations where we can assume $\dim(V)$ or $\dim(W)$ arbitrarily large, we can conclude the statement in general, since for a fixed ρ, the dimensions of $\eta^V_{W,B}(\rho)$ and $\phi^V_{W,B}(\rho)$ both form polynomials of q^N (resp. the dimensions of $\zeta^{W,B}_V(\rho)$ and $\psi^{W,B}_V(\rho)$ both form polynomials of q^n). Since the semisimple part and the sign data of $\eta^V_{W,B}(\rho)$ or $\zeta^{W,B}_V(\rho)$ are already determined by considering the restriction of the oscillator representation to the general linear group, this suffices to prove the representations themselves match.

4.2. The statement of the construction. The purpose of this subsection is to describe concretely our proposed eta and zeta correspondences. For the logic of our proof, as we described above, we give these proposed correspondences their own notation: To investigate the eta correspondence, we define our proposal

$$\phi^{W,B}_V : \widehat{\mathrm{O}(W,B)} \hookrightarrow \widehat{\mathrm{Sp}(V)}$$

(which we will apply for pairs $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic stable range), while for the zeta correspondence, we define our proposal

$$\psi^V_{W,B} : \widehat{\mathrm{Sp}(V)} \hookrightarrow \widehat{\mathrm{O}(W,B)}$$

(which we will apply for pairs $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the orthogonal stable range).

In every case of stable pairs $(\mathrm{Sp}(V), \mathrm{O}(W,B))$, the construction of the relevant choice of $\phi^{W,B}_V$ and $\psi^V_{W,B}$ in terms of (extended) classification data proceeds according to isolating certain *alterable* factors of the semisimple and unipotent components of any input data, corresponding to the eigenvalue $(-1)^{dim(W)}$.

In paragraph 4.2.1, we define $\phi_V^{W,B}$ for $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic stable range where the dimension of W is odd. In paragraph 4.2.2, we define $\phi_V^{W,B}$ for a pair in the symplectic stable range where the dimension of W is even. In paragraph 4.2.3, we define $\psi_{W,B}^V$ for a reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the orthogonal stable range where the dimension of W is odd. In paragraph 4.2.4, we define $\psi_{W,B}^V$ for pairs in the orthogonal stable range where the dimension of W is even.

4.2.1. *The odd symplectic stable case.* Consider a choice of type I reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ such that W is odd-dimensional. Write $dim(V) = 2N$ and $dim(W) = 2m+1$. (In this case, the symplectic stable range condition would require that $N \geq 2m+1$.)

Consider an irreducible representation π of $\mathrm{O}(W,B) = \mathrm{O}_{2m+1}(\mathbb{F}_q)$. Recalling Subsection 3.5, these irreducible representations are indexed by extended classification data, and we can write

$$\pi = r^{\mathrm{O}_{2m+1}(\mathbb{F}_q)}[(s), u]^{\alpha}$$

for a conjugacy class of a semisimple element $s \in \mathrm{SO}^*_{2m+1}(\mathbb{F}_q) = \mathrm{Sp}_{2m}(\mathbb{F}_q)$, an irreducible unipotent representation u of the centralizer of s, and a certain choice of sign $\alpha = \pm 1$. As discussed in Subsection 3.2, s is conjugate to a sum of blocks

$$s \sim (A_1)^{\oplus p} \oplus (A_{-1})^{\oplus \ell} \oplus \bigoplus_{i=1}^{n} (A_{\lambda_i})^{\oplus j_i} \tag{4.2.1}$$

for some distinct choices of $\lambda_i \in \mu_{q^{r_i} \pm 1}$ not equal to ± 1 and multiplicities p, ℓ, and j_i of the blocks, such that the rank condition gives

$$m = p + \ell + \sum_{i=1}^{n} j_i r_i \tag{4.2.2}$$

(we allow ℓ and m to be 0, to cover the case when s has no 1 or -1 eigenvalues).

We must define an irreducible representation $\phi_V^{W,B}(\pi)$ of $\mathrm{Sp}(V) = \mathrm{Sp}_{2N}(\mathbb{F}_q)$. We proceed by altering π's original extended classification data to produce $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-classification data defining $\phi_V^{W,B}(\pi)$, which must consist of a conjugacy class of a semisimple element

$$\phi^{\alpha}(s) \in \mathrm{SO}_{2N+1}(\mathbb{F}_q) = \mathrm{Sp}^*_{2N}(\mathbb{F}_q),$$

a unipotent representation

$$\phi^\alpha(u) \in Z_{\widehat{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}}(\phi^\alpha(s))_u,$$

and a central sign needed if $\phi^\alpha(s)$ has -1 eigenvalues (which it will turn out to always have in this case) and the corresponding factor of $\phi^\alpha(u)$ is a non-degenerate symbol.

We begin with describing the semisimple component of the data $(\phi^\alpha(s))$. Recalling the notation introduced in Definition 3.2.2, we consider the element $\sigma^\alpha_{N-m} \in \mathrm{SO}_{2(N-m)+1}(\mathbb{F}_q)$ and define $\phi^\alpha(s)$ to be the element

$$\begin{aligned}\phi^\alpha(s) := (A_1)^{\oplus p} \oplus (A_{-1})^{\oplus \ell} \oplus \bigoplus_{i=1}^n (A_{\lambda_i})^{\oplus j_i} \oplus \sigma^\alpha_{N-m} = \\ (A_1)^{\oplus p} \oplus \bigoplus_{i=1}^n (A_{\lambda_i})^{\oplus j_i} \oplus \sigma^\alpha_{N-m+\ell}\end{aligned} \tag{4.2.3}$$

This forms a semisimple element of

$$(\mathrm{SO}_2^+(\mathbb{F}_q))^p \times \prod_{i=1}^n (\mathrm{SO}_2^\pm(\mathbb{F}_{q^{r_i}}))^{j_i} \times \mathrm{SO}_{2(N-m+\ell)+1}(\mathbb{F}_q),$$

which embeds as a subgroup of $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$ with top rank N.

Now we consider the dual of the centralizer of $\phi^\alpha(s)$ in $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$. Recalling Subsection 3.3, we find that the identity component of the centralizer of this chosen element $\phi^\alpha(s)$ in $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$ is

$$\prod_{i=1}^n \mathrm{U}^\pm_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}_{2p+1}(\mathbb{F}_q) \times \mathrm{SO}^\pm_{2(N-m+\ell)}(\mathbb{F}_q). \tag{4.2.4}$$

The unipotent component of the new classification data must form an irreducible unipotent representation of this group.

For simplicity, let us consider the factors not corresponding to -1 eigenvalues separately, defining

$$H = \prod_{i=1}^n \mathrm{U}^\pm_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}_{2p+1}(\mathbb{F}_q) \tag{4.2.5}$$

Comparing with (3.3.3), we find

$$Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s) = H^* \times \mathrm{Sp}_{2\ell}(\mathbb{F}_q), \tag{4.2.6}$$

while our above work gives

$$Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(\phi^\alpha(s))^\circ = H \times \mathrm{SO}^\pm_{2(N-m+\ell)}(\mathbb{F}_q).$$

Now let us describe the unipotent representation $\phi^\alpha(u)$. Considering the original choice of u, defining the input representation of

$\mathrm{O}(W,B)$, it forms an irreducible representation of (4.2.6), and hence we can factor it into a tensor product of the form

$$u = u_{H^*} \otimes \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$$

where u_{H^*} is an irreducible unipotent representation of H^* and $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ forms a symbol defining an irreducible unipotent representation of $\mathrm{Sp}_{2\ell}(\mathbb{F}_q)$. The defect $a-b$ of the symbol is odd, so we may switch rows to assume without loss of generality that $a-b$ is 1 mod 4. Write

$$N'_\pi := N - m + \frac{a+b-1}{2} \tag{4.2.7}$$

(note that by the symplectic stable range condition, we automatically have $N'_\pi \geq N-m \geq m+1$). If $N'_\pi > \lambda_a$ or $N'_\pi > \mu_b$ (which we note would both always hold if $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ was assumed to be in the symplectic stable range), then we may concatenate N'_π onto the end the first or second row of the symbol $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$, obtaining

$$\phi^+(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}) = \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b < N'_\pi},$$

describing a unipotent representation corresponding to a symbol of $\mathrm{SO}^+_{2(N-m+\ell)}(\mathbb{F}_q)$ and

$$\phi^-(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}) = \binom{\lambda_1 < \cdots < \lambda_a < N'_\pi}{\mu_1 < \cdots < \mu_b},$$

describing a unipotent representation corresponding to a symbol of $\mathrm{SO}^-_{2(N-m+\ell)}(\mathbb{F}_q)$. We then can define

$$\phi^\alpha(u) := \widetilde{u_{H*}} \otimes \phi^\alpha(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}), \tag{4.2.8}$$

giving a unipotent representation of the group $H \times \mathrm{SO}^\alpha_{2(N-m+\ell)}(\mathbb{F}_q) = Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(\phi^\alpha(s))^\circ$ (recalling our notation (3.1.1)).

Finally, we need a central sign to complete the classification data (4.2.9) when $\phi^\alpha(s)$ has -1 eigenvalues (which always occurs) and when the symbol produced by $\phi^\alpha(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})$ is non-degenerate. We put the discriminant $disc(B)$ to be the central sign of (4.2.9).

To summarize: *For a pair $(Sp(V), O(W,B))$ in the symplectic stable range with dim $(V) = 2N$, $dim(W) = 2m+1$, for a choice of extended classification data $[(s), u]$, α, the $Sp(V)$-representation obtained from applying $\phi^{W,B}_V$ to the irreducible $O(W,B)$-representation*

$r^{O(W,B)}[(s),u]^\alpha$ *is*

(4.2.9)
$$\phi_V^{W,B}(r^{O(W,B)}[(s),u]^\alpha) := r^{Sp(V)}[(\phi^\alpha(s)),\phi^\alpha(u),disc(B)],$$

with $\phi^\alpha(s)$ *defined by (4.2.3) and* $\phi^\alpha(u)$ *defined by (4.2.8) (ommiting the central sign data in the case when the altered symbol factor of* $\phi^\alpha(u)$ *is degenerate.*

4.2.2. *The even symplectic stable case.* Consider a choice of type I reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ such that W is even dimensional. We write $dim(V) = 2N$ and $dim(W) = 2m$. (In this case, the symplectic stable range condition is that $2m \leq N$.) Let us fix the sign α, corresponding to whether the form B is fully split or not, so that

$$\mathrm{O}(W,B) = \mathrm{O}^\alpha_{2m}(\mathbb{F}_q).$$

Consider an irreducible $O^\alpha_{2m}(\mathbb{F}_q)$-representation π. As described in Subsection 3.5, its extended classification data consists of the conjugacy class of a semisimple element $s \in \mathrm{SO}^\alpha_{2m}(\mathbb{F}_q)$ (up to conjugacy considered in the full orthogonal group $\mathrm{O}^\pm_{2m}(\mathbb{F}_q)$), a unipotent representation u of the centralizer of s in $\mathrm{SO}^\alpha_{2m}(\mathbb{F}_q)$ (whose factors corresponding to ± 1 eigenvalues, if present, correspond to full D or ${}^2D$-type symbols), and possible extension sign data γ consisting of an $(\alpha(s)+\beta(s))$-tuple of signs (recalling (3.5.4)). We write

$$\pi = r^{\mathrm{O}^\alpha_{2m}(\mathbb{F}_q)}[(s),u]^\gamma.$$

Again, by the discussion in Subsection 3.2, s is conjugate to a sum of blocks (4.2.1) for distinct choices of eigenvalues $\lambda_i \in \mu_{q^{r_i}\pm 1}$ not equal to ± 1 and multiplicities p, ℓ, and j_i satsifying (4.2.2).

We must define an irreducible representation $\phi_V^{W,B}(\pi)$ of $\mathrm{Sp}(V) = \mathrm{Sp}_{2N}(\mathbb{F}_q)$. As in the case of symplectic stable reductive dual pairs involving odd orthogonal groups discussed in Subsubsection 4.2.1 above, we proceed by altering π's extended classification data to produce new representation $\phi_V^{W,B}(\pi)$'s $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-classifcation data, consisting of a conjugacy class of a semisimple element

$$\phi(s) \in \mathrm{SO}_{2N+1}(\mathbb{F}_q) = \mathrm{Sp}^*_{2N}(\mathbb{F}_q),$$

a unipotent irreducible representation $\phi^\gamma(u)$ of $\phi(s)$'s centralizer in $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$ (with the factor corresponding to -1 eigenvalues corresponding to a full symbol), and a central sign needed precisely when $\phi(s)$ has a non-zero multiplicity of -1 eigenvalues and the symbol factor of $\phi^\gamma(u)$ corresponding to these -1 eigenvalues is non-degenerate.

We begin by describing the semisimple component $(\phi(s))$ of the new data. In this case, where our given reductive dual pair involves an even orthogonal group, we produce $\phi(s)$ by adding only 1 eigenvalue blocks to s, i.e. by adding an identity matrix of size $2(N-m)+1$:

$$\begin{aligned}\phi(s) = (A_1)^{\oplus p} \oplus (A_{-1})^{\oplus \ell} \oplus \bigoplus_{i=1}^{n} (A_{\lambda_i})^{\oplus j_i} \oplus I_{2(N-m)+1} = \\ (A_{-1})^{\oplus \ell} \oplus \bigoplus_{i=1}^{n} (A_{\lambda_i})^{\oplus j_i} \oplus I_{2(N-m+p)+1}\end{aligned} \tag{4.2.10}$$

(Note that each different class (s) considered as a conjugacy class in $\mathrm{O}(W,B)$ corresponds to a different $\phi(s)$, whereas if we only considered (s) as a conjugacy class in $\mathrm{SO}(W,B)$, in cases with eigenvalues not equal to ± 1, there would be another $\mathrm{SO}(W,B)$-conjugacy class (s') with $(\phi(s)) = (\phi(s'))$ in $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$.) This forms a semisimple element of

$$(\mathrm{SO}_2^+(\mathbb{F}_q))^\ell \times \prod_{i=1}^{n} (\mathrm{SO}_2^\pm(\mathbb{F}_{q^{r_i}}))^{j_i} \times \mathrm{SO}_{2(N-m+p)+1}(\mathbb{F}_q),$$

which embeds as a subgroup of $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$ with top rank N.

Next we consider the centralizer of this element $\phi(s)$ in $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$. Recalling again Subsection 3.3, we find that its identity component is

$$\prod_{i=1}^{n} \mathrm{U}_{j_i}^\pm(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}_{2\ell}^\pm(\mathbb{F}_q) \times \mathrm{SO}_{2(N-m+p)+1}(\mathbb{F}_q). \tag{4.2.11}$$

The unipotent component $\phi^\gamma(u)$ must form a unipotent representation of this group. For simplicity, we again separate the factors of $\phi(s)$'s centralizer not corresponding to 1 eigenvalues

$$H = \prod_{i=1}^{n} \mathrm{U}_{j_i}^\pm(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}_{2\ell}^\pm(\mathbb{F}_q) \tag{4.2.12}$$

(Note that in this case H is self-dual, so that $H^* = H$). We find

$$Z_{\mathrm{SO}_{2m}^\alpha(\mathbb{F}_q)}(s)^\circ = H \times \mathrm{SO}_{2p}^\pm(\mathbb{F}_q), \tag{4.2.13}$$

while our above work gives

$$Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(\phi(s)) = H^\circ \times \mathrm{SO}_{2(N-m+p)+1}(\mathbb{F}_q). \tag{4.2.14}$$

To describe $\phi^\gamma(u)$, consider the original choice of unipotent data u defining π. Then u can be factored according to (4.2.13) as a tensor product of the form

$$u = u_H \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

where u_H is an irreducible unipotent representation of H and $\binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ forms a (possibly degenerate) symbol of $\mathrm{SO}^{\pm}_{2p}(\mathbb{F}_q)$. We note that split or non-split $\mathrm{SO}^{\pm}_{2p}(\mathbb{F}_q)$ may both possible for some choice of s. To attempt to consider the cases as concisely as possible, we do not separate these cases yet, and hence only assume $a-b$ is even, and the rank of the symbol $\binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ is r. Write

$$N'_\rho = N - m + \frac{a+b}{2}.$$

First, suppose the symbol $\binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$ is non-degenerate (i.e. the rows of increasing integers are distinct). Then we may pick the minimal i such that $\lambda_{a-i} \neq \mu_{b-i}$ and switch rows so that $\lambda_{a-i} > \mu_{b-i}$. We note that we always have $N'_\pi > \lambda_a$ or $N'_\pi > \mu_b$ since $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ was assumed to be in the symplectic stable range. The symbols

$$\begin{pmatrix} \lambda_1 < \dots < \lambda_a < N'_\rho \\ \mu_1 < \dots < \mu_b \end{pmatrix}, \qquad \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b < N'_\rho \end{pmatrix}$$

both define symbols for $\mathrm{SO}_{2(N-m+p)+1}(\mathbb{F}_q)$. We put

$$\phi^-(\begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}) = \begin{pmatrix} \lambda_1 < \dots < \lambda_a < N'_\rho \\ \mu_1 < \dots < \mu_b \end{pmatrix}$$
$$\phi^+(\begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}) = \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b < N'_\rho \end{pmatrix}.$$

Then taking

$$\phi^{\pm}(u) = u_H \otimes \phi^{\pm}(\begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}) \tag{4.2.15}$$

defines an irreducible representation of the centralizer of $\phi(s)$ (4.2.14).

Now suppose the symbol of $\mathrm{SO}^{\pm}_{2p}(\mathbb{F}_q)$ appearing in u is degenerate, so that

$$u = u_H \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \lambda_1 < \dots < \lambda_a \end{pmatrix}$$

(we also count the case of the trivial representation $1 = \binom{\emptyset}{\emptyset}$ of trivial group $\mathrm{SO}^+_0(\mathbb{F}_q)$ in this case). Then we put

$$\phi(u) = u_H \otimes \begin{pmatrix} \lambda_1 < \dots < \lambda_a < N'_\rho \\ \lambda_1 < \dots < \lambda_a \end{pmatrix} \tag{4.2.16}$$

Finally, to define an output irreducible $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representation, we may need to also choose output central sign data if $\phi(s)$ has -1 eigenvalues and the corresponding factor of $\phi^{\pm}(u)$ or $\phi(u)$ is a non-degenerate symbol. By definition, $\phi(s)$ has the same number of -1

eigenvalues as s and the corresponding factor of the unipotent data was not altered. Therefore, in this case, the -1 component of the extension sign data γ of the original $\mathrm{O}(W,B)$-classification data supplies the central sign data precisely when it is needed.

To summarize: *For a pair $(Sp(V), O(W,B))$ in the symplectic stable range with dim $(V) = 2N$, dim$(W) = 2m$, for a choice of extended classification data $[(s), u], \gamma$, writing γ_α for the sign associated to* 1 *eigenvalues of s (if present) and γ_β for the sign associated to -1 eigenvalues (if present), the $Sp(V)$-representation obtained from applying $\phi_V^{W,B}$ to the irreducible $O(W,B)$-representation $r^{O(W,B)}[(s),u]^\gamma$ is*

$$\phi_V^{W,B}(r^{O(W,B)}[(s),u]^\gamma) := r^{Sp(V)}[(\phi(s)), \phi^{\gamma_\alpha}(u), \gamma_\beta], \tag{4.2.17}$$

with $\phi(s)$ defined by (4.2.10), $\phi^{\gamma_\alpha}(u)$ defined by the appropriate case of (4.2.15) and (4.2.16) (ommiting sign data if it is not present).

4.2.3. *The odd orthogonal stable case.* Suppose $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ a type I reductive dual pair where W is odd dimensional. Write $\dim(V) = 2N$ an $\dim(W) = 2m+1$. In this case, the orthogonal stable range require $m \geq 2N$.

Consider an irreducible representation ρ of $\mathrm{Sp}(V) = \mathrm{Sp}_{2N}(\mathbb{F}_q)$. Its classification data consists of a conjugacy class of a semisimple element in the dual group $s \in \mathrm{SO}_{2N+1}(\mathbb{F}_q) = \mathrm{Sp}^*_{2N}(\mathbb{F}_q)$, an irreducible unipotent representation u of the centralizer of s in $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$, and possible central sign data we denote by $\alpha = \pm 1$, in the case when -1 is an eigenvalue of s. We write $\rho = r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, \alpha]$. Again, s is conjugate to a sum of blocks (4.2.1) with an additional 1 inserted, for distinct choices of eigenvalues $\lambda_i \in \mu_{q^{r_i}\pm 1}$ not equal to ± 1 and multiplicites p, ℓ, and j_i satisfying (4.2.2).

Our goal now is to define an irreducible representation $\psi^V_{W,B}(\rho)$ of $\mathrm{O}(W,B) = \mathrm{O}_{2m+1}(\mathbb{F}_q)$. As in the case of the symplectic stable range construction of $\phi_V^{W,B}$ described in Subsubsections 4.2.1 and 4.2.2 above, we still proceed by altering ρ's classification data to construct the new representation $\psi^V_{W,B}(\rho)$'s extended $\mathrm{O}_{2m+1}(\mathbb{F}_q)$-classification data, consisting of a conjugacy class of a semisimple element

$$\psi(s) \in \mathrm{Sp}_{2m}(\mathbb{F}_q) = \mathrm{SO}^*_{2m+1}(\mathbb{F}_q),$$

a unipotent irreducible representation $\psi^\alpha(u)$ of the dual of $\psi(s)$'s centralizer in $\mathrm{Sp}_{2m}(\mathbb{F}_q)$, and an extension sign if needed.

We begin with describing the semisimple component $(\psi(s))$ of the new data. In this case, where the reductive dual pair again involves an odd orthogonal group, similarly as in Subsubsection 4.2.1, we produce $\phi(s)$ by adding only -1 eigenvalues to s (while removing a single 1 eigenvalue)

$$\begin{aligned}\psi(s) = (A_1)^{\oplus p} \oplus (A_{-1})^{\oplus \ell} \oplus \bigoplus_{i=1}^{n} (A_{\lambda_i})^{\oplus j_i} \oplus (-I)_{2(m-N)} \\ (A_1)^{\oplus p} \oplus \bigoplus_{i=1}^{n} (A_{\lambda_i})^{\oplus j_i} \oplus (-I)_{2(m-N+\ell)}\end{aligned} \tag{4.2.18}$$

specifies a semisimple element of

$$(\mathrm{SO}_2^+(\mathbb{F}_q))^p \times \prod_{i=1}^{n} (\mathrm{SO}_2^\pm(\mathbb{F}_{q^{r_i}}))^{j_i} \times \mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)$$

which embeds as a subgroup of $\mathrm{Sp}_{2m}(\mathbb{F}_q) = \mathrm{SO}^*_{2m+1}(\mathbb{F}_q)$ with top rank m.

Next we consider the (connected) centralizer of this element $\psi(s)$ in $\mathrm{Sp}_{2m}(\mathbb{F}_q)$, which is

$$\prod_{i=1}^{n} \mathrm{U}^\pm_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q) \times \mathrm{Sp}_{2p}(\mathbb{F}_q). \tag{4.2.19}$$

The unipotent component $\psi^\alpha(u)$ must form an irreducible unipotent representation of this group. We again separate the factors of $\psi(s)$'s centralizer not corresponding to the altered -1 eigenvalues

$$H = \prod_{i=1}^{n} \mathrm{U}^\pm_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{Sp}_{2p}(\mathbb{F}_q) \tag{4.2.20}$$

We find

$$Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)^\circ}(s) = H^* \times \mathrm{SO}^\pm_{2\ell}(\mathbb{F}_q), \tag{4.2.21}$$

while our above work gives

$$Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(\psi(s)) = H \times \mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q). \tag{4.2.22}$$

To describe $\psi^\alpha(u)$, consider the irreducible unipotent representation u of the dual of (4.2.21) forming the unipotent classification data of ρ. Yet again, we express it as a tensor product of an irreducible unipotent representation of H^* with a symbol of $\mathrm{SO}^\pm_{2\ell}(\mathbb{F}_q)$

$$u = u_{H^*} \otimes \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}.$$

We first treat the case where the symbol $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ is non-degenerate. Switch rows so that for the minimal i with $\lambda_{a-i} \neq \mu_{b-i}$, we have

$\lambda_{a-i} < \mu_{b-i}$. Write

$$m'_\rho = m - N + \frac{a+b}{2}.$$

Since $m'_\rho > \lambda_a$ and $m'_\rho > \mu_b$ which is ensured for $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the orthogonal stable range, we may concatenate m'_ρ to the end of one of the rows of the symbol, producing symbols

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < m'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < m'_\rho \end{pmatrix} \tag{4.2.23}$$

which define two distinct irreducible unipotent representations of the factor $\mathrm{Sp}_{2(m-N+\ell)}(\mathbb{F}_q)$ of (4.2.22). We put

$$\psi^-(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}) = \begin{pmatrix} \lambda_1 < \cdots < \lambda_a < m'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

$$\psi^+(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}) = \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < m'_\rho \end{pmatrix}.$$

Then put

$$\psi^\alpha(u) = \widetilde{u_{H^*}} \otimes \psi^\alpha(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}) \tag{4.2.24}$$

Now suppose the symbol of $\mathrm{SO}^\pm_{2\ell}(\mathbb{F}_q)$ appearing in u is degenerate, so that

$$u = u_{H^*} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \lambda_1 < \cdots < \lambda_a \end{pmatrix}$$

(we also count the case of the trivial representation $1 = \binom{\emptyset}{\emptyset}$ of $\mathrm{SO}^+_0(\mathbb{F}_q)$ in this case). Then we do not have central sign data and put

$$\psi(u) = \widetilde{u_{H^*}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a < m'_\rho \\ \lambda_1 < \cdots < \lambda_a \end{pmatrix} \tag{4.2.25}$$

To summarize: *For a pair $(Sp(V), O(W,B))$ in the orthogonal stable range with $dim(V) = 2N$, $dim(W) = 2m+1$, for a choice of classification data $[(s), u, \alpha]$ where we omit the central sign data α in the case when s has no -1 eigenvalues, the $O(W,B)$-representation obtained from applying $\psi^V_{W,B}$ to the irreducible $Sp(V)$-representation $r^{Sp(V)}[(s), u, \alpha]$ is*

$$\psi^V_{W,B}(r^{Sp(V)}[(s), u, \alpha]) := r^{O(W,B)}[(\psi(s)), \psi^\alpha(u)]^{disc(B)} \tag{4.2.26}$$

with $\psi(s)$ defined by (4.2.18), $\psi^\alpha(u)$ defined by the appropriate case of (4.2.24) and (4.2.25).

4.2.4. *The even orthogonal stable case.* Suppose $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is a type I reductive dual pair where W is of even dimension. Write $\dim(V) = 2N$ and $\dim(W) = 2m$ and write σ for the sign so that $\mathrm{O}(W, B) = \mathrm{O}^{\sigma}_{2m}(\mathbb{F}_q)$. In this case, the orthogonal stable range requires that the maximal isotropic subspace of W with respect to B is of dimension greater than or equal to $2N$.

Consider an irreducible representation ρ of $\mathrm{Sp}(V) = \mathrm{Sp}_{2N}(\mathbb{F}_q)$. We want to produce $\mathrm{O}_{2m}(\mathbb{F}_q)^{\sigma}$-classification data, which we recall consists of a semisimple conjugacy class $(s) \in \mathrm{O}^{\sigma}_{2m}(\mathbb{F}_q)$, a unipotent part u which can be considered to consist of a unipotent irreducible represen-taion of the (dual) of the centralizer of s in $\mathrm{SO}^{\sigma}_{2m}(\mathbb{F}_q)$, and central sign data. We note that since $\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[V \otimes W])$ is the permutation representation $\mathbb{C}W$ tensored with the representation $\epsilon(det)$ (corresponding to the sign representation of $\mathrm{O}(W, B)/\mathrm{SO}(W, B)$), part of the central sign data is already forced. Specifically, as in the case of the symplectic group, we will only need to choose central sign data for the output representation corresponding to -1-eigenvalues. Broadly, we will construct the new semisimple and unipotent parts of the $\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)$-classification data

$$(\psi(s)), \psi(u)$$

by adding 1 eigenvalues to s and altering the symbol of the affect factor of the unipotent part by adding a single new coordinate to one of the rows to achieve the new needed rank and defect.

To be more specific, write (s) with $s \in \mathrm{SO}_{2N+1}(\mathbb{F}_q) = \mathrm{Sp}^*_{2N}(\mathbb{F}_q)$ for the semisimple part of the classification data for the input $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representation ρ. In this case, where the reductive dual pair again involves an odd orthogonal group, similarly as above, we produce $\psi(s)$ by adding only 1 eigenvalues to s (while also removing the single forced 1 eigenvalue)

$$\begin{aligned}\psi(s) &= (A_{-1})^{\oplus \ell} \oplus (A_1)^{\oplus p} \oplus \bigoplus_{i=1}^{n} (A_{\lambda_i})^{\oplus j_i} \oplus I_{2(m-N)} \\ &= (A_{-1})^{\oplus \ell} \oplus \bigoplus_{i=1}^{n} (A_{\lambda_i})^{\oplus j_i} \oplus I_{2(m-N+p)}\end{aligned} \tag{4.2.27}$$

specifying a semisimple element of

$$(\mathrm{SO}^{+}_{2}(\mathbb{F}_q))^{\ell} \times \prod_{i=1}^{n} (\mathrm{SO}^{\pm}_{2}(\mathbb{F}_{q^{r_i}}))^{j_i} \times \mathrm{SO}^{\beta}_{2(N-m+p)}(\mathbb{F}_q)$$

which embeds as a subgroup of $\mathrm{SO}_{2m}(\mathbb{F}_q)$ with top rank m (note that the sign β is determined by the total sign σ of $\mathrm{O}^{\sigma}_{2m}(\mathbb{F}_q)$. As in Subsection 4.2.2, each distinct $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$-conjugacy class (s) gives a distinct $\mathrm{O}^{\sigma}_{2m}(\mathbb{F}_q)$-conjugacy class $\psi(s)$.

The identity component of the centralizer of $\psi(s)$ in $\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)$ is

$$\prod_{i=1}^{n} \mathrm{U}^{\pm}_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q) \times \mathrm{SO}^{\beta}_{2(N-m+p)}(\mathbb{F}_q). \tag{4.2.28}$$

We again separate out the factors of the centralizer of $\psi(s)$ corresponding to the eigenvalues not equal to 1, writing

$$H = \prod_{i=1}^{n} \mathrm{U}^{\pm}_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q)$$

(which is self-dual $H = H^*$), so that

$$Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(s)^{\circ} = H \times \mathrm{SO}_{2p+1}(\mathbb{F}_q)$$

and

$$Z_{\mathrm{O}^{\sigma}_{2m}(\mathbb{F}_q)}(\psi(s))^{\circ} = H \times \mathrm{SO}^{\beta}_{2(N-m+p)}(\mathbb{F}_q). \tag{4.2.29}$$

To construct the unipotent part of the $\mathrm{O}^{\sigma}_{2m}(\mathbb{F}_q)$-classification data $\psi(u)$, consider the symbol $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ and the irreducible unipotent H-representation u_H such that

$$u = u_H \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}.$$

Switch the symbol rows so that the defect $a - b$ is 1 mod 4 (which is possible since this symbol has odd defect). Let us write

$$m'_{\rho} = m - N + \frac{a+b-1}{2}.$$

Then, if $\beta = +$, if $\mu_b < m'_{\rho}$, putting

$$\psi(u) = u_H \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < m'_{\rho} \end{pmatrix} \tag{4.2.30}$$

gives a unipotent representation of (4.2.29). Similarly, if $\beta = -$, if $\lambda_a < m'_{\rho}$, putting

$$\psi(u) = u_H \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a < \mu'_{\rho} \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \tag{4.2.31}$$

gives a unipotent representation of (4.2.29). Again, note that both $m'_{\rho} > \lambda_a$ and $m'_{\rho} > \mu_b$ are ensured if $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ is in the orthogonal stable range.

The extension data corresponding to 1 eigenvalues of $\psi(s)$ is taken to be $+$. Now, as in Subsection 4.2.2, (s) and $(\psi(s))$ have the same multiplicity of -1 eigenvalues. Therefore, the undetermined extension sign data needed to describe $\zeta_V^{W,B}(\rho)$ corresponding to -1 eigenvalues of $\psi(s)$ can be taken to be the central sign data is given in ρ's original $\mathrm{Sp}_{2N}(\mathbb{F}_q)$- classification data.

To summarize: *Suppose we are given the above notation. We define $\zeta_V^{W,B}(\rho)$ to be the irreducible $O(W,B)$-representation with $O(W,B)$-classification data $[\psi(s),\psi(u),\alpha]$, where the final sign is the central sign α of ρ arising if s has -1 eigenvalues and where we omit it if s has no such eigenvalues*

$$\begin{aligned}&\psi^V_{W,B}(r^{Sp(V)}[(s),u,\alpha]) :=\\ &r^{O(W,B)}[(\psi(s)),\psi(u)]^{(+,\alpha)}\end{aligned} \tag{4.2.32}$$

with $\psi(s)$ defined by (4.2.27) and $\psi(u)$ defined by (4.2.30) or (4.2.31).

We also define the following terminology, in order to more easily refer to the constructions described in this section later.

4.2.5. Definition. *Given the above notation, we define the* alterable data *of an input irreducible representation ρ for an η of ζ correspondence to consist of*

1. *the* alterable semisimple data (s_{alt}). *If the dimension of the orthogonal space W is odd, we take s_{alt} to be sum of all blocks A_{-1} appearing as a factor of s, forming a negative identity matrix of even size. If s has no -1 eigenvalues, we write that the alterable semisimple data is $\emptyset$. Similarly, if the dimension of the orthogonal space W is even, we take s_{alt} to be sum of all blocks A_1 appearing as a factor of s, forming an identity matrix (of even or odd size).*
2. *the* alterable unipotent data u_{alt}, *which we take to be the irreducible unipotent representation tensor factor of the original irreducible unipotent representation u arising in ρ's classification data, corresponding to the factor of the (dual) of the centralizer of s corresponding to the factor s_{alt}.*

4.3. Combinatorics. As we found in Subsections 2.2 and 2.3, a key step in decomposition the restriction of an oscillator representation $\omega[V\otimes W]$ to $\mathrm{Sp}(V)\times \mathrm{O}(W,B)$ is to separate off its "top part," which specifically singles out summands arising from the eta or zeta correspondence with source corresponding to the appropriate full-rank orthogonal or symplectic group, respectively. In the symplectic stable range, we write

$$\omega[V \otimes W]^{\mathrm{top}} := \bigoplus_{\rho \in \mathrm{O}(W,B)} \eta^{V}_{W,B}(\rho) \otimes \rho, \tag{4.3.1}$$

and call it the *top part* of $\omega[V \otimes W]$ (with respect to the eta correspondence). Similarly, in the orthogonal stable range, we write

$$\omega[V \otimes W]^{\mathrm{top}'} := \bigoplus_{\rho \in \mathrm{O}(W,B)} \rho \otimes \zeta^{V}_{W,B}(\rho), \tag{4.3.2}$$

and call it the *top part* of $\omega[V \otimes W]$ (with respect to the zeta correspondence).

From here, the proof of Theorem 4.1.1 separates into two key steps: A combinatorial verification that the dimension of the direct sum of matches the dimension of the top part of $\omega[V \otimes W]$, and an inductive argument showing that the claimed correspondence in Theorem 4.1.1 is the only possible one. The first step is the goal of this section.

4.3.1. THEOREM. *Fix a reductive dual pair $(Sp(V), O(W, B))$.*

(1) *If $(Sp(V), O(W, B))$ is in the symplectic stable range, the dimension of the top part of the restriction of $\omega[V \otimes W]$ matches the sum of products of the dimensions of irreducible representations of $O(W, B)$ and their $\phi^{W,B}_{V}$ correspondences:*

$$dim(\omega[V \otimes W]^{top}) = \sum_{\pi \in \widehat{O(W,B)}} dim(\pi) \cdot dim(\phi^{W,B}_{V}(\pi)). \tag{4.3.3}$$

(2) *Similarly, if $(Sp(V), O(W, B))$ is in the orthogonal stable range, the dimension of the top part of the restriction of $\omega[V \otimes W]$ matches the sum of products of the dimensions of irreducible representations of $Sp(V)$ and their $\psi^{V}_{W,B}$ correspondences:*

$$dim(\omega[V \otimes W]^{top'}) = \sum_{\rho \in \widehat{Sp(V)}} dim(\rho) \cdot dim(\psi^{V}_{W,B}(\rho)). \tag{4.3.4}$$

In this subsection, we focus on describing the proof of part (1) of Theorem 4.3.1. In Subsection 4.4, we discuss how the combinatorics can be modified for the orthogonal stable range to obtain part (2). We begin by processing the left hand side of (4.3.3). Knowing the dimensions of the full oscillator representations, the decompositions obtained in Theorem 2.1.4 can be used to obtain a linear system of equations for the dimensions of the top parts of the oscillator representations. Recursively solving for the top parts gives the following

4.3.2. PROPOSITION. *Fix a reductive dual pair $(Sp(V), O(W, B))$ in the symplectic stable range.*

(1) *If the orthogonal space W is odd-dimensional, then writing $dim(V) = 2N$ and $dim(W) = 2m + 1$, the dimension of the top part of the restriction of $\omega[V \otimes W]$ is*

$$\sum_{i=0}^{m} (-1)^{m-i} \cdot q^{\binom{m-i}{2}} \cdot \binom{m}{i}_q \cdot \prod_{k=i+1}^{m} (q^k + 1) \cdot q^{(2i+1)N} \tag{4.3.5}$$

(2) *If W is even-dimensional and the orthogonal from B is totally split, then writing $dim(V) = 2N$ and $dim(W) = 2m$, the dimension of the top part of the restriction of $\omega[V \otimes W]$ is*

$$\sum_{i=0}^{m} (-1)^{m-i} q^{\binom{m-i}{2}} \binom{m}{i}_q \cdot \prod_{j=i}^{m-1} (q^j + 1) \cdot q^{2iN} \tag{4.3.6}$$

(3) *If W is even-dimensional and the orthogonal from B is totally split, then writing $dim(V) = 2N$ and $dim(W) = 2m$, the dimension of the top part of the restriction of $\omega[V \otimes W]$ is*

$$\sum_{i=1}^{m} (-1)^{m-i} q^{\binom{m-i}{2}} \binom{m-1}{i}_q \cdot \prod_{j=i+1}^{m} (q^j + 1) \cdot q^{2iN} \tag{4.3.7}$$

PROOF. We focus on the case of odd dimensional orthogonal space W. Both cases of even dimensional W work entirely similarly. We recall the sizes $|\mathrm{O}(W,B)/P_\ell^{\mathrm{O}(W,B)}|$ of the quotients of an orthogonal group by a maximal parabolic subgroup (2.4.10). Let X_m denote the dimension of the top part $\omega[V \otimes W]^{\text{ top}}$, where W is a $2m+1$-dimensional $\mathbb{F}_q$-space. Write, for $j < i$

$$\begin{gathered} C_{i,j} := -\binom{i}{j}_q \cdot \prod_{k=j+1}^{i} (q^k + 1) = \\ \frac{(q^{2i} - 1)(q^{2(i-1)} - 1) \dots (q^{2(j+1)} - 1)}{(q^{i-j} - 1)(q^{i-j-1} - 1) \dots (q - 1)} \end{gathered} \tag{4.3.8}$$

Taking the dimension of the decomposition (2.1.11) then gives the recursive equation

$$X_m = q^{(2m+1)N} + \sum_{i=0}^{m-1} C_{m,i} \cdot X_i. \tag{4.3.9}$$

Our goal to prove (4.3.5) is to re-express the right-hand side of (4.3.9) in terms of a sum of $q^{(2i+1)N}$ for $0 \le i \le m$ some lower coefficient.

Now, iteratively applying (4.3.9), we find that

$$X_m = \sum_{i=0}^{m} \left(\sum_{i=\ell_1<\cdots<\ell_j=m} \prod_{k=1}^{j-1} C_{\ell_{k+1},\ell_k} \right) q^{(2i+1)N}.$$

It suffices to prove

$$\sum_{i=\ell_1<\cdots<\ell_j=m} \prod_{k=1}^{j-1} C_{\ell_{k+1},\ell_k} = C_{m,i} \cdot q^{\binom{i}{2}}. \tag{4.3.10}$$

Using (4.3.8), each term $\prod_{k=1}^{j-1} C_{\ell_{k+1},\ell_k}$ where $i = \ell_1 < \cdots < \ell_j = m$, factors as

$$\frac{(q^{2m}-1)(q^{2(m-1)}-1)\dots(q^{2(m-i+1)}-1)}{\displaystyle\prod_{k=1}^{j-1} \prod_{r=1}^{\ell_{k+1}-\ell_k} (q^r-1)},$$

which can be simplified as

$$C_{m,i} \cdot \frac{(q^{m-i}-1)(q^{m-i-1}-1)\dots(q-1)}{\displaystyle\prod_{k=1}^{j-1} \prod_{r=1}^{\ell_{k+1}-\ell_k} (q^r-1)},$$

reducing the claim to

$$q^{\binom{m-i}{2}} = \sum_{i=\ell_1<\cdots<\ell_j=m} \frac{(q^{m-i}-1)(q^{m-i-1}-1)\dots(q-1)}{\displaystyle\prod_{k=1}^{j-1} \prod_{r=1}^{\ell_{k+1}-\ell_k} (q^r-1)}. \tag{4.3.11}$$

The right-hand side of (4.3.11) can also be written as

$$\sum_{0=\ell'_1<\cdots<\ell'_j=m-i} \binom{\ell'_j}{\ell'_{j-1}}_q \binom{\ell'_{j-1}}{\ell'_{j-2}}_q \cdots \binom{\ell'_2}{\ell'_1}_q,$$

by substituting $\ell'_j = \ell_j - i$, so (4.3.11) follows from a q-version of the multinomial theorem.

□

We re-express (4.3.5), (4.3.6), and (4.3.7) again as follows, to separate it into terms which group according to "levels" of input representations corresponding to the size of the alterable factor of the semismple classification data (as defined in Definition 4.2.5):

4.3.3. PROPOSITION. *Consider a reductive dual pair* $(Sp(V), O(W,B))$ *in the symplectic stable range.*

(1) *If the orthogonal space W is odd-dimensional, then writing $dim(V) = 2N$ and $dim(W) = 2m+1$, the top dimension of $\omega[V \otimes W]^{top}$ can be re-expressed as*

$$\sum_{\ell=0}^{m}(-1)^{\ell}\, q^{N+(m-\ell)(m-\ell-1)+\ell^2} \binom{m}{\ell}_{q^2} \prod_{j=0}^{m-\ell-1} (q^{2(N-i)}-1). \tag{4.3.12}$$

(2) *If the orthogonal space W is even-dimensional and B is totally split, then writing $dim(V) = 2N$ and $dim(W) = 2m$, the dimension of $\omega[V \otimes W]^{top}$ can be re-expressed as*

$$\sum_{\ell=0}^{m}(-1)^{\ell} q^{\ell(\ell-1)+(m-\ell)(m-\ell-1)}. \binom{m}{\ell}_{q^2} \frac{(q^{m-\ell}+q^{\ell})}{(q^m+1)} \prod_{j=0}^{m-\ell-1} (q^{2(N-j)}-1). \tag{4.3.13}$$

(3) *If the orthogonal space W is even-dimensional and B is totally split, then writing $dim(V) = 2N$ and $dim(W) = 2m$, the dimension of $\omega[V \otimes W]^{top}$ can be re-expressed as*

$$\sum_{\ell=0}^{m-1}(-1)^{\ell} q^{\ell(\ell-1)+(m-\ell)(m-\ell-1)}. \binom{m}{\ell}_{q^2} \frac{(q^{m-\ell}-q^{\ell})}{(q^m-1)} \prod_{j=0}^{m-\ell-1} (q^{2(N-j)}-1). \tag{4.3.14}$$

PROOF. Again, we treat the case of odd-dimensional orthogonal spaces now. Both cases of even-dimensional W follow completely similarly. The cases Substituting $i = m - \ell$, (4.3.5) can be re-written as

$$\sum_{\ell=0}^{m}(-1)^{\ell} q^{\binom{\ell}{2}} \binom{m}{\ell}_{q^2} \Big(\prod_{j=1}^{\ell}(q^j+1)\Big) q^{(2(m-\ell)+1)+N}. \tag{4.3.15}$$

Now in (4.3.12), using

$$(m-\ell-1)(m-\ell) = \sum_{j=0}^{m-\ell-1} 2k,$$

we have

$$q^{(m-\ell-1)(m-\ell)} \prod_{j=0}^{m-\ell-1} (q^{2(N-i)}-1) = \prod_{j=0}^{m-\ell-1} (q^{2N}-q^{2i}).$$

Hence, (4.3.12) reduces to

$$\sum_{\ell=0}^{m}(-1)^{\ell}q^{N+\ell^2}\binom{m}{\ell}_{q^2}\prod_{i=0}^{m-\ell-1}(q^{2N}-q^{2i}).$$

Finally, at each ℓ,

$$\prod_{i=0}^{m-\ell-1}(q^{2N}-q^{2i})=\sum_{j=0}^{m-\ell}q^{2Nj}\cdot\sum_{1\leq i_1<\cdots<i_{m-\ell-j}\leq m-\ell-1}q^{2(i_1+\dots i_{m-\ell-j})}=$$

$$\sum_{j=0}^{m-\ell}q^{2Nj}\cdot\binom{m-\ell}{j}_{q^2}.$$

Therefore, the coefficient of $q^{(2(m-\ell)+1)N}$ in (4.3.15) for each ℓ is

$$\sum_{k=0}^{\ell}(-1)^{k}q^{k^2}\binom{m}{m-k}_{q^2}\binom{m-k}{m-\ell}_{q^2}$$

(identifying $\binom{m}{k}_{q^2}$ with $\binom{m}{m-k}_{q^2}$). Hence, the claim reduces to verifying that

$$\sum_{k=0}^{\ell}(-1)^{k}q^{k^2}\binom{m}{m-k}_{q^2}\binom{m-k}{m-\ell}_{q^2}= \tag{4.3.16}$$
$$q^{\binom{\ell}{2}}\binom{m}{m-\ell}_{q^2}\prod_{j=1}^{\ell}(q^j+1)$$

Further, we have

$$\binom{m}{m-k}_{q^2}\binom{m-k}{m-\ell}_{q^2}=\binom{m}{m-\ell}_{q^2}\binom{\ell}{\ell-k}_{q^2}=$$
$$\binom{m}{m-\ell}_{q^2}\binom{\ell}{k}_{q^2},$$

reducing (4.3.16) again to a q-multinomial theorem.

□

The purpose of re-writing the dimension of the top part of $\omega[V\otimes W]$ as (4.3.12) is because, for each ℓ, the prime to q part of the ℓth term of (4.3.12) is

$$\binom{m}{\ell}_{q^2} \prod_{i=0}^{m-\ell-1} (q^{2(N-i)} - 1) = \tag{4.3.17}$$

$$\frac{|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(m-\ell)}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2\ell}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}},$$

the prime to q part of the ℓth term of (4.3.13) is

$$\binom{m}{\ell}_{q^2} \frac{(q^{m-\ell}+q^{\ell})}{(q^m+1)} \prod_{i=0}^{m-\ell-1} (q^{2(N-j)} - 1) =$$

$$\frac{1}{2}\left(\frac{|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{O}^+_{2(m-\ell)}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{SO}^+_{2\ell}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}} - \right. \tag{4.3.18}$$

$$\left. \frac{|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{O}^-_{2(m-\ell)}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{SO}^-_{2\ell}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}\right),$$

and the prime to q part of the ℓth term of (4.3.14) is

$$\binom{m}{\ell}_{q^2} \frac{(q^{m-\ell}-q^{\ell})}{(q^m-1)} \prod_{i=0}^{m-\ell-1} (q^{2(N-j)} - 1) =$$

$$\frac{1}{2}\left(\frac{|\mathrm{O}^-_{2m}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{O}^-_{2(m-\ell)}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{SO}^+_{2\ell}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}} - \right. \tag{4.3.19}$$

$$\left. \frac{|\mathrm{O}^-_{2m}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{O}^+_{2(m-\ell)}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{SO}^-_{2\ell}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}\right).$$

We use Proposition 4.3.3 to conclude (4.3.3) by approximating the right hand recursively by considering terms $dim(\pi)dim(\phi_{W,B}(\pi))$ separately for $\pi \in \widehat{\mathrm{O}(W,B)}$ arising from a conjugacy class of a semisimple element of the dual group $\mathrm{Sp}_{2m}(\mathbb{F}_q)$, which is singular of type $(m-\ell,\ell)$ (i.e. has -1 as an eigenvalue with multiplicity 2ℓ), using the elementary fact that the sum of the squares of the dimensions of all irreducible representations of a group G recover its group order. This gives that the "level ℓ" approximation of the right hand side of (4.3.3) (which counts correctly the terms from π arising from conjugacy classes of semisimple elements $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ with eigenvalue -1 of multiplicity less than or equal to 2ℓ, and miss-counts the terms from π arising from conjugacy classes with eigenvalue -1 of multiplicity more than 2ℓ) is the sum of the first ℓ terms of (4.3.12).

More formally:

PROOF OF THEOREM 4.3.1, PART (1). Let us specifically consider the case W is an $\mathbb{F}_q$-vector space of dimension $2m+1$ with symmetric bilinear form B (the even dimensional cases are analogous).

An irreducible representation π of $\mathrm{O}(W,B)$ can be expressed as $r[(s),u]^{\pm 1}$, corresponding to a conjugacy class of a semisimple element $s \in \mathrm{Sp}_{2m}(\mathbb{F}_q)$, a unipotent representation u of s's centralizer expressible as a product of symbols, and a extension data sign ± 1. In this case, since the dimension of $r^{\mathrm{O}(W,B)}[(s),u]^{\pm 1}$ is equal to the dimension of the $\mathrm{SO}(W,B)$-representation $r^{\mathrm{SO}(W,B)}[(s),u]$, we may re-express the right hand side of (4.3.3) as the sum over all choices of $(s) \in \mathrm{Sp}_{2m}(\mathbb{F}_q)$, $u \in \widehat{Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s)}_u$ of terms

$$\begin{gathered}\dim(r^{\mathrm{SO}(W,B)}[(s),u])\cdot \\ (\dim(\phi(r^{\mathrm{SO}(W,B)}[(s),u]^+)) + \dim(\phi(r^{\mathrm{SO}(W,B)}[(s),u]^-))).\end{gathered} \tag{4.3.20}$$

It suffices to prove by Proposition 4.3.3 that adding up these terms gives (4.3.12).

We organize these terms by considering, for $\ell = 0, \dots, m$, the set $\mathscr{S}_\ell$ of semisimple conjugacy classes (s) such that the multiplicity of -1 as an eigenvalue of s is 2ℓ. In this case, we may express the centralizer of s as a product

$$Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s) = (Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s))_{\neq -1} \times \mathrm{Sp}_{2\ell}(\mathbb{F}_q) \tag{4.3.21}$$

of $\mathrm{Sp}_{2\ell}(\mathbb{F}_q)$ with a subgroup $(Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s))_{\neq -1} \subseteq \mathrm{Sp}_{2(m-\ell)}(\mathbb{F}_q)$, obtained as the centralizer of a semisimple element $(s)_{\neq -1}$ of $\mathrm{Sp}_{2(m-\ell)}(\mathbb{F}_q)$ with no -1 eigenvalues. Note that $s = (s)_{\neq -1} \oplus -I_{2\ell}$.

To be more specific, first let us focus on choices data $[(s),u]$ with $(s) \in \mathscr{S}_0$. The dimension of the corresponding $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$-representation can be expressed as

$$\dim(r^{\mathrm{SO}_{2m+1}(\mathbb{F}_q)}[(s),u]) = \frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}}{|(Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s))_{\neq -1}|_{q'}}\dim(u).$$

The centralizer of the semisimple data $\phi^\pm(s) \in \mathrm{SO}_{2N+1}(\mathbb{F}_q)$ defining $\phi(r^{\mathrm{O}_{2m+1}(\mathbb{F}_q)}[(s),u]^\pm)$ is

$$Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(\phi^\pm(s)) = (Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s))^*_{\neq -1} \times \mathrm{SO}^\pm_{2(N-m)}(\mathbb{F}_q)$$

and $\phi^{\pm}(u) \cong u \otimes 1$. Therefore, we can express the dimension of the image of these representations under our proposed construction of the eta correspondence as

$$\dim(\phi(r^{\mathrm{O}_{2m+1}(\mathbb{F}_q)}[(s),u]^{\pm})) = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{2\cdot|(Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s))^*_{\neq -1}\times \mathrm{SO}^{\pm}_{2(N-m)}(\mathbb{F}_q)|_{q'}}\dim(u) = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{2\cdot|\mathrm{SO}^{\pm}_{2(N-m)}(\mathbb{F}_q)|_{q'}\cdot|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}}\dim(r^{\mathrm{SO}_{2m+1}(\mathbb{F}_q)}[(s),u]).$$

Therefore, the sum of terms (4.3.20) over all choices of $(s)\in\mathscr{S}_0$ and $u\in\widehat{Z_{\mathrm{Sp}_2m(\mathbb{F}_q)}(s)}_u$ can be re-expressed as $|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}/|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}$ times the sum of terms

$$(\frac{1}{2|\mathrm{SO}^{+}_{2(N-m)}(\mathbb{F}_q)|_{q'}}+\frac{1}{2|\mathrm{SO}^{-}_{2(N-m)}(\mathbb{F}_q)|_{q'}})\dim(r^{\mathrm{SO}_{2m+1}(\mathbb{F}_q)}[(s),u])^2$$

which can be simplified as

$$\frac{q^{N-m}}{|\mathrm{Sp}_{2(N-m)}(\mathbb{F}_q)|_{q'}}\dim(r^{\mathrm{SO}_{2m+1}(\mathbb{F}_q)}[(s),u])^2 \tag{4.3.22}$$

If we took the sum of these terms over all choices of $(s)\in\mathrm{Sp}_{2m}(\mathbb{F}_q)$ and $u\in\widehat{Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s)}_u$ instead of only those with $(s)\in\mathscr{S}_0$, we could simplify this expression as

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}}\frac{q^{N-m}}{|\mathrm{Sp}_{2(N-m)}(\mathbb{F}_q)|_{q'}}|\mathrm{Sp}_{2m}(\mathbb{F}_q)| = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(N-m)}(\mathbb{F}_q)|_{q'}}q^{N+m^2-m}. \tag{4.3.23}$$

Therefore, the contribution of the terms (4.3.20) to the dimension of the top part of the oscillator representation can be estimated as the main term (4.3.23), summed with an error term of $-|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}/|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}$ times the sum of terms (4.3.22) for all $(s)\notin\mathscr{S}_0$, and u. Note that (4.3.23) is exactly the $\ell=0$ term of (4.3.12).

Now let us consider the terms (4.3.20) for $(s)\in\mathscr{S}_1$ and any u. Considering (4.3.21) for $\ell=1$, we may express u as $u_{\neq -1}\otimes\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ for a symbol $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}\in\{\binom{1}{\emptyset},\ \binom{0<1}{1}\}=\widehat{\mathrm{Sp}_2(\mathbb{F}_q)}_u$. Using similar methods as above, we find that we can compare the dimension of the representation corresponding to this data with the $\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)$-representation corresponding to $(s)_{\neq -1}$ and $u_{\neq -1}$ to obtain

$$\dim(r^{\mathrm{SO}_{2m+1}(\mathbb{F}_q)}[(s),u]) = \frac{|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'} \cdot \dim(r^{\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}])}{|\mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_2(\mathbb{F}_q)|_{q'}} \cdot \dim(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}).$$

Similarly, the dimension of $\phi(r^{\mathrm{O}_{2m+1}(\mathbb{F}_q)}[(s),u]^{\pm 1})$ can be expressed as

$$\frac{|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'} \cdot \dim(r^{\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}])}{|\mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)|_{q'}|\mathrm{SO}^{\pm}_{2(N-m+1)}(\mathbb{F}_q)|_{q'}} \cdot \dim(\phi^{\pm}\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}).$$

Recall that in this case

$$\phi^+\tbinom{1}{\emptyset} = \tbinom{1}{N-m}, \quad \phi^-\tbinom{1}{\emptyset} = \tbinom{1<N-m}{\emptyset}$$
$$\phi^+\tbinom{0<1}{1} = \tbinom{0<1}{1<N-m+1}, \quad \phi^-\tbinom{0<1}{1} = \tbinom{0<1<N-m+1}{1}.$$

Therefore, the terms of (4.3.20) corresponding to $(s) \in \mathscr{S}_1$ and u can be expressed as the product of the sum over all choices of $(s)_{\neq -1} \in \mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)$ with no -1 eigenvalues and unipotent representations $u_{\neq -1}$ of the centralizer of $(s)_{\neq -1}$ of

$$\frac{|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'} \cdot |\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)|^2_{q'} \cdot |\mathrm{Sp}_2(\mathbb{F}_q)|_{q'}} \dim(r^{\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}])^2, \tag{4.3.24}$$

multiplied with $1/2$ times the sum

$$\begin{aligned} &\frac{\dim(\binom{1}{\emptyset})}{|\mathrm{Sp}_2(\mathbb{F}_q)|_{q'}} \frac{\dim(\binom{1}{N-m})}{|\mathrm{SO}^+_{2(N-m+1)}(\mathbb{F}_q)|_{q'}} + \frac{\dim(\binom{0<1}{1})}{|\mathrm{Sp}_2(\mathbb{F}_q)|_{q'}} \frac{\dim(\binom{0<1}{1<N-m+1})}{|\mathrm{SO}^+_{2(N-m+1)}(\mathbb{F}_q)|_{q'}} + \\ &\frac{\dim(\binom{1}{\emptyset})}{|\mathrm{Sp}_2(\mathbb{F}_q)|_{q'}} \frac{\dim(\binom{1<N-m}{\emptyset})}{|\mathrm{SO}^-_{2(N-m+1)}(\mathbb{F}_q)|_{q'}} + \frac{\dim(\binom{0<1}{1})}{|\mathrm{Sp}_2(\mathbb{F}_q)|_{q'}} \frac{\dim(\binom{0<1<N-m+1}{1})}{|\mathrm{SO}^+_{2(N-m+1)}(\mathbb{F}_q)|_{q'}}. \end{aligned} \tag{4.3.25}$$

Let us also take into account the error terms from our above estimation of the contribution of $\mathscr{S}_0$. We must subtract, which can be simplified as

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'} q^{N-m}}{|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2(N-m)}(\mathbb{F}_q)|_{q'}} \cdot \frac{|\mathrm{Sp}_{2m}(\mathbb{F}_q)|^2_{q'}}{|\mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)^2_{q'}|\mathrm{Sp}_2(\mathbb{F}_q)|^2_{q'}} \dim(r^{\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}])^2 \cdot (1+q^2) \tag{4.3.26}$$

(since $1+q^2 = \dim(\binom{1}{\emptyset})^2 + \dim(\binom{0<1}{1})^2$). We can estimate the contribution of (4.3.24) (multiplied by 1/2 of (4.3.25)) and (4.3.26) by summing over all $\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)$-representations, instead of only those corresponding to semisimple data $(s)_{\neq -1}$ with no -1 eigenvalues. Doing this allows us to simplify the contribution as the product of a total coefficient

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)|^2_{q'}}|\mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)| = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)|_{q'}}q^{(m-1)^2}$$

multiplied by the sum of 1/2 times (4.3.25) and

$$-(\frac{\dim(\binom{1}{\emptyset})^2}{|\mathrm{Sp}_2(\mathbb{F}_q)|^2_{q'}} + \frac{\dim(\binom{0<1}{1})^2}{|\mathrm{Sp}_2(\mathbb{F}_q)|^2_{q'}}),$$

This sum can ultimately be reduced to

$$-\frac{q^{N-(m-1)+1}}{|\mathrm{Sp}_2(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2(N-m+1)}(\mathbb{F}_q)|_{q'}}.$$

In total, we can that this estimation is the $\ell = 1$ term of (4.3.12). However, we can see that the above estimation (working with all representations of $\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)$ instead of those only corresponding to semisimple data with no -1 eigenvalues) creates two kinds of error terms: We must subtract the terms obtained from products of (4.3.24) and 1/2 of (4.3.25) with $r^{\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}]$ replaced by any $r^{\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)}[(s'), u']$ for $(s') \in \mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)$ with -1 as an eigenvalue. We must also add back the terms of (4.3.26), again with the representations $r^{\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}]$ replaced by $r^{\mathrm{SO}_{2(m-1)+1}(\mathbb{F}_q)}[(s'), u']$ for $(s') \in \mathrm{Sp}_{2(m-1)}(\mathbb{F}_q)$ with -1 as an eigenvalue.

Let us continue in this fashion inductively, processing at the ℓth step the contribution of the terms (4.3.20) to the right hand side of (4.3.3) for representations corresponding to $(s) \in \mathscr{S}_\ell$ for general $0 \leq \ell \leq m$. As in the above cases, our goal is to prove that the contribution of these terms, summed with all error terms arising from previous steps contributing terms obtained from level ℓ representations, produces the ℓth term of (4.3.12), summed with error corresponding to representations of level strictly greater than ℓ.

Consider $(s) \in \mathscr{S}_\ell$ and $u \in \widehat{Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s)}_u$. Let us express

$$u = u_{\neq -1} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

for a symbol $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b} \in \widehat{\mathrm{Sp}_{2\ell}(\mathbb{F}_q)}_u$. Generalizing the above cases, we can again express the dimension of $r^{\mathrm{SO}_{2m+1}(\mathbb{F}_q)}[(s), u]$ as

$$\frac{|\mathrm{Sp}_{2m}|_{q'}\dim(r^{\mathrm{SO}_{2(m-\ell)+1}}[(s)_{\neq -1}, u_{\neq -1}])}{|\mathrm{Sp}_{2(m-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2\ell}(\mathbb{F}_q)|_{q'}}\dim(\begin{pmatrix}\lambda_1 < \cdots < \lambda_a\\ \mu_1 < \cdots < \mu_b\end{pmatrix})$$

and the dimension of $\phi(r^{\mathrm{O}_{2m+1}(\mathbb{F}_q)}[(s), u]^{\pm 1})$ is

$$\frac{|\mathrm{Sp}_{2N}|_{q'}\dim(r^{\mathrm{SO}_{2(m-\ell)+1}}[(s)_{\neq -1}, u_{\neq -1}])}{|\mathrm{Sp}_{2(m-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{SO}^{\pm}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}\dim(\phi^{\pm}\begin{pmatrix}\lambda_1 < \cdots < \lambda_a\\ \mu_1 < \cdots < \mu_b\end{pmatrix}).$$

Taking a product of these dimensions gives the product of the sum

$$\frac{|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(m-\ell)}(\mathbb{F}_q)|^2_{q'}}\dim(r^{\mathrm{SO}_{2(m-\ell)+1}}[(s)_{\neq -1}, u_{\neq -1}])^2 \tag{4.3.27}$$

over all $r^{\mathrm{SO}_{2(m-\ell)+1}}[(s)_{\neq -1}, u_{\neq -1}]$ for any $(s)_{\neq -1} \in \mathrm{Sp}_{2(m-\ell)}(\mathbb{F}_q)$ with no -1 eigenvalues, with the sum

$$\frac{\dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{|\mathrm{Sp}_{2\ell}(\mathbb{F}_q)|_{q'}}(\frac{\dim(\phi^+\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{|\mathrm{SO}^+_{2(N-m+\ell)}(\mathbb{F}_q)} + \frac{\dim(\phi^-\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{\mathrm{SO}^-_{2(N-m+\ell)}(\mathbb{F}_q)}) \tag{4.3.28}$$

over all symbols $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b} \in \widehat{\mathrm{Sp}_{2\ell}(\mathbb{F}_q)}_u$.

We must also consider the error terms arising from previous levels of representations. Specifically, consider a choice of $0 < k = k_1 + \cdots + k_j \leq \ell$. We must consider the error terms contributed from the k_1th step, replacing $r^{\mathrm{SO}_{2(m-k_1)+1}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}]$ by $\mathrm{SO}_{2(m-k_1)+1}(\mathbb{F}_q)$-representations corresponding to semisimple data with -1 as an eigenvalue of multiplicity $2k_2$. This contributes and error term we estimate at step $k_1 + k_2$, and we may consider its error by replacing $r^{\mathrm{SO}_{2(m-(k_1+k_2))+1}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}]$ by $\mathrm{SO}_{2(m-(k_1+k_2))+1}(\mathbb{F}_q)$-representations corresponding to semisimple data with -1 as an eigenvalue of multiplicity $2k_3$. By iterating this process we may finally consider corresponding error terms at the ℓth step, consisting of the product of the following factors: the sum of

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(m-\ell)}(\mathbb{F}_q)|^2_{q'}}\dim(r^{\mathrm{SO}_{2(m-\ell)+1}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}])^2, \tag{4.3.29}$$

the sum over $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b} \in \widehat{\mathrm{Sp}_{2(\ell-k)}(\mathbb{F}_q)}_u$ of terms

$$\frac{\dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{|\mathrm{Sp}_{2(\ell-k)}(\mathbb{F}_q)|_{q'}}(\frac{\dim(\phi^+\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{|\mathrm{SO}^+_{2(N-m+\ell-k)}(\mathbb{F}_q)} + \frac{\dim(\phi^-\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{\mathrm{SO}^-_{2(N-m+\ell-k)}(\mathbb{F}_q)}),$$

and the products

$$\prod_{i=1}^{j}\Big(\sum_{\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}\in\widehat{\mathrm{Sp}_{2k_i}(\mathbb{F}_q)}_u}\frac{\dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})^2}{|\mathrm{Sp}_{2k_i}(\mathbb{F}_q)|_{q'}^2}\Big)).$$

The sign of this error term is $(-1)^j$. The main term can be obtained by summing these terms over all representations $r^{\mathrm{SO}_{2(m-\ell)+1}(\mathbb{F}_q)}[(s'),u']$ instead of only the representations $r^{\mathrm{SO}_{2(m-\ell)+1}(\mathbb{F}_q)}[(s)_{\neq -1},u_{\neq -1}]$ for $(s)_{\neq -1}$ with no -1 eigenvalues. This creates new error terms affecting later steps. Using the fact that the sum of the squares of the dimensions of $r^{\mathrm{SO}_{2(m-\ell)+1}(\mathbb{F}_q)}[(s'),u']$ is $|\mathrm{SO}_{2(m-\ell)+1}(\mathbb{F}_q)|$, we may reduce these estimations of the terms (4.3.27) and (4.3.29).

By reducing and adding up, we obtain the product of a total coefficient

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2m}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(m-\ell)}(\mathbb{F}_q)|_{q'}}q^{(m-\ell)^2} \tag{4.3.30}$$

times the sum over each $j=0,\dots\ell$ of terms

$$\begin{gathered}(-1)^j\sum_{0\le k=k_1+\cdots+k_j\le\ell}\ \prod_{i=1}^{j}\Big(\sum_{\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}\in\widehat{\mathrm{Sp}_{2k_i}(\mathbb{F}_q)}_u}\frac{\dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})^2}{|\mathrm{Sp}_{2k_i}(\mathbb{F}_q)|_{q'}^2}\Big)\cdot\\ \Big(\sum_{\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}\in\widehat{\mathrm{Sp}_{2(\ell-k)}(\mathbb{F}_q)}_u}\frac{\dim\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}}{|\mathrm{Sp}_{2(\ell-k)}(\mathbb{F}_q)|_{q'}}\Big(\frac{\phi^+\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}}{2|SO^+_{2(N-m+\ell-k)}(\mathbb{F}_q)|_{q'}}+\\ \frac{\phi^-\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}}{2|SO^-_{2(N-m+\ell-k)}(\mathbb{F}_q)|_{q'}}\Big)\Big).\end{gathered} \tag{4.3.31}$$

Our goal is to reduce this to the ℓth term of (4.3.12). By comparing the ℓth term of (4.3.12) with (4.3.30) using (4.3.17), we find that we have reduced the problem of reducing (4.3.31) to

$$(-1)^\ell\frac{q^{N-(m-\ell)+\ell^2}}{|\mathrm{Sp}_{2\ell}(\mathbb{F}_q)|_{q'}\cdot|\mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}.$$

Using the inclusion-exclusion principle, this is equivalent to the claim that the sum of terms (4.3.28) over all symbols $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}\in$

$\widehat{\mathrm{Sp}_{2\ell}(\mathbb{F}_q)}_u$ is equal to the sum over $k = 0, \ldots, \ell$ of

$$(4.3.32) \quad \begin{gathered} \Big(\sum_{\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b} \in \widehat{\mathrm{Sp}_{2k}(\mathbb{F}_q)}_u} \frac{\dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})^2}{|\mathrm{Sp}_{2k}(\mathbb{F}_q)|_{q'}^2} \Big) \cdot \\ \Big((-1)^{\ell-k} \frac{q^{N-(m-(\ell-k))+(\ell-k)^2}}{|\mathrm{Sp}_{2(\ell-k)}(\mathbb{F}_q)|_{q'} |\mathrm{Sp}_{2(N-m+\ell-k)}(\mathbb{F}_q)|_{q'}} \Big). \end{gathered}$$

This can be concluded since $\phi^{\pm}\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ are defined as

$$(4.3.33) \quad \binom{\lambda_1<\cdots<\lambda_a<N-m+\frac{a+b-1}{2}}{\mu_1<\cdots<\mu_b}, \quad \binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b<N-m+\frac{a+b-1}{2}}$$

and by recalling that the dimensions of symbols can be computed according to (3.4.9). We may use the sign change in the dimensions of (4.3.33) and comparison with symbols of lower rank by reducing the entry values to obtain that the sum of terms (4.3.32) matches (4.3.28). For example, we can see the Steinberg representation term of (4.3.28) is equal to the term of (4.3.32) for $\ell = k$ contributed by the trivial symbol.

A similar argument can also be applied to cases of even-dimensional orthogonal spaces (W, B). Let us consider the fully split case $\mathrm{O}(W, B) = \mathrm{O}^+_{2m}(\mathbb{F}_q)$ (the case of $\mathrm{O}(W, B) = \mathrm{O}^-_{2m}(\mathbb{F}_q)$ is entirely similar). An irreducible representation π of $\mathrm{O}(W, B) = \mathrm{O}^+_{2m}(\mathbb{F}_q)$ can be expressed as $r[(s), u]^\gamma$ where (s) is a semisimle conjugacy class $(s) \in \mathrm{O}^+_{2m}(\mathbb{F}_q)$ with $\det(s) = 1$, u is a unipotent representation of $Z_{\mathrm{O}^+_{2m}(\mathbb{F}_q)}(s)^\circ$ expressible as a tensor product of symbols, and γ consists of an element in $\{\pm 1\}^{\alpha(s)+\beta(s)}$. Our goal is to prove that the right hand side of (4.3.3), i.e. the sum of all terms

$$(4.3.34) \quad \dim(r^{\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)}[(s), u]^\gamma) \cdot \dim(\phi(r^{\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)}[(s), u]^\gamma))$$

over such choices of data $(s), u$ and γ, gives (4.3.13).

As in the odd case, for $\ell = 0, \ldots, m$, we consider sets $\mathscr{S}_{\ell,\pm}$ of $(s) \in \mathrm{O}^+_{2m}(\mathbb{F}_q)$ with $\det(s) = 1$ satisfying the additional condition that the multiplicity of -1 as an eigenvalue of s is 2ℓ and the orthogonal group obtained from restricting B to -1's eigensapce is $\mathrm{O}^{\pm}_{2\ell}(\mathbb{F}_q)$. (Let us put $\mathscr{S}_{0,-} = \emptyset$.) For $(s) \in \mathscr{S}_{\ell,\pm}$, we can express

$$Z_{\mathrm{O}^+_{2m}(\mathbb{F}_q)}(s)^\circ = (Z_{\mathrm{O}^+_{2m}(\mathbb{F}_q)}(s)^\circ)_{\neq 1} \times \mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q),$$

where $Z_{\mathrm{O}^+_{2m}(\mathbb{F}_q)}(s)^\circ)_{\neq 1} \subseteq \mathrm{SO}^{\pm}_{2(m-\ell)}(\mathbb{F}_q)$ (with sign matching the sign $\mathscr{S}_{\ell,\pm}$) is identity component of the centralizer of a semisimple element

$(s)_{\neq 1}$ in $\mathrm{SO}^{\pm}_{2(m-\ell)}(\mathbb{F}_q)$ with no -1 eigenvalues. We may also then express a unipotent representation u of the identity component of s's centralizer as a tensor product $u_{\neq 1} \otimes \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ for a $\mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q)$-symbol $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$.

Now let us consider the terms (4.3.34) corresponding to data with $(s) \in \mathscr{S}_{0,+}$. (Note that this forces $\alpha(s) = 0$.) We find

$$\dim(r^{\mathrm{O}^+_{2m}(\mathbb{F}_q)}[(s), u]^\gamma) = \frac{|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'}}{2^{\beta(s)}|Z_{\mathrm{O}^+_{2m}(\mathbb{F}_q)}(s)^\circ|_{q'}} \dim(u)$$

and, since $Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(\phi(s))^\circ = Z_{\mathrm{O}^+_{2m}(\mathbb{F}_q)}(s)^\circ \times \mathrm{SO}_{2(N-m)+1}(\mathbb{F}_q)$ for these (s), we also obtain

$$\dim(\phi(r^{\mathrm{O}^+_{2m}(\mathbb{F}_q)}[(s), u]^\gamma)) = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{2^{\beta(s)}|Z_{\mathrm{O}^+_{2m}(\mathbb{F}_q)}(s)^\circ|_{q'}|\mathrm{SO}_{2(N-m)+1}(\mathbb{F}_q)|_{q'}} \dim(u).$$

In particular, we may express $\dim(\phi(r^{\mathrm{O}^+_{2m}(\mathbb{F}_q)}[(s), u]^\gamma))$ as

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'}|\mathrm{SO}_{2(N-m)+1}(\mathbb{F}_q)|_{q'}} \dim(r^{\mathrm{O}^+_{2m}(\mathbb{F}_q)}[(s), u]^\gamma).$$

Therefore, the terms (4.3.34) for $(s) \in \mathscr{S}_{0,+}$ and corresponding u and γ can be expressed as

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'}|\mathrm{SO}_{2(N-m)+1}(\mathbb{F}_q)|_{q'}} \dim(r^{\mathrm{O}^+_{2m}(\mathbb{F}_q)}[(s), u]^\gamma)^2. \tag{4.3.35}$$

We may estimate the sum of these terms by summing over terms (4.3.35) over all $\mathrm{O}^+_{2m}(\mathbb{F}_q)$-representations $r^{\mathrm{O}^+_{2m}(\mathbb{F}_q)}[(s), u]^\gamma$, not only those arising from $(s) \in \mathscr{S}_{0,+}$. Doing this, we are able to collect the squares of all $\mathrm{O}^+_{2m}(\mathbb{F}_q)$-representations into the group order, giving an esitmate of sum of terms

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'}|\mathrm{SO}_{2(N-m)+1}(\mathbb{F}_q)|_{q'}} |\mathrm{O}^+_{2m}(\mathbb{F}_q)| = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(N-m)}(\mathbb{F}_q)|_{q'}} q^{m(m-1)}. \tag{4.3.36}$$

We observe that this is precisely the $(\ell = 0)$th term of (4.3.13). The error of this estimate is that we must subtract again the terms (4.3.35) for $\mathrm{O}^+_{2m}(\mathbb{F}_q)$-representations obtained from $(s) \in \mathscr{S}_{\ell,\pm}$ for $\ell > 0$. As in the odd case, these will taken into account later at the ℓth step when we consider the contibution of terms (4.3.34) originating from $r^{\mathrm{O}^+_{2m}(\mathbb{F}_q)}[(s), u]^\gamma$ for $(s) \in \mathscr{S}_{\ell,\pm}$.

From here, the process continues precisely the same way as it did in the case of odd orthogonal groups. At step ℓ, considering terms (4.3.34) contributed by representations corresponding to semisimple data $(s) \in \mathscr{S}_{\ell,\pm}$. In this case, we obtain

$$\dim(r^{\mathrm{O}^+_{2m}(\mathbb{F}_q)}[(s),u]^\gamma) = \frac{|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'}\dim(r^{\mathrm{O}^\pm_{2(m-\ell)}(\mathbb{F}_q)}[(s)_{\neq 1},u_{\neq 1}]^{\gamma_{\neq 1}})}{2|\mathrm{O}^\pm_{2(m-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{SO}^\pm_{2\ell}(\mathbb{F}_q)|_{q'}}\dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}) \tag{4.3.37}$$

and

$$\dim(\phi(r^{\mathrm{O}^+_{2m}(\mathbb{F}_q)}[(s),u]^\gamma)) = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}\dim(r^{\mathrm{O}^\pm_{2(m-\ell)}(\mathbb{F}_q)}[(s)_{\neq 1},u_{\neq 1}]^{\gamma_{\neq 1}})}{|\mathrm{O}^\pm_{2(m-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{SO}_{2(N-m+\ell)+1}(\mathbb{F}_q)|_{q'}}\dim(\phi^{\gamma_\alpha}\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}) \tag{4.3.38}$$

where all the $\pm$ signs are equal to the sign with $(s) \in \mathscr{S}_{\ell,\pm}$. We can collect the sum of the products of (4.3.37) and (4.3.38) according to the representations $r^{\mathrm{O}^\pm_{2(m-\ell)}}[(s)_{\neq 1},u_{\neq 1}]^{\gamma_{\neq 1}}$ to give the product of the sum of terms

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'}}{2|\mathrm{O}^\pm_{2(m-\ell)}(\mathbb{F}_q)|^2_{q'}}\dim(r^{\mathrm{O}^\pm_{2(m-\ell)}}[(s)_{\neq 1},u_{\neq 1}]^{\gamma_{\neq 1}})^2, \tag{4.3.39}$$

with

$$\sum_{\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}\in\widehat{\mathrm{SO}^\pm_{2\ell}}_u}\frac{\dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{|\mathrm{SO}^\pm_{2\ell}|_{q'}}(\frac{\dim(\phi^+\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})+\dim(\phi^-\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{|\mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}})). \tag{4.3.40}$$

The sum of (4.3.39) can be estimated by summing over every $\mathrm{O}^\pm_{2(m-\ell)}(\mathbb{F}_q)$-representation instead of only $r^{\mathrm{O}^\pm_{2(m-\ell)}}[(s)_{\neq 1},u_{\neq 1}]^{\gamma_{\neq 1}}$, reducing to the product of the coefficient

$$\frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'}}{2|\mathrm{O}^\pm_{2(m-\ell)}(\mathbb{F}_q)|_{q'}}q^{(m-\ell)(m-\ell-1)} \tag{4.3.41}$$

and (4.3.40) and generates error terms that must be considered at later steps. Using a similar reasoning, as in the odd case, the error terms that must be considered at step ℓ correspond to choices of $0 < k_1+\cdots+k_j = k \leq \ell$ and can be estimated as terms $(-1)^j$ times (4.3.41) times

$$\sum_{\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}\in\widehat{\mathrm{SO}^\pm_{2(\ell-k)}}_u}\frac{\dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{|\mathrm{SO}^\pm_{2(\ell-k)}|_{q'}}(\frac{\dim(\phi^+\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})+\dim(\phi^-\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{|\mathrm{Sp}_{2(N-m+\ell-k)}(\mathbb{F}_q)|_{q'}}).$$

$$\prod_{i=1}^{j}(\sum_{\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}\in\widehat{\mathrm{SO}^+_{2k_i}}_u}\frac{\dim(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})^2}{|\mathrm{SO}^\pm_{2k_i}(\mathbb{F}_q)|^2_{q'}})$$

(again, with this estimation causing error terms at later steps). This can be calculated again using an inclusion-exclusion principle, the definition of $\phi^{\pm}\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$, and (3.4.9), obtaining (when all signs are considered)

$$(-1)^{\ell}\frac{q^{\ell(\ell-1)}}{|\mathrm{SO}^{+}_{2\ell}|_{q'}|\mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}+(-1)^{\ell-1}\frac{q^{\ell(\ell-1)}}{|\mathrm{SO}^{-}_{2\ell}|_{q'}|\mathrm{Sp}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}}.$$

By combining this factor with (4.3.41) and comparing with (4.3.18), we see this recovers the ℓth term of (4.3.13). □

4.4. Modifications of the combinatorics for the orthogonal stable range. We now discuss the analogous results for the case of reductive dual pairs in the orthogonal stable range. The methods applied above to conclude part (1) of Theorem 4.3.1 work completely similarly for this case, but we still make note of the analogous formulae used in the main steps.

Again, the first step is to recursively calculate the top part (4.3.2) of an oscillator representation restricted to a reductive dual pair in the orthogonal stable range. Unlike in the analogous calculations in Proposition 4.3.2 for the symplectic stable range, we will note obtain separate formulae depending on the parity of the dimension of W.

4.4.1. PROPOSITION. *Consider an orthogonal stable range reductive dual pair $(Sp(V), O(W,B))$. The dimension of the top part of $\omega[V\otimes W]$ is*

$$\sum_{i=0}^{N}(-1)^{N-i}\cdot q^{\binom{N-i}{2}}\cdot\binom{N}{i}_q\cdot\prod_{j=i+1}^{N}(q^j+1)\cdot q^{i\cdot dim\,(W)}. \tag{4.4.1}$$

Again, the next step is to re-express the formula for $\dim(\omega[V\otimes W]^{\mathrm{top}'}$ in terms which will be more easily interpretable from the perspective of representations:

4.4.2. PROPOSITION. *For a reductive dual pair $(Sp(V), O(W,B))$ in the orthogonal stable range.*

(1) *If W is odd-dimensional, then writing $dim(V)=2N$, $dim(W)=2m+1$, the dimension of the top part of the oscillator representation $\omega[V\otimes W]^{top'}$ can be re-expressed as*

$$dim(\omega[V\otimes W]^{top'}) =$$

$$\sum_{\ell=0}^{N}(-1)^{\ell}\cdot q^{(N-\ell)^2+\ell(\ell-1)}\cdot\binom{N}{\ell}_{q^2}\cdot\prod_{i=0}^{N-\ell-1}(q^{2(m-i)}-1). \tag{4.4.2}$$

(2) *If W is even-dimensional, then writing dim(V) = 2N, dim(W) = 2m, the dimension of the top part of the oscillator representation $\omega[V \otimes W]^{top'}$ can be re-expressed as*

$$\omega[V \otimes W]^{top'} = \sum_{\ell=0}^{N} q^{\ell^2+(N-\ell)(N-\ell-1)} \binom{N}{\ell}_{q^2} \cdot \prod_{i=1}^{N-\ell} (q^{2(m-N+\ell+i)} - 1) \tag{4.4.3}$$

Proof of Theorem 4.3.1, part (2). The proof largely follows the exact same structure as the proof of part (1). We treat the case of reductive dual pairs involving an odd-dimensional orthogonal space W first. Write $\dim(V) = 2N$ and $\dim(W) = 2m+1$. Our goal is to prove that the sum over all $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representations expressible as $r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, \pm 1]$ (omitting the ± 1 sign data if it is not needed) where (s) is a semisimple conjugacy class in $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$ and u is a unipotent representation of its centralizer expressible as a tensor product of symbols) of terms

$$\dim(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, \pm 1]) \cdot \dim(\psi(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, \pm 1])) \tag{4.4.4}$$

matches (4.4.2).

As in the case of the eta correspondence, we begin by re-expressing (4.4.2) in terms of group orders. To work with the ℓth term of (4.4.2), we note that

$$\binom{N}{\ell}_{q^2} \prod_{i=1}^{N-\ell-1} (q^{2(m-i)} - 1) = \frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'} |\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(N-\ell)}|_{q'} |\mathrm{Sp}_{2(m-N+\ell)}(\mathbb{F}_q)|_{q'}} \left(\frac{1}{|\mathrm{O}^+_{2\ell}(\mathbb{F}_q)|_{q'}} - \frac{1}{|\mathrm{O}^-_{2\ell}(\mathbb{F}_q)|_{q'}}\right). \tag{4.4.5}$$

Let us partition the semisimple conjugacy classes $(s) \in \mathrm{SO}_{2N+1}(\mathbb{F}_q)$ into sets $\mathscr{S}_{\ell,\pm}$ indexed by $\ell = 0, \dots, N$ and a choice of sign, consisting of (s) where the multiplicity of -1 as an eigenvalue is 2ℓ and the restriction of the symmetric bilinear form ddefining $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$ to the -1-eigenspace is split (in the case of $\mathscr{S}_{\ell,+}$) or non-split (in the case of $\mathscr{S}_{\ell,-}$). For $\mathscr{S}_{\ell,\pm}$, the identity component of its centralizer can be expressed as a product

$$(Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(s))_{\neq -1} \times \mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q),$$

where $(Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(s))_{\neq -1}$ is the (connected) centralizer of a semisimple element $(s)_{\neq -1}$ with no -1 eigenvalues in $\mathrm{SO}_{2(N-\ell)+1}(\mathbb{F}_q)$. We can factor the unipotent representation u then as a tensor product

$$u_{\neq -1} \otimes \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$$

for a symbol $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ of $\mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q)$. (Put $\mathscr{S}_{0,-} = \emptyset$.)

First, for $(s) \in \mathscr{S}_{0,+}$ and corresponding u (note that sign data cannot be needed in this case), we have

$$\dim(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s),u]) = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|(Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(s))_{\neq -1}|_{q'}}\dim(u)$$

and

$$\dim(\psi(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s),u])) = \frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}}{|(Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(s))_{\neq -1}|_{q'}|\mathrm{Sp}_{2(m-N)}(\mathbb{F}_q)|_{q'}}\dim(u),$$

so

$$\dim(\psi(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s),u])) = \frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}\dim(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s),u])}{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2(m-N)}(\mathbb{F}_q)|_{q'}}.$$

Therefore, the corresponding term (4.4.4) is expressible as

$$\frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2(m-N)}(\mathbb{F}_q)|_{q'}}\dim(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s),u])^2. \tag{4.4.6}$$

We estimate the contribution of these terms by summing terms (4.4.6) over all irreducible $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representations instead of only those obtained as $r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s),u]$ for $(s) \in \mathscr{S}_{0,+}$, with the error of then subtracting back away all of the contributions (4.4.6) with $r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s),u]$ replaced by some general $r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s),u,\pm 1]$ (ommiting the sign data if it is unnecessary) for $(s) \in \mathscr{S}_{\ell,\pm}$. This estimation can be calculated using the fact that the sum of squares of all dimensions of irreducible $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representations recovers the group order, therefore reducing to

$$\frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2(m-N)}(\mathbb{F}_q)|_{q'}}|\mathrm{Sp}_{2N}(\mathbb{F}_q)| = \frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2(m-N)}(\mathbb{F}_q)|_{q'}}q^{N^2}.$$

We note that this exactly matches the $(\ell = 0)$th term of (4.4.2).

Proceeding similarly as in the eta correspondence case, we work inductively, calculating that at an ℓth step, we can estimate the terms (4.4.4) corresponding to $(s) \in \mathscr{S}_{\ell,\pm}$ and the error arising from previous steps' (possibly iterated) estimations affecting this level as the ℓth term of (4.4.2), with error affecting later steps. All the steps can be followed through using the exact same methods as in the eta correspondence case, so we will only summarize the key steps now. First, we find that for $r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, \alpha]$ for $(s) \in \mathscr{S}_{\ell,\pm}$, we may express

$$\dim(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, \alpha]) = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'} \dim(r^{\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}])}{2|\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q)|_{q'}} \dim(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}) \tag{4.4.7}$$

and

$$\dim(\psi(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, \alpha])) = \frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'} \dim(r^{\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}])}{|\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{SO}_{2(m-N+\ell)+1}(\mathbb{F}_q)|_{q'}} \dim(\psi^{\alpha}\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}), \tag{4.4.8}$$

while if no sign data α occurs, we find

$$\dim(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u]) = \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'} \dim(r^{\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}])}{|\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q)|_{q'}} \dim(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}) \tag{4.4.9}$$

and

$$\dim(\psi(r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u])) = \frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'} \dim(r^{\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}])}{|\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{SO}_{2(m-N+\ell)+1}(\mathbb{F}_q)|_{q'}} \dim(\psi\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}). \tag{4.4.10}$$

Ultimately, their product can be estimated (by summing over general $\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)$-representations instead of only $r^{\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}]$ and collecting the sum of square dimensions as the group order) as

$$\frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)|_{q'}} q^{(N-\ell)^2} \tag{4.4.11}$$

times the sum over all irreducible $\mathrm{O}^{\pm}_{2\ell}(\mathbb{F}_q)$-represnetations u of terms

$$\frac{\dim(u)}{|\mathrm{O}^{\pm}_{2\ell}(\mathbb{F}_q)|_{q'}}(\frac{\dim(\psi^{+}\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}) + \dim(\psi^{-}\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b})}{|\mathrm{SO}_{2(m-N+\ell)+1}(\mathbb{F}_q)|_{q'}}) \tag{4.4.12}$$

where $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ is the (possible degenerate) $\mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q)$-symbol such that $u \subseteq \mathrm{Ind}^{\mathrm{O}^{\pm}_{2\ell}(\mathbb{F}_q)}_{\mathrm{SO}^{\pm}_{2\ell}(\mathbb{F}_q)}(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b})$. The error terms from previous steps are indexed by choices of $0 < k_1 + \cdots + k_j \leq \ell$ and are estimated (again by summing in over general $\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)$-representations instead of only $r^{\mathrm{Sp}_{2(N-\ell)}(\mathbb{F}_q)}[(s)_{\neq -1}, u_{\neq -1}]$ and collecting the sum of square dimensions as the group order) the form $(-1)^j$ times the product of (4.4.11) and the sum

$$\frac{\dim(u)}{|\mathrm{O}^{\pm}_{2(\ell-k)}(\mathbb{F}_q)|_{q'}}\left(\frac{\dim(\psi^+\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})+\dim(\psi^-\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})}{|\mathrm{SO}_{2(m-N+\ell-k)+1}(\mathbb{F}_q)|_{q'}}\right) \tag{4.4.13}$$

where $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ is the (possible degenerate) $\mathrm{SO}^{\pm}_{2(\ell-k)}(\mathbb{F}_q)$-symbol such that $u\subseteq \mathrm{Ind}^{\mathrm{O}^{\pm}_{2(\ell-k)}(\mathbb{F}_q)}_{\mathrm{SO}^{\pm}_{2(\ell-k)}(\mathbb{F}_q)}(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})$, times

$$\prod_{i=1}^{j}\sum_{u\in\widehat{\mathrm{O}^{\pm}_{2k_i}(\mathbb{F}_q)}_u}\frac{\dim(u)^2}{|\mathrm{O}^{\pm}_{2k_i}(\mathbb{F}_q)|^2_{q'}}. \tag{4.4.14}$$

Our goal is to calculate that the sum of the main estimate and the estimated error terms described above is equal to the ℓth term of (4.4.2). Factoring out (4.4.11) and comparing with (4.4.5), it suffices to show the sum of (4.4.12) with all products of $(-1)^j$ times (4.4.13) and (4.4.14) gives

$$(-1)^{\ell}\frac{q^{\ell(\ell-1)}}{|\mathrm{Sp}_{2(m-N+\ell)}(\mathbb{F}_q)|_{q'}}\left(\frac{1}{|\mathrm{O}^+_{2\ell}(\mathbb{F}_q)|_{q'}}-\frac{1}{|\mathrm{O}^-_{2\ell}(\mathbb{F}_q)|_{q'}}\right)$$

This can be approached as before using the inclusion-exclusion principle and (3.4.9).

For the even case, the argument is modified exactly in the same way as in part (1). For $\mathrm{O}(W,B)=\mathrm{O}^+_{2m}(\mathbb{F}_q)$, the analogue of (4.4.5) is

$$\frac{1}{4}\left(\sum_{\epsilon_1,\epsilon_2=\pm}\epsilon_2\frac{|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{O}^{\epsilon_1}_{2(N-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{O}^{\epsilon_2}_{2\ell}(\mathbb{F}_q)|_{q'}|\mathrm{SO}^{\epsilon_1}_{2(m-N+\ell)}(\mathbb{F}_q)|_{q'}}\right).$$

For $\mathrm{O}(W,B)=\mathrm{O}^+_{2m}(\mathbb{F}_q)$, it is

$$\frac{1}{4}\left(\sum_{\epsilon_1,\epsilon_2=\pm}\epsilon_2\frac{|\mathrm{O}^+_{2m}(\mathbb{F}_q)|_{q'}|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{O}^{\epsilon_1}_{2(N-\ell)}(\mathbb{F}_q)|_{q'}|\mathrm{O}^{\epsilon_2}_{2\ell}(\mathbb{F}_q)|_{q'}|\mathrm{SO}^{-\epsilon_1}_{2(m-N+\ell)}(\mathbb{F}_q)|_{q'}}\right).$$

We proceed by organizing the terms (4.4.4) according to the multiplicity of 1 as an eigenvalue of s and whether or not the restriction of the $2N+1$-dimensional symmetric bilinear form is split on the -1-eigenspace instead of by -1 eigenvalues. All other steps are analogous to the odd case.

□

4.5. Determining the semisimple data and concluding the stable classification theorem. In this subsection, we conclude the statement of Theorem 4.1.1. First, we note that the toral characters of the eta and zeta correspondence are determined inductively, by examining the restriction of the oscillator representations to finite general linear groups. This confirms that the semisimple and cetnral sign data of $\eta^V_{W,B}(\rho)$ (resp. $\zeta^{W,B}_V(\rho)$) matches that of $\phi^V_{W,B}(\rho)$ (resp. $\psi^{W,B}_V(\rho)$).

It then remains in all cases to confirm the unipotent part of $\eta^V_{W,B}(\rho)$ (resp. $\zeta^{W,B}_V(\rho)$) matches that of $\phi^V_{W,B}(\rho)$ (resp. $\psi^{W,B}_V(\rho)$). First, we prove Proposition 4.1.2, and conclude that for $N >> n$, we have

$$dim(\eta^V_{W,B}(\rho)) = dim(\phi^V_{W,B}(\rho)) \tag{4.5.1}$$

(and similarly, for $n >> N$, we have

$$dim(\zeta^V_{W,B}(\rho)) = dim(\psi^V_{W,B}(\rho))). \tag{4.5.2}$$

We may view these dimensions as polynomials of q^N (resp. q^n). The results of Chapter 2 can be used to see that in either stable range, the idempotent in the endomorphism algebra picking out any summand of the eta (resp. zeta) correspondence does not depend on N (resp. n). Therefore, we can apply the description from Chapter 2 to see that (4.5.11) and (4.5.2) both hold for any choice of N, n in the symplectic and orthogonal stable ranges. Therefore, since each unipotent representation corresponding to a different symbol has a different dimension, we find that our claimed construction is the only possible choice. Hence, we conclude Theorem 4.1.1.

For the remainder of this section, we restrict attention to the case of the eta correspondence and $\phi^V_{W,B}$, since the case of the zeta correspondence and $\psi^{W,B}_V$ can be done completely similarly.

The first order of business in this subsection is to prove that the semisimple part (and sign data) of the $\mathrm{Sp}(V)$-classification data of the representation obtained by applying an eta correspondence $\eta^V_{W,B}(\rho)$ matches the semisimple part (and sign data) of our constructed representation $\phi^V_{W,B}(\rho)$ (and the similar statement for $\zeta^{W,B}_V$ and $\psi^{W,B}_V$).

Broadly, this can be concluded since, considering $GL_N(\mathbb{F}_q) \subseteq \mathrm{Sp}(V)$, the restriction of the oscillator representation is

$$\mathrm{Res}_{GL_N(\mathbb{F}_q)}(\omega[V]) \cong \epsilon(det) \otimes \mathbb{C}\mathbb{F}_q^N.$$

Now we also have the restriction

$$\mathrm{Res}_{GL(V)}(\omega[V \otimes W]) \cong (\mathrm{Res}_{GL(V)}(\omega[V]))^{\otimes W}$$

where $\otimes W$ denotes a degree $\dim(W)$ tensor product of oscillator representations $\omega[V]$. Since characters are matched exactly in the premutation representation factors, for example in the case of odd-dimensional W, we know the underlying toral character and the sign data. We now restrict attention to the case of comparing $\eta^V_{W,B}$ and $\phi^V_{W,B}$, for $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic stable range. The case of comparing $\zeta^{W,B}_V$ and $\psi^{W,B}_V$ for the orthogonal stable range is similar.

4.5.1. Proposition. *Suppose* $(Sp(V), O(W,B))$ *is in the symplectic stable range. If* $dim(W) = 2m+1$ *is odd, for* ρ *an irreducible representation of* $SO_{2m+1}(\mathbb{F}_q)$ *arising from the conjugacy class of a semisimple element* $s \in SO_{2m+1}(\mathbb{F}_q)$ *and a unipotent representation* u *of its centralizer, then in the classification data of* $\eta^V_{W,B}((\pm 1)\otimes\rho)$*, its semisimple part is*

$$(\phi^{\pm}(s)) = (s \oplus \sigma^{\pm}_{N-m}).$$

If $dim(W) = 2m$ *is even, for* ρ *an irreducible representation of* $O^{\pm}_{2m}(\mathbb{F}_q)$ *arising from an* $SO^{\pm}_{2m}(\mathbb{F}_q)$*-representation corresponding to the conjugacy class of a semisimple element* $s \in SO^{\pm}_{2m}(\mathbb{F}_q)$ *and a unipotent representation* u *of its centralizer, then in the Jordan decomposition of* $\eta^V_{W,B}(\rho)$*, its semisimple part is*

$$(\phi(s)) = (s \oplus I_{2(N-m)+1}).$$

Proof. Suppose $dim(W) = 2m+1$. Let us begin by considering

$$\underbrace{\mathrm{SO}^{\pm}_2(\mathbb{F}_q) \times \cdots \times \mathrm{SO}^{\pm}_2(\mathbb{F}_q)}_{m} \tag{4.5.3}$$

as a torus of $\mathrm{SO}(W,B)$. Fix a character

$$\chi_{a_1} \otimes \cdots \otimes \chi_{a_m},$$

corresponding to $a_1, \ldots, a_m \in \mu_{q\mp 1} \cong \mathrm{SO}^{\pm}_2(\mathbb{F}_q)$. Consider the maximal parabolic subgroup with Levi (4.5.3) (i.e. the Borel subgroup) $B(W,B) \subseteq \mathrm{SO}(W,B)$. Then, for an irreducible representation ρ with this character, i.e.

$$\rho \subseteq \mathrm{Ind}_{\mathrm{O}(W,B)}(\chi_{a_1} \otimes \cdots \otimes \chi_{a_m}),$$

we need to prove that $\eta_{W,B}(\rho)$ corresponds to a toral character

$$\chi_{a_1} \otimes \cdots \otimes \chi_{a_m} \otimes (\epsilon)^{\otimes N-m} \tag{4.5.4}$$

in $\underbrace{\mathrm{SO}^{\pm}_2(\mathbb{F}_q) \times \cdots \times \mathrm{SO}^{\pm}_2(\mathbb{F}_q)}_{N} \subseteq \mathrm{Sp}_{2N}(\mathbb{F}_q)$ (considering ϵ as the quadratic character of $\mu_{q\mp 1} = \mathrm{SO}^{\pm}_2(\mathbb{F}_q)$.

Consider the inclusion of the product of this torus with $\mathrm{Sp}(V)$

$$\underbrace{\mathrm{SO}_2^{\pm}(\mathbb{F}_q) \times \cdots \times \mathrm{SO}_2^{\pm}(\mathbb{F}_q)}_{m} \times \mathrm{Sp}(V) \subseteq \mathrm{SO}(W, B) \times \mathrm{Sp}(V) \subseteq \mathrm{Sp}(V \otimes W). \tag{4.5.5}$$

Pick the ith factor $\mathrm{SO}_2^{\pm}(\mathbb{F}_q)$ in (4.5.5), taking the inclusion

$$\mathrm{SO}_2^{\pm}(\mathbb{F}_q) \times \mathrm{Sp}(V) \subseteq \mathrm{Sp}(V \otimes W) \tag{4.5.6}$$

Restricting $\omega[V \otimes W]$ along (4.5.6) gives a restriction

$$\mathrm{Res}_{\mathrm{SO}_2^{\pm}(\mathbb{F}_q) \times \mathrm{Sp}(V)}(\omega[V \otimes \mathbb{F}_q^2]) \otimes \mathbb{C}^{q^{(2m-1)N}} \tag{4.5.7}$$

considering $\mathbb{F}_q^2$ with the split and non-split symmetric bilinear form, respectively, (and taking the trivial action on $\mathbb{C}^{q^{(2m-1)N}}$. Recalling the results of Chapter 2, in each factor (4.5.7), it decomposes as a $\mathrm{SO}_2^{\pm}(\mathbb{F}_q) \times \mathrm{Sp}(V)$-representation pairing every χ_{a_i}-type $\mathrm{SO}_2^{\pm}(\mathbb{F}_q)$-representation with a representation $\mathrm{Sp}(V)$ in the induction

$$\mathrm{Ind}^{\mathrm{SO}_2^{\pm}(\mathbb{F}_q)}(\chi_{a_i}),$$

considering $\mathrm{SO}_2^{\pm}(\mathbb{F}_q)$ as a factor of a torus in $\mathrm{Sp}(V)$ Since this holds for every i, it also holds in the restriction of $\omega[V \otimes W]$ along (4.5.5): in

$$\mathrm{Res}_{\mathrm{SO}_2^{\pm}(\mathbb{F}_q) \times \cdots \times \mathrm{SO}_2^{\pm}(\mathbb{F}_q) \times \mathrm{Sp}(V)}(\omega[V \otimes W]),$$

the character $\chi_{a_1} \otimes \cdots \otimes \chi_{a_m}$ as a representation of $\mathrm{SO}_2^{\pm}(\mathbb{F}_q) \times \cdots \times \mathrm{SO}_2^{\pm}(\mathbb{F}_q)$ is paired with a representation of $\mathrm{Sp}(V)$ in that character's induction, viewing the copies of $\mathrm{SO}_2^{\pm}(\mathbb{F}_q)$'s as blocks in a torus of $\mathrm{Sp}(V)$.

The remaining factors of ϵ in (4.5.4) corresponding to the remaining $N - m$ factors in a torus of $\mathrm{Sp}(V)$ arise since the restriction of $\mathrm{Sp}(V)$ to a representation of

$$GL_{N-m}(\mathbb{F}_q) \subseteq GL(\Lambda) \subseteq \mathrm{Sp}(V)$$

is $\epsilon(det)$ tensored with a permutation representation.

A similar argument applies to both even-dimensional cases.

□

Now we can prove Propostion 4.1.2 by induction. Again, we restrict attention to the case of $N >> n$, since the case of $n >> N$ is completely similar.

First, we begin by observing the following

4.5.2. LEMMA. *Fix n, and consider $N >> n$. Every irreducible representation of $Sp_{2N}(\mathbb{F}_q)$ with N-rank n is constructed by applying $\phi^V_{W,B}$ to an irreducible representation of $O(W,B)$ for n-dimensional orthogonal space (W,B).*

PROOF. First suppose $dim(W) = n = 2m+1$. Writing out the definition of $\phi_{W,B}$, we find that the statement is equivalent to the claim that every irreducible representation of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ of N-rank $2m+1$ arises from a conjugacy class (s) of a semisimple element of $\mathrm{SO}_{2N+1}(\mathbb{F}_q)$ with centralizer

$$\prod_{i=1}^{t} U^{\pm}_{j_i}(\mathbb{F}_{q^{r_i}}) \times \mathrm{SO}_{2\ell+1}(\mathbb{F}_q) \times \mathrm{SO}^{\pm}_{2(N-m+p)}(\mathbb{F}_q) \tag{4.5.8}$$

and a unipotent representation u, whose $\mathrm{SO}^{\pm}_{2(N-m+p)}(\mathbb{F}_q)$-representation tensor factor $u_{\mathrm{SO}^{\pm}_{2(N-m+p)}}$ corresponds to a symbol

$$\begin{pmatrix} \alpha_1 < \cdots < \alpha_a \\ \beta_1 < \cdots < \beta_b \end{pmatrix}$$

such that either $\alpha_a = N - m + p + \frac{a+b-1}{2}$ or $\beta_b = N - m + p + \frac{a+b-1}{2}$.

First note the prime to q part of the group orders

$$|\mathrm{SO}_{2N+1}(\mathbb{F}_q)|_{q'} = \prod_{i=1}^{N}(q^{2i}-1), \quad |\mathrm{SO}_{2\ell+1}(\mathbb{F}_q)|_{q'} = \prod_{i=1}^{\ell}(q^{2i}-1),$$

for the groups of type B

$$|\mathrm{SO}^{\pm}_{2(N-m+p)}(\mathbb{F}_q)|_{q'} = (q^{N-m+p} \mp 1) \prod_{i=1}^{N-m+p-1} (q^{2i}-1),$$

for the froup of type D, and

$$|U^{+}_{j_i}(\mathbb{F}_q)|_{q'} = \prod_{u=1}^{j_i}(q^u - 1) \text{ for } i = 1, \dots, r$$

$$|U^{-}_{k_i}(\mathbb{F}_q)|_{q'} = \prod_{u=1}^{k_i}(q^u - (-1)^u) \text{ for } i = 1, \dots, t.$$

Therefore, the total top degree of q in the quotient of prime to q parts of the quotient of group orders

$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|\prod_{i=1}^{r} U^{+}_{j_i}(\mathbb{F}_{q^{n'_i}}) \times \prod_{i=1}^{t} U^{-}_{k_i}(\mathbb{F}_{q^{n''_i}}) \times SO^{\pm}_{2\ell}(\mathbb{F}_q) \times Sp_{2p}(\mathbb{F}_q)|_{q'}} \tag{4.5.9}$$

is

$$\sum_{i=1}^{N} 2i - (\sum_{i=1}^{\ell} 2i + (N-m+p) + \sum_{i=1}^{N-m+p-1} 2i + \sum_{i=1}^{r}\sum_{u=1}^{j_i} u + \sum_{i=1}^{t}\sum_{u=1}^{k_i} u),$$

which can be simplified as

$$N(N+1) - (\ell(\ell+1) + (N-m+p)^2 + \sum_{i=1}^{r} \frac{j_i(j_i+1)}{2} + \sum_{i=1}^{t} \frac{k_i(k_i+1)}{2}). \tag{4.5.10}$$

The terms not involving N (arising from $\mathrm{SO}_{2\ell+1}(\mathbb{F}_q)$ and the unitary groups) do not affect the N-rank of the final $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representation, since

$$\ell + \sum_{i=1}^{r} j_i + \sum_{i=1}^{t} k_i \leq m < \frac{N}{2}.$$

The remaining terms of (4.5.10) are

$$N \cdot (1 - 2(m-p)) + (m-p)^2.$$

Therefore, no smaller factor of type D can occur than those allowed by (4.5.8).

The condition on the symbol arises since otherwise the factor

$$\frac{|SO^{\pm}_{2p}(\mathbb{F}_q)|_{q'}}{2^{(a+b-2)/2}}$$

of (3.4.9) contributes additional copies of N, unless it is cancelled by the denominator of (3.4.9), which can only occur if the rank $N-m+p+(a+b-1)/2$ occurs as an entry in the symbol itself.

A similar argument applies to even cases of $n = dim(W)$.

□

The case of Proposition 4.1.2 for $N >> n$ then follows by induction.

Proof of Proposition 4.1.2, part (1). First we consider the case of W with odd dimensions, and proceed by induction. Suppose for every $m' < m$, we know that the disjoint union of the images of the two eta correspondences $\eta^V_{W,B}$ such that $dim(W) = 2m'+1$ is exactly the set of all irreducible representations of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ with N-rank $2m'+1$, for $N >> m$.

Suppose (W, B) forms an orthogonal space of dimension $2m + 1$. By the definition of $\eta_V^{W,B}$, the sum

$$\bigoplus_{\rho \in \widehat{\mathrm{O}(W,B)}} \rho \otimes \eta_V^{W,B}(\rho)$$

is the top summand of $\omega[V \otimes W]$. In particular, its dimension less than or equal to

$$\dim(\omega[V \otimes W]) = q^{(2m+1)N},$$

so all $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representations of higher N-rank cannot occur in the image of $\eta_V^{W,B}$. Additionally, the images of the different η-correspondences are all disjoint. Therefore, by the induction hypothesis, no irreducible representations of lesser odd N-rank may occur in the image of $\eta^{W,B}$.

To conclude Theorem 4.1.1, note that the pairing $\phi_{W,B}$ obtains the maximal possible dimension

$$\dim(\bigoplus_{\rho \in \mathrm{O}(W,B)} \rho \otimes \phi^{W,B}(\rho)).$$

If a representation of $\mathrm{O}(W, B)$ were paired by $\eta_{W,B}$ with a $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representation of lesser N-rank, it would waste dimensions in

$$\dim(\bigoplus_{\rho \in \widehat{\mathrm{O}(W,B)}} \rho \otimes \eta^{W,B}(\rho)),$$

which would be impossible to get back, by Theorem 4.3.1, since no other representations of N-rank $2m + 1$ exist by Proposition 4.5.2. $\square$

Now we have set-up enough to conclude Theorem 4.1.1. First, we conclude that for every $\rho \in \widehat{\mathrm{O}(W, B)}$,

$$dim(\eta_{W,B}^V(\rho)) = dim(\phi_{W,B}^V(\rho)). \tag{4.5.11}$$

for V of dimension $2N$ and W of dimension n, with $N >> n$. In our construction, for a fixed choice of (W, B) and ρ, for every $N \geq n$, the dimension of our constructed representation $\phi_{W,B}^V(\rho)$ for $dim(V) = 2N$ can be expressed as a polynomial of q^N (see (4.5.12) below). On the other hand, we recall the results of Chapter 2, which allow us to consider the eta correspondence on the level of idempotents. By the stable description of the endomorphism algebra of an oscillator representation given in Chapter 2, we also know the dimensions of $\eta_{W,B}^V(\rho)$ for a fixed ρ and (W, B) must be polynomial in q^N. Therefore, 4.5.11 must in fact hold for every $N \geq n$. Combining this with the results of the previous subsection, we conclude that

$$\eta^V_{W,B}(\rho) = \phi^V_{W,B}(\rho),$$

since all symbols have different dimensions.

First, combining Proposition 4.5.1, Proposition 4.1.2, and Theorem 4.3.1 allows us to conclude (4.5.11) for $N >> n$: Our construction $\phi^V_{W,B}$ satisfies the condition that, for representations $\rho, \pi \in \widehat{\mathrm{O}(W,B)}$ such that $\dim(\rho) < \dim(\pi)$, we have

$$\dim(\phi^{W,B}_V(\rho)) < \dim(\phi^{W,B}_V(\pi)).$$

Therefore, $\phi^{W,B}_V$ is an injective correspondence from which maximizes the dimension sum

$$\sum_{\rho \in \widehat{\mathrm{O}(W,B)}} \dim(\rho) \cdot \dim(\phi^{W,B}(\rho)),$$

which we know numerically matches with

$$\sum_{\rho \in \widehat{\mathrm{O}(W,B)}} \dim(\rho) \cdot \dim(\eta^{W,B}(\rho))$$

by Theorem 4.3.1. Therefore, for $N >> n$, we must have that the dimensions of $\eta^V_{W,B}(\rho)$ match the dimensions of $\phi^V_{W,B}(\rho)$. It remains to prove that this holds for every $N \geq n$, from which we can conclude that the unipotent parts of their classification data agree in general. We do this now, concluding Theorem 4.1.1, art (1). The proof of Part (2) is similar, using the analogue of Proposition 4.5.1 for the zeta correspondence, and the orthogonal stable cases of Proposition 4.1.2, and Theorem 4.3.1.

Proof of Theorem 4.1.1. We restrict attention to the case of W odd dimensional in the symplectic stable range. The even dimensional case and the cases of the orthogonal stable range proceed similarly. Fix an orthogonal space (W, B) of dimesion $n = 2m + 1$, and fix an irreducible representation ρ of $\mathrm{O}(W, B)$. Considering $\mathrm{O}(W, B) = \mathbb{Z}/2 \times SO_{2m+1}(\mathbb{F}_q)$, write ρ as a tensor product

$$\rho = (\alpha) \otimes r(s), u$$

for α denoting a sign specifying a $\mathbb{Z}/2$-action, and $[(s), u]$ denoting the $SO_{2m+1}(\mathbb{F}_q)$-classification data corresponding to the restriction of ρ to $SO_{2m+1}(\mathbb{F}_q)$. Let us consider the symbol $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ associated to the factor of u corresponding to the -1 eigenvalues of s, as in the construction of $\phi^V_{W,B}(\rho)$. Recall the notation $N'_\rho = N - m + \frac{a+b-1}{2}$. For

every V of dimension $2N$ with $N \geq 2m+1$, the dimension of $\phi^V_{W,B}(\rho)$ is then equal to
(4.5.12)

$$\frac{dim(\rho) \cdot \prod_{i=N'_\rho+1}^{N}(q^{2i}-1) \cdot \prod_{i=1}^{a}(q^{N'_\rho}+\alpha \cdot q^{\lambda_i}) \cdot \prod_{i=1}^{b}(q^{N'_\rho}-\alpha \cdot q^{\mu_i})}{2 \cdot q^{(a+b-1)(a+b+1)/4} \cdot |\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}}.,$$

which is a polynomial expression applied to q^N.

On the other hand, let us consider the values of $dim(\eta^V_{W,B}(\rho))$ for V of dimension $2N$ as a function of N. We recall the description of endomorphism algebra of $\omega[V \otimes W]$ over $\mathrm{Sp}(V)$ given in Chapter 2: Considering the Schrödinger model of the oscillator representation, there is an isomorphism between the endomorphism algebra and the space of $\mathrm{Sp}(V)$-fixed points in $\mathbb{C}(V \otimes W)$

$$(\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W]), \circ) \cong (\mathbb{C}(V \otimes W)^{\mathrm{Sp}(V)}, \star), \tag{4.5.13}$$

where $\star$ is defined by

$$(v_1 \otimes w_1) \star (v_2 \otimes w_2) = \psi(\frac{S(v_1,v_2) \cdot B(w_1,w_2)}{2}) \cdot (v_1 \otimes w_1 + v_2 \otimes w_2)$$

(here ψ denotes the non-trivial additive character corresponding to $1 \in \mathbb{F}_q^\times$, under our identification of $\mathbb{F}_q$ with its Pontrjagin dual). To consider the eta correspondence $\eta^V_{W,B}$, recall from Chapter 2 that we consider $\omega[V \otimes W]$ as a degree $dim(W)$ tensor product of oscillator representations $\omega_{a_1}[V] \otimes \cdots \otimes \omega_{a_n}[V]$ (considering B to be equivalent to the symmetric bilinear form corresponding to a diagonal matrix with entries $a_1, \ldots, a_n$). This essential corresponds to writing out $V \otimes W$ as a direct sum of n copies of V. Therefore we also view (4.5.13) as describing

$$\mathrm{End}_{\mathrm{Sp}(V)}(\omega_{a_1}[V] \otimes \cdots \otimes \omega_{a_n}[V])). \tag{4.5.14}$$

We note that as long as $N \geq n$, the right hand side of (4.5.13), as an algebra, is stable and does not depend on N. Therefore the same linear combination of n-tuples of V vectors in the right hand side of (4.5.13) describes the idempotent with image $\eta^V_{W,B}(\rho)$ for any choice of $N \geq n$. In particular, the dimension of $\eta^V_{W,B}(\rho)$ (expressible as the trace of this idempotent in (4.5.14) for V of dimension $2N$, is also polynomial in q^N, since, considering one tensor factor at a time, trace of a linear combination of V-vectors (v) as an endomorphism of $\omega_{a_i}[V]$

is computed according to

$$tr((v)) = \begin{cases} 0 \text{ if } v \neq 0 \\ q^N \text{ if } v = 0 \end{cases} .$$

Hence, since this polynomial agrees with the polynomial (4.5.12) for infinitely many values i.e., when applied to q^N for N large enough, they must in fact always agree. Therefore, we obtain (4.5.11) for every $N \geq n$.

Combining this with the results of the previous subsection which confirm that the semisimple and sign parts of the classification data for $\eta^V_{W,B}(\rho)$ and $\phi^V_{W,B}(\rho)$ always match, we obtain that the unipotent parts must match also (since every symbol has a different dimension). Therefore, we obtain that

$$\eta^V_{W,B}(\rho) = \phi^V_{W,B}(\rho),$$

by Lusztig's classification of irreducible representations, as claimed.

□

CHAPTER 5

Interpolated representation theory

While we were able to describe Howe duality in the stable ranges by directly studying endomorphism algebras of representations, these is a rather large gap consisting of the pairs $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ which are in neither of the stable ranges and therefore the methods used so far do not give precise results about them. To treat these cases, we bring in a different method, namely interpolated categories of representations. This theory was discovered by P. Deligne [**7, 8**] and continued by many other authors [**26, 34, 36**]. The basic idea is to start, say with the categories $\mathrm{Rep}(S_t)$ of finite dimensional complex representations of finite symmetric groups S_t. It turns out, however [**7**], that one can give a precise meaning to $\mathrm{Rep}(S_t)$ with t a complex number. This is a part of the theory of tensor categories [**17**]. The kind of categories we obtain here are called (semisimple) pre-Tannakian categories. A category $\mathscr{C}$ being pre-Tannakian means that it is a locally finite, $\mathbb{C}$-linear symmetric tensor category with strong duality such that $\mathrm{End}(1) = \mathbb{C}$.

In our present context, we are interested in categories of the form $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$, $\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))$, for $t \in \mathbb{C}$ [**34, 35**]. A special feature is a category $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ which also incorporates the oscillator representation. This can be interpreted as the category of representations of the "interpolated metaplectic group," which trivializes for $t = n + \frac{\pi i k}{\ln(q)}$ for $n \in \mathbb{N}$, $k \in \mathbb{Z}$. To construct these categories, we will introduce the formalism of T-algebras [**36**].

5.1. Categorical background. The purpose of this subsection is to give some background of the context in which we consider interpolated representation categories. For more information, the reader is refered to [**7, 8, 17**]. To properly capture representation theory, we want to consider $\mathbb{C}$-linear additive categories with an appropriate form of tensor product and duality. The most restrictive form of this context is semisimple *pre-Tannakian categories*, which include the classical categories of representations (over $\mathbb{C}$). However, the methods of construction we discuss in this chapter for interpolated representation categories may, a priori, not yield such categories. We briefly recall the significance of objects with non-integer dimension. We also begin recalling certain constructions, such as the pseudo-abelian envelope and semisimplification, which appear in the construction of interpolated categories and can potentially improve a given category to be closer to semisimple pre-Tannakian.

We will be working with $\mathbb{C}$-linear additive categories. By $\mathbb{C}$-linear, we mean that Hom-sets are $\mathbb{C}$-vector spaces with bilinear composition; by additive, we mean that in addition finite products (equivalently coproducts) exist. The most general categories we are interested in here are $\mathbb{C}$-linear additive categories $\mathscr{C}$, with an associative, commutative, unital (ACU) tensor product $\otimes$ (in the symmetric monoidal category sense) writing 1 for the unit object, and strong duality (or rigidity), which we recall means that for every object X of $\mathscr{C}$, there is a dual object $X^\vee$ with evaluation and coevaluation maps

$$\mathrm{ev}_X : X^\vee \otimes X \to 1$$
$$\mathrm{coev}_X : 1 \to X \otimes X^\vee$$

satisfying the triangle identities, which require that the compositions

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{coev}_X \otimes \mathrm{Id}_X} & X \otimes X^\vee \otimes X & \xrightarrow{\mathrm{Id}_X \otimes \mathrm{ev}_X} & X \\ X^\vee & \xrightarrow{\mathrm{Id}_{X^\vee} \otimes \mathrm{coev}_X} & X^\vee \otimes X \otimes X^\vee & \xrightarrow{\mathrm{ev}_X \otimes \mathrm{Id}_{X^\vee}} & X^\vee \end{array} \tag{5.1.1}$$

are Id_X and $\mathrm{Id}_{X^\vee}$, respectively (we note the implicit use of the associativity of the tensor product in the middle of the compositions (5.1.1)). Since we only consider categories where the tensor product is commutative, we make no distinction between right and left duals (for a description of the following story without assuming commutativity, we refer to Section 2.10 of [**17**]).

For such a category $\mathscr{C}$, we also consider the category of super objects $s\mathscr{C}$ which is that category of $\mathbb{Z}/2$-graded objects in $\mathscr{C}$, with the commutativity isomorphism for tensor product given by

$$x \otimes y \to (-1)^{\deg(x)\deg(y)} y \otimes x.$$

Given such a category, we may construct trace operations on morphisms. For objects $X, Y, Z, \in \mathrm{Obj}(\mathscr{C})$, we recall that strong duality gives natural identifications between the Hom-spaces

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X \otimes Y, Z) &\xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}(Y, X^\vee \otimes Z) \\ f &\longmapsto (\mathrm{Id}_{X^\vee} \otimes f) \circ (\mathrm{ev}_X \otimes \mathrm{Id}_Y) \end{aligned}$$

In particular, we may construct an operation

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(X \otimes Y, X \otimes Z) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathscr{C}}(Y, X^\vee \otimes X \otimes Z) \\ & & \downarrow \qquad f \mapsto (\mathrm{ev}_X \otimes \mathrm{Id}_Z) \circ f \\ & & \mathrm{Hom}_{\mathscr{C}}(Y, Z) \end{array} \tag{5.1.2}$$

which we call the *partial trace* on the X-coordinate. These partial trace operations, in combination with tensor product of morphisms and permutation, recover morphism composition: For objects X, Y, Z of $\mathscr{C}$, the chain of operations

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(X, Y) \otimes \mathrm{Hom}_{\mathscr{C}}(Y, Z) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(X \otimes Y, Y \otimes Z) \\ & \swarrow & \\ \mathrm{Hom}_{\mathscr{C}}(Y \otimes X, Y \otimes Z) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(X, Z) \end{array} \tag{5.1.3}$$

is precisely morphism composition, with the first arrow given by tensor product of morphisms, the second arrow given by composition with the switch morphism $Y \otimes Z \to Z \otimes Y$, and the third arrow given by the partial trace on the Y-coordinate.

If we further assume the natural embedding

$$\mathbb{C} \to \mathrm{End}_{\mathscr{C}}(1) \tag{5.1.4}$$

is an isomorphism, then for any object X of $\mathscr{C}$ and endomorphism $f \in \mathrm{End}_{\mathscr{C}}(X)$, we may consider the composition

$$1 \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{Id}_{X^\vee}} X \otimes X^\vee \xrightarrow{\mathrm{ev}_{X^\vee}} 1 \, .$$

We call the corresponding complex number under (5.1.4) the *trace* of f and denote it by $\mathrm{tr}_{\mathscr{C}}(f)$ (we omit the subscript from the notation when the category is clear). For an object X of $\mathscr{C}$, we call the number

$$\dim_{\mathscr{C}}(X) = \mathrm{tr}(\mathrm{Id}_X) \tag{5.1.5}$$

the *(categorical) dimension* of X in $\mathscr{C}$.

5.1.1. Definition. *A* quasi-pre-Tannakian category *is a category satisfying all the above properties, i.e. a $\mathbb{C}$-linear additive category with an associative commutative unital bilinear tensor product which satisfies strong duality such that (5.1.4) is an isomorphism.*

If $\mathscr{C}$ is an abelian category (i.e. if kernels and cokernels exist, and every monomorphism is a kernel and every epimorphism is a cokernel), we may introduce the concept of the *length* of an object $X \in \mathrm{Obj}(\mathscr{C})$ as the length n of a Jordan-Hölder series

$$0 = X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X \tag{5.1.6}$$

such that every quotient X_i/X_{i-1} is a simple object in $\mathscr{C}$, if such a series exists. (It can be shown for objects X of finite length that the number n does not depend on the choice of the Jordan-Hölder series.)

A $\mathbb{C}$-linear abelian category $\mathscr{C}$ is called *locally finite* if for every two objects $X, Y \in Obj(\mathscr{C})$, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a finite-dimensional $\mathbb{C}$-vector space and if every object of $\mathscr{C}$ has finite length. A locally finite abelian quasi-pre-Tannakian category is called a *pre-Tannakian category*.

Note that none of the above conditions on the category $\mathscr{C}$ require that the dimensions (5.1.5) of the objects of $\mathscr{C}$ must be integers. In a classical context, one more commonly considers categories where this is required. More specifically, one may consider *(neutral) Tannakian categories*, which can be defined as pre-Tannakian categories which admit an exact, faithful functor of $\mathbb{C}$-linear tensor categories

$$\mathscr{C} \to \mathrm{Vect},$$

called a *fiber functor*. This is not the context we primarily will consider in this chapter, and we note that conditions requiring a category's objects to be of integer dimensions can force some unexpected strong properties in the categorical structure.

For example, one way to categorically enforce that the objects have integer dimensions is to demand that $\mathscr{C}$ admits an exact symmetric tensor functor to the category sVect of super vector spaces. It turns out that this condition is equivalent to the cateogry having subexponential growth:

5.1.2. Theorem (P. Deligne, [**8**]). *Over a field of characteristic 0 (e.g. $\mathbb{C}$), a pre-Tannakian category $\mathscr{C}$ admits an exact symmetric tensor functor*

$$\mathscr{C} \to sVect$$

if and only if for every object X in $\mathscr{C}$, the length of $X^{\otimes n}$, as a function of $n \in \mathbb{N}$ is sub-exponential, meaning that there exists a constant $a_X \in \mathbb{R}$ such that

$$length(X^{\otimes n}) \leq a_X^n$$

for every n (in which case, we say $\mathscr{C}$ is of sub-exponential growth*).*

Since interpolated representation categories are obtained, roughly speaking, by replacing the dimension of a basic object by a generally complex number, this theorem in particular implies that pre-Tannakian categories obtained from interpolating the representation theory of a family of groups are (generically) never of sub-exponential growth. This discussion is somewhat tangential to our present purpose of using interpolated representation categories to recover genuine representation information, and we only note it here for completeness.

We do, however, hope to obtain categories which are abelian from interpolation constructions. While this may not always be possible, there are some standard constructions which can we applied to "improve" a given category. First, we may consider the *pseudo-abelian* (or *Karoubian*) *envelope* of a category, which is obtained by introducing objects so that every idempotent in the original category has an image object. (In particular, applying this to a pre-additive category gives a pseudo-abelian category.) While this will be enough to construct a semisimple pre-Tannakian in certain special cases (such as in the construction of the interpolation of the representations of the general linear group), it is not enough in general. There is another construction we apply to a $\mathbb{C}$-linear additive category with associative, commutative, unital tensor product and strong duality called *semisimplification*.

5.2. Semisimplicity and the Jacobson radical. The purpose of this subsection is to describe the *semisimplification* construction and clarify when it outputs a genuinely semisimple category. (For more details, we refer to [**18**] and [**7**], Section 6.1.)

5.2.1. Definition. *In a $\mathbb{C}$-linear additive category $\mathscr{C}$ with associative, commutative, unital tensor product and strong duality, for objects X, Y of $\mathscr{C}$, we say a morphism $f : X \to Y$ is* negligible *when for ever morphism $g : Y \to X$ the trace of the composition of f with g is 0:*

$$tr(f \circ g) = 0.$$

The negligible morphisms of a $\mathbb{C}$-linear additive category $\mathscr{C}$ with associative, commutative, unital tensor product and strong duality form a tensor ideal (meaning that any composition involving a negligible morphism is also negligible, and any tensor product of a negligiblle morphism with another morphism is also negligible). We therefore may define the semisimplification $\mathscr{S}(\mathscr{C})$ by quotienting the negligible morphisms out of each Hom-space, i.e. putting the objects of $\mathscr{S}(\mathscr{C})$ to be the same as the objects of $\mathscr{C}$ and putting, for every objects X, Y, the Hom-space $\mathrm{Hom}_{\mathscr{S}(\mathscr{C})}(X, Y)$ to be the quotient of $Hom_{\mathscr{C}}(X, Y)$ by negligible morphisms. Consider also the resulting semisimplification quotient functor

$$\mathscr{S} : \mathscr{C} \to \mathscr{S}(\mathscr{C}) \tag{5.2.1}$$

(preserving tensor product and trace).

Now, even semisimplification need not necessarily output a semisimple, or even abelian category. The most telling example of this is the "mise en garde" category described in Subsection 5.8 of [**7**], where a $\mathbb{C}$-linear additive category (with associative, commutative, unital tensor product and strong duality) is constructed such that the basic generating object X has an endomorphism $\theta : X \to X$ such that $\theta \circ \theta = 0$, but it has trace $\mathrm{tr}(\theta) = 1$. In any tensor category, the trace of a nilpotent endomorphism must be 0, and therefore the semisimplification quotient functor (5.2.1) cannot land in a tensor category. In particular, semisimplifcation does not even give an abelian category in this case.

This example suggests that specifically nilpotent endomorphisms with non-zero trace are an obstruction to the semisimplicity of the output of semisimplification. In fact, a slight strengthening of this precisely characterizes when semisimplification will be semisimple:

5.2.2. Lemma. *The semisimplification of a $\mathbb{C}$-linear additive category with strong duality and an associative, commutative, unital tensor product generated by a basic object X is semisimple if and only if for every endomorphism $f \in End(X^{\otimes n})$, if the trace of f is non-zero, then for every n, there exists an $m > n$ such that*

$$tr(f^{\circ m}) \neq 0.$$

Proof of Lemma 5.2.2. To prove sufficiency, consider an endomorphism f of some tensor power $X^{\otimes n}$ is non-negligible, i.e. there exists some morphism $g \in \mathrm{End}(X^{\otimes n})$ such that the trace of $f \circ g$ is non-zero. The trace condition then gives that for every n, there exists a $m > n$ such that

$$tr((f \circ g)^{\circ m}) \neq 0,$$

and hence, f is not an element of the Jacobian ideal of the endomorphism algebra $\mathrm{End}(X^{\otimes n})$, and in particular, is not nilpotent. Therefore, the semisimplification of the category is semisimple.

Necessity follows from the general result that in a semisimple $\mathbb{C}$-algebra (e.g. the endomorphism algebra of $X^{\otimes n}$ in the semisimplification), if some general trace operation (i.e. a linear combination of trace on each factor, consider the endomorphism algebra as a product of matrix algebras) is non-zero on an element f, then for every n, there exists an $m > n$ such that the trace operation is non-zero on $f^{\circ m}$. (This follows, for example, by considering the Vanermonde determinant.)

□

5.3. T-algebras and first examples of interpolated representation theories. Now that we have discussed some general categorical context and constructions, we can discuss some concrete examples of interpolated categories. In this subsection, we recall one method of explicit construction of the interpolated representation categories. More specifically, a $\mathbb{C}$-linear additive category $\mathscr{C}$ with a $\mathbb{C}$-bilinear associative, commutative, unital tensor product and strong duality which is generated by a "basic" object X can be axiomatized by a certain universal algebra construction we call a *T-algebra* (see [**37, 36**], following [**7**], Chapter 10). We use this to describe the construction of the original interpolations of the general linear groups and the symmetric groups introduced in [**10, 7, 8**], which we denote by $\mathfrak{Rep}(\mathrm{GL}_c)$ and $\mathfrak{Rep}(S_t)$ in this book (using $\mathfrak{Rep}$ to always distinguish when we are considering a category produced by interpolation).

In rough terms, the data of a T-algebra consists of a system of $\mathbb{C}$-vector spaces captuing the Hom-spaces between tensor powers of the generating object X

$$\mathcal{T}_{S,T} = \mathrm{Hom}_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T}),$$

with enough additional data to capture the category structure of composition and tensor product. Considering (5.1.3), we need not separately define composition, and it suffices to capture the data of "partial trace" operations and the tensor product (with appropriate axioms). We give the formal definition of this structure now:

5.3.1. Definition. *A* T-algebra $\mathcal{T}$ *is a universal algebra structure which consists of the data of vector spaces* $\mathcal{T}_{S,T}$ *corresponding (functorially) to pairs of finite sets* S, T*, along with the data of* partial trace *operations*

$$\boldsymbol{\tau}_\phi : \mathcal{T}_{S,T} \to \mathcal{T}_{S\smallsetminus S', T\smallsetminus T'} \tag{5.3.1}$$

corresponding (functorially) to bijections $\phi : S' \to T'$ *for subsets* $S' \subseteq S$*,* $T' \subseteq T$*, the data of* product *operations*

$$\boldsymbol{\pi} : \mathcal{T}_{S_1,T_1} \otimes \mathcal{T}_{S_2,T_2} \to \mathcal{T}_{S_1 \amalg S_2, T_1 \amalg T_2}$$

for finite sets S_1, S_2, T_1, T_2*, and the data of a product unit* $1 \in \mathcal{T}_{\emptyset,\emptyset}$ *and an "identity" element* $\iota \in \mathcal{T}_{\{1\},\{1\}}$ *satisfying the following axioms:*

(1) Associativity, commutativity, and unitality of products. *For disjoint sets* S_1, S_2, S_3 *and* T_1, T_2, T_3*, the usual diagrams*

$$\begin{array}{ccc}
\mathcal{T}_{S_1,T_1} \otimes \mathcal{T}_{S_2,T_2} \otimes \mathcal{T}_{S_3,T_3} & \xrightarrow{Id_{\mathcal{T}_{S_1,T_1}} \otimes \boldsymbol{\pi}} & \mathcal{T}_{S_1,T_1} \otimes \mathcal{T}_{S_2 \amalg S_3, T_2 \amalg T_3} \\
\downarrow {\scriptstyle \boldsymbol{\pi} \otimes Id_{\mathcal{T}_{S_3,T_3}}} & & \downarrow {\scriptstyle \boldsymbol{\pi}} \\
\mathcal{T}_{S_1 \amalg S_2, T_1 \amalg T_2} \otimes \mathcal{T}_{S_3,T_3} & \xrightarrow[\boldsymbol{\pi}]{} & \mathcal{T}_{S_1 \amalg S_2 \amalg S_3, T_1 \amalg T_2 \amalg T_3}
\end{array}$$

$$\begin{array}{ccc}
\mathcal{T}_{S_1,T_1} \otimes \mathcal{T}_{S_2,T_2} & \xrightarrow{\boldsymbol{\pi}} & \mathcal{T}_{S_1 \amalg S_2, T_1 \amalg T_2} \\
\downarrow {\scriptstyle \sigma} & & \downarrow {\scriptstyle =} \\
\mathcal{T}_{S_2,T_2} \otimes \mathcal{T}_{S_1,T_1} & \xrightarrow[\boldsymbol{\pi}]{} & \mathcal{T}_{S_1 \amalg S_2, T_1 \amalg T_2}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{T}_{S_1,T_1} & \xrightarrow{Id_{\mathcal{T}_{S_1,T_1}} \otimes 1} & \mathcal{T}_{S_1,T_1} \otimes \mathcal{T}_{\emptyset,\emptyset} \\
& \searrow {\scriptstyle =} & \downarrow {\scriptstyle \boldsymbol{\pi}} \\
& & \mathcal{T}_{S_1,T_1}
\end{array}$$

commute (writing σ *for the switch of tensor factors in the second diagram).*

(2) Commutation of trace and products. *For finite sets* $S' \subseteq S_1 \amalg S_2$*,* $T' \subseteq T_1 \amalg T_2$ *and a bijection* $\phi : S' \to T'$*, the following diagram commutes*

$$\begin{array}{ccc}
\mathcal{T}_{S_1,T_1} \otimes \mathcal{T}_{S_2,T_2} & \xrightarrow{\qquad \boldsymbol{\pi} \qquad} & \mathcal{T}_{S_1 \amalg S_2, T_1 \amalg T_2} \\
\downarrow {\scriptstyle \boldsymbol{\tau}_{\phi|_{S_1 \amalg T_1}} \otimes \boldsymbol{\tau}_{\phi|_{S_2 \amalg T_2}}} & & \downarrow {\scriptstyle \boldsymbol{\tau}_\phi} \\
\mathcal{T}_{S_1 \smallsetminus S', T_1 \smallsetminus T'} \otimes \mathcal{T}_{S_2 \smallsetminus S', T_2 \smallsetminus T'} & \xrightarrow[\boldsymbol{\pi}]{} & \mathcal{T}_{(S_1 \amalg S_2) \smallsetminus S', (T_1 \amalg T_2) \smallsetminus T'}
\end{array}$$

(3) Composition with the identity morphism. *For every element* $f \in \mathcal{T}_{\{2\},\{2\}}$*, consider* $f\boldsymbol{\pi}\iota \in \mathcal{T}_{\{1,2\},\{1,2\}}$*. Write* σ *for the switch of* 1 *and* 2 *in* $\Sigma_{\{1,2\}}$*. We then require*

$$\boldsymbol{\tau}_{Id_{\{1\}}}(\sigma(f\boldsymbol{\pi}\iota)) = \boldsymbol{\tau}_{Id_{\{1\}}}(\sigma(\iota\boldsymbol{\pi}f)) = f \in \mathcal{T}_{\{2\},\{2\}}.$$

As we described above, a $\mathbb{C}$-linear additive category $\mathscr{C}$ with associative, commutative, unital tensor product and strong duality, generated by a basic object X, is equivalent to the data of a T-algebra: Given such a category $\mathscr{C}$, we may form its corresponding T-algebra $\mathcal{T}^{\mathscr{C}}$ by putting, for finite sets S, T,

$$\mathcal{T}^{\mathscr{C}}_{S,T} = \operatorname{Hom}_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T})$$

with $\boldsymbol{\tau}$ defined by the partial trace operations (5.1.2) and $\boldsymbol{\pi}$ defined by the tensor product. Given a T-algebra $\mathcal{T}$, we produce the corresponding category $\mathscr{C}[\mathcal{T}]$ as follows: First we construct a category $\mathscr{C}[\mathcal{T}]_0$ with formal objects $X^{\otimes S} \otimes (X^\vee)^{\otimes T}$, putting

$$\begin{aligned} Hom_{\mathscr{C}[\mathcal{T}]_0}(X^{\otimes S_1} \otimes (X^\vee)^{\otimes T_1}, X^{\otimes S_2} \otimes (X^\vee)^{\otimes T_2}) = \\ Hom_{\mathscr{C}[\mathcal{T}]_0}(X^{\otimes S_1 \amalg T_2}, X^{\otimes S_2 \amalg T_1}) = \mathcal{T}_{S_1 \amalg T_2, S_2 \amalg T_1}. \end{aligned}$$

Composition can be defined from partial trace and product, as in (5.1.3). We produce $\mathscr{C}[\mathcal{T}]$ by adding formal direct sums to $\mathscr{C}[\mathcal{T}]_0$ and then taking a pseudo-abelian envelope.

5.3.2. Proposition. *If the semisimplification of the category $\mathscr{C}[\mathcal{T}]$ for a T-algebra $\mathcal{T}$ is semisimple (and thus abelian and pre-Tannakian), then for every sub-T-algebra $\mathcal{T}' \subseteq \mathcal{T}$, the (pseudo-abelian envelope of the) semisimplification of $\mathscr{C}[\mathcal{T}']$ is semisimple pre-Tannakian.*

Proof. This follows by Lemma 5.2.2: Suppose that the semisimplification of $\mathscr{C}[\mathcal{T}']$ is not semisimple. Then the condition given in Lemma 5.2.2 can be stated in the T-algebra language by saying that there exists an element $f \in \mathcal{T}'_{S,S}$ corresponding to an endomorphism of $X^{\otimes S}$ in $\mathscr{C}[\mathcal{T}']$ (denoting the basic object of $\mathscr{C}[\mathcal{T}']$ by X), such that the trace

$$\boldsymbol{\tau}_{\mathrm{Id}_S}(f) \neq 0 \tag{5.3.2}$$

and for every $m > |S|$

$$\boldsymbol{\tau}_{\mathrm{Id}_S}(f^{\circ m}) = 0. \tag{5.3.3}$$

If $\mathcal{T}'$ is a sub-T-algebra of $\mathcal{T}$, then both (5.3.2) and (5.3.3) also remain true in $\mathcal{T}$, preventing $\mathscr{C}[\mathcal{T}]$ from being semisimple. □

The definition of T-algebra (and the associated categories) is based on P. Deligne's construction of the category of representations of a general linear group of non-integer rank, which we denote by $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{C}))$ for $t \in \mathbb{C} \smallsetminus \mathbb{Z}$, and which can be considered the first (and, in fact, universal) example of an "inteprolated representation theory" (see [**7, 10**]). First, let us write X_N for the standard N-dimensional representation of $\mathrm{GL}_N(\mathbb{C})$. We note that X_N tensor generates the category $\mathrm{Rep}(\mathrm{GL}_N(\mathbb{C}))$ (meaning that every object can of the category can be obtained as a subquotient of a finite direct sum of tensor powers of X_N). In essence, this means that the categorical structure of $\mathrm{Rep}(\mathrm{GL}_N(\mathbb{C}))$ is fully captured in the structure of the Hom-spaces between tensor powers of X_N. We also note that for N very large compared to the tensor degrees, these structures are, as algebras (in the case of endomorphisms) and modules (in the case of general Hom-spaces), stable.

Recall that

$$\mathrm{End}_{\mathrm{GL}_N(\mathbb{C})}(X_N^{\otimes n}) \cong \mathbb{C}\Sigma_n \tag{5.3.4}$$

for every $N >> n$. We then form the category $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{C}))$ so that it is tensor-generated by a basic object X_t of dimension t. In other words, we consider the T-algebra $\mathcal{T}^{\mathrm{GL}_t(\mathbb{C})}$ defined by putting, for finite sets S, T,

$$\mathcal{T}_{S,T}^{\mathrm{GL}_t(\mathbb{C})} = \mathbb{C}\{f : S \to T \text{ bijective}\}$$

(so it $|S| \neq |T|$, we have $\mathcal{T}_{S,T}^{\mathrm{GL}_t(\mathbb{C})} = 0$). We define product by disjoint union on bijections, taking the unit and "identity" elements to be

$$1 = \mathrm{Id}_\emptyset \in \mathcal{T}_{\emptyset,\emptyset}^{\mathrm{GL}_t(\mathbb{C})}, \quad \iota = \mathrm{Id}_{\{1\}} \in \mathcal{T}_{\{1\},\{1\}}^{\mathrm{GL}_t(\mathbb{C})}.$$

In the case of $|S| = |T|$, let us diagrammatically represent a bijection $S \to T$ by two rows of dots corresponding to the elements of S and T, and connecting the dots in S to the dots representing their images in T under the bijection. We use the convention of drawing the row corresponding to S above the row corresponding to T. Partial trace can then be defined diagrammatically, by additionally drawing lines between the dots corresponding to the coordinates matched during the partial trace, then composing where possible, removing loops and multiplying the resulting bijection by t to the power of the number of loops. In formal terms, the only non-zero case of a partial trace operation is for sets S, T with $|S| = |T|$ and a bijection $\phi : S' \to T'$ between subsets $S' \subseteq S$, $T' \subseteq T$, in which case we define

$$\boldsymbol{\tau}_\phi : \mathcal{T}_{S,T}^{\mathrm{GL}_t(\mathbb{C})} \to \mathcal{T}_{S\smallsetminus S', T\smallsetminus T'}^{\mathrm{GL}_t(\mathbb{C})}$$

by sending a bijection $f : S \to T$ to the bijection $S \smallsetminus S' \to T \smallsetminus T'$ defined by sending an element $x \in S \smallsetminus S'$ to its image under f, followed by as many applications of $f|_{S'} \circ \sigma^{-1}$ as needed to give an output in $T \smallsetminus T'$, multiplied by the $\mathbb{C}$-coefficient of t to the power of the number of equivalence classes

$$|\{x \in S' \mid \exists n \text{ such that } (f|_{S'} \circ \sigma^{-1})^{\circ n}(x) = x\}/\sim|$$

putting $x \sim y$ when there exists an m such that $(f|_{S'} \circ \sigma^{-1})^{\circ m}(x) = y$. This is the only consistent choice of partial trace operation so that the categorical dimension of the basic object is

$$\boldsymbol{\tau}_{\mathrm{Id}_{\{1\}}}(\iota) = \boldsymbol{\tau}_{\mathrm{Id}_{\{1\}}}(\mathrm{Id}_{\{1\}}) = t.$$

We shall write $\mathfrak{Rep}(GL_t(\mathbb{C}))$ for the category $\mathscr{C}[\mathcal{T}^{\mathrm{GL}_t}]$ obtained by taking the pseudo-abelian envelope of the category diagrammatic category corresponding to $\mathcal{T}^{\mathrm{GL}_t}$.

5.3.3. THEOREM (P. Deligne, [**7**]). *For $t \in \mathbb{C} \smallsetminus \mathbb{Z}$, the category $\mathfrak{Rep}(GL_t(\mathbb{C}))$ is a semisimple pre-Tannakian category. For $t = n \in \mathbb{Z}$, the semisimplification of the category $\mathfrak{Rep}(GL_{t=n}(\mathbb{C}))$ is semisimple, and is equivalent to the classical category of representations*

$$\mathscr{S}(\mathfrak{Rep}(GL_{t=n}(\mathbb{C}))) = Rep(GL_n(\mathbb{C})). \tag{5.3.5}$$

In fact, though we will not make use of this result in our study of oscillator representations, we also note that the data of a T-algebra with basic object of trace $t \in \mathbb{C}$ is equivalent to the data of a $\mathfrak{Rep}(GL_t)$-algebra.

Another classical interpolated representation category introduced in [**7**] is the representations of the symmetric group S_t for non-integer values of $t \in \mathbb{C}$. We denote these categories by $\mathfrak{Rep}(S_t)$. We define the T-algebra $\mathcal{T}^{S_t}$ by putting, for finite sets S, T,

$$\mathcal{T}^{S_t}_{S,T} = \mathbb{C}\{\{U_1, \dots, U_n\} \mid \emptyset \neq U_i \text{ disjoint, } S \amalg T = U_1 \amalg \cdots \amalg U_n\}$$

(alternatively, $\mathcal{T}^{S_t}_{S,T}$ can be described as the free $\mathbb{C}$-vector space generated by equivalence relations on $S \amalg T$). Take unit and identity elements to be

$$1 = \emptyset \in \mathcal{T}^{S_t}_{\emptyset,\emptyset}, \quad \iota = \{\{1_1, 1_2\}\} \in \mathcal{T}^{S_t}_{\{1_1\},\{1_2\}}.$$

For finite sets S, T and a bijection $\phi : S' \to T'$ for subsets $S' \subseteq S$, $T' \subseteq T$, we define the partial trace $\boldsymbol{\tau}_\phi$ as follows. For a choice of $\{U_1, \dots, U_n\}$, consider the sets

$$U_i \smallsetminus (S' \amalg T') \tag{5.3.6}$$

We take $\boldsymbol{\tau}_\phi(\{U_1, \dots, U_n\})$ to be the partition of $(S \amalg T) \smallsetminus (S' \amalg T')$ consisting of the sets (5.3.6) (disregarding empty ones), with $\mathbb{C}$-coefficient

$$\prod_{i=n-k+1}^{n} (t - (i-1)), \tag{5.3.7}$$

where k is the number of the sets (5.3.6) which are empty.

To define the product $\boldsymbol{\pi}$ of a partitions $\{U_1, \dots, U_n\} \in \mathcal{T}^{S_t}_{S,T}$ and $\{U'_1, \dots, U'_m\} \in \mathcal{T}^{S_t}_{S',T'}$ is defined as the sum, indexed by "gluing" surjective maps on the sets of indices of the form

$$f : \{1, \dots, m\} \amalg \{1, \dots, m'\} \twoheadrightarrow \{1, \dots, \ell\}$$

(for $\ell \leq m + m'$) whose restriction to $\{1, \dots, m\}$ and restriction to $\{1, \dots, m'\}$ are both injective, of the associated partitions of $S \amalg S' \amalg T \amalg T'$ with ℓ components with the component corresponding to $i \in \{1, \dots, \ell\}$ given by the disjoint union of U_j if $f(j) = i$ and U'_k if $f(k) = i$ (at least one of which must exist by the surjectiveness of f).

5.3.4. THEOREM (P. Deligne, [**7**]). *For $t \in \mathbb{C} \setminus \mathbb{N}$, the category $\mathfrak{Rep}(S_t)$ is a semisimple pre-Tannakian category. For $t = n \in \mathbb{N}$, the semisimplification of the category $\mathfrak{Rep}(S_{t=n})$ is semisimple, and is equivalent to the classical category of representations Rep(S_n).*

5.4. The standard interpolation categories for algebraic groups over a finite field. Now that we have seen some first constructions of interpolated representation categories, the purpose of this subsection is to recall a few more examples of interpolated categories which are closer to the context: specifically, the category $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ of the representations of general linear groups over $\mathbb{F}_q$, $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ of representations of the symplectic groups over $\mathbb{F}_q$, and $\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))$ of representations of orthogonal groups over $\mathbb{F}_q$, originally due to Knop [**34, 35**]. All of these interpolated models use the vector representation as the basic standard object. We call these the "standard" interpoaltions.

It actually turns out that this interpolation of the representations of the symplectic group is not suitable for our study of the oscillator representation. In fact, there will be no object of dimension q^t corresponding to the oscillator representation in $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$. We will instead need a "finer" interpolation, which we denote by $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$, of the representations of the finite symplectic groups, as a category generated by the oscillator representation, which we define in the next subsection.

We again discuss the construction of the categories $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$, $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$, and $\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))$ by describing their corresponding T-algebra.

First we note that, for these particular categories, it is possible to describe these T-algebras by taking them to correspond to the structure of the *Hom*-spaces over the corresponding finite group of Lie type in high enough rank.

The standard interpolation $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ is defined to be tensor-generated by a basic "standard representation" of dimension q^t written as $\mathbb{C}\mathbb{F}_q^t$ (corresponding to the standard representation $\mathbb{C}\mathbb{F}_q^N$ of $\mathrm{GL}_N(\mathbb{F}_q)$). The category satisfies

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))}((\mathbb{C}\mathbb{F}_q^t)^{\otimes m}, (\mathbb{C}\mathbb{F}_q^t)^{\otimes n}) = \\ \mathrm{Hom}_{\mathrm{Rep}(\mathrm{GL}_N(\mathbb{F}_q))}((\mathbb{C}\mathbb{F}_q^N)^{\otimes m}, (\mathbb{C}\mathbb{F}_q^N)^{\otimes n}), \end{aligned} \tag{5.4.1}$$

for a large enough $N >> m, n$.

Write V_N for the $2N$-dimensional underlying symplectic $\mathbb{F}_q$-space of the group $\mathrm{Sp}(V_N) = \mathrm{Sp}_{2N}(\mathbb{F}_q)$. The standard interolated category

$\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ is defined to be tensor-generated by a basic "vector representation" object $\mathbb{C}V_t$ of dimension q^{2t} (corresponding to the vector representation $\mathbb{C}V_N$ of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$). The category satisfies

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))}((\mathbb{C}V_t)^{\otimes m}, (\mathbb{C}V_t)^{\otimes n}) =\\ \mathrm{Hom}_{\mathrm{Rep}(\mathrm{Sp}_{2N}(\mathbb{F}_q))}((\mathbb{C}V_N)^{\otimes m}, (\mathbb{C}V_N)^{\otimes n}),\end{aligned} \tag{5.4.2}$$

for a large enough $N >> m, n$. However, recalling Proposition 1.3.3, we note that no idempotent corresponding to an oscillator representation summand $\omega_a[V_N]^{\pm}$ ever appears in any Hom-space on the right hand side of (5.4.2) for any possible N. In particular, there are no objects of $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ corresponding to the simple summands of an oscillator representation. In particular, though the semisimplification of $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$

Finally, we may also define a (signed) standard interpolated category of orthogonal group representations $\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))$ with basic object corresponding to the signed standard representation

$$\mathbb{C}W^- = \mathbb{C}W \otimes \epsilon(\det) \tag{5.4.3}$$

of an orthogonal group $\mathrm{O}(W, B)$, writing $\epsilon(\det)$ for the sign representation on the center $\mathbb{Z}/2$ corresponding to det. Denoting the basic object $\mathbb{C}W_t^-$ of $\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))$, we yet again define the category so that the Hom-spaces between its tensor powers to be equal to what they would be in an orthogonal group of high enough rank (which may be chosen to be even or odd)

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))}((\mathbb{C}W_t^-)^{\otimes m}, (\mathbb{C}W_t^-)^{\otimes n}) =\\ \mathrm{Hom}_{\mathrm{Rep}(\mathrm{O}(W,B))}((\mathbb{C}W^-)^{\otimes m}, (\mathbb{C}W^-)^{\otimes n}),\end{aligned} \tag{5.4.4}$$

for $\dim(W) >> n, m$, as in (5.4.2), (5.4.4). Unlike in the symplectic case, modelling the (twisted) permutation representation as the basic generating object in $\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))$ will be enough for the purpose of considering Howe duality, recalling that the restriction of an oscillator representation $\omega[V \otimes W]$, when restricted to the $\mathrm{O}(W, B)$ component of the reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is isomorphic to

$$\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[V \otimes W]) \cong (\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]))^{\otimes N},$$

considering $\mathbb{F}_q^2$ as the symplectic space with the standard symplectic form, and we further have

$$\mathrm{Res}_{\mathrm{O}(W,B)}(\omega[\mathbb{F}_q^2 \otimes W]) \cong \mathbb{C}W^-$$

(which can be seen since by considering first the restriction of $\omega[\mathbb{F}_q^2 \otimes W]$ to $\mathrm{GL}(W)$, since W plays the role of the Lagrangian in $\omega[\mathbb{F}_q^2 \otimes W]$'s Schrödinger model).

We also note that the construction (5.4.3) can also be done for GL_t. This produces again the category $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$. The point is that the signed action of $\mathrm{GL}_n(\mathbb{F}_q)$ for $n >> 0$ does not affect the sign of permutation of tensor factors. In particular, the sign does not lead to odd objects in $s\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$, whose dimension would have the opposite sign.

Now, both $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ and $\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))$ are known to, after semisimplification when needed at natural number values of t, give semisimple categories. For examples of the methods that can be used to prove this genre of result, see [**7, 8, 36**]. The semisimplification of the standard interpolation category $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ of the symplectic group at $t = N$ is known to give a semisimple pre-Tannakian category. However, it is not equivalent to the genuine representation category $\mathrm{Rep}(\mathrm{Sp}_{2N}(\mathbb{F}_q))$ (since, for example, there is again no object of dimension q^N in the semisimplification $\mathscr{S}(\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q)))$ corresponding to the oscillator representation).

While (5.4.1), (5.4.2), and (5.4.4) can be used to define T-algebras (after checking that the partial trace operations, tensor products, and permutation action on tensor factor coordinates are also stable under increasing rank), it is also instructive to also give more combinatorial descriptions of the models of $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$, $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$, and $\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))$, for example to more clearly connect with our construction of $\mathfrak{Rep}(S_t)$ in Subsection 5.3.

We describe the category $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$.Let us write X for the basic object of $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ of dimension q^t (interpolating $\mathbb{C}\mathbb{F}_q^N$). The T-algebra $\mathcal{T}^{\mathrm{GL}_t(\mathbb{F}_q)}$ corresponding to $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ generated by X can be combinatorially described by identifying for finite sets S, T the vector space

$$\begin{aligned} \mathcal{T}_{S,T}^{\mathrm{GL}_t(\mathbb{F}_q)} &= \mathrm{Hom}(X^{\otimes S}, X^{\otimes T}) \\ &= \mathrm{Hom}_{\mathrm{Rep}(\mathrm{GL}_N(\mathbb{F}_q))}((\mathbb{C}\mathbb{F}_q^N)^{\otimes S}, (\mathbb{C}\mathbb{F}_q^N)^{\otimes T}) \quad \text{for } N >> |S|, |T|, \end{aligned} \tag{5.4.5}$$

with the free $\mathbb{C}$-vector space generated by equivalence classes of quotients

$$f : \mathbb{F}_q^{S \amalg T} = \mathbb{F}_q\{e_i \mid i \in S \amalg T\} \twoheadrightarrow V \tag{5.4.6}$$

for any $\mathbb{F}_q$-spaces V with the equivalence relation that f is equivalent to any composition of f with an isomorphism of the target.

Let us first describe partial trace operations. For a bijection $\phi : S' \to T'$, we must describe $\boldsymbol{\tau}_\phi(f)$ for a quotient map f as in (5.4.6). If there exists an $i \in S'$ such that $f(e_i) \neq f(e_{\phi(i)})$, then we put $\boldsymbol{\tau}_\phi(f) = 0$.

Suppose now for every $i \in S'$

$$f(e_i) = f(e_{\phi(i)})$$

holds. Take $\boldsymbol{\tau}_\phi(f)$ to be a multiple of the restriction of $f|_{\mathbb{F}_q^{(S \amalg T) \setminus (S' \amalg T')}}$, where the coefficient is determined by the difference of dimensions

$$k = dim(V) - dim(Im(f|_{\mathbb{F}_q^{(S \amalg T) \setminus (S' \amalg T')}})),$$

by being 1 if $k = 0$, and

$$\prod_{i=\dim(V)-k+1}^{\dim(V)} (q^t - q^{i-1}) \tag{5.4.7}$$

if $k \neq 0$. We note that this formula for general t matches the formula obtained by polynomially interpolating (in q^t) the respective formulas for $N >> 0$. Note also that (5.4.7) can be interpreted as a q-version of the formula (5.3.7).

Again, like in the definition of $\mathfrak{Rep}(S_t)$ the product $\boldsymbol{\pi}$ is described according to certain "gluings" of the target vector space. Appling $\boldsymbol{\pi}$ to a pair of quotient maps

$$(f : \mathbb{F}_q^{S \amalg T} \twoheadrightarrow V) \in \mathcal{T}_{S,T}^{\mathrm{GL}_t(\mathbb{F}_q)}$$
$$(f' : \mathbb{F}_q^{S' \amalg T'} \twoheadrightarrow V') \in \mathcal{T}_{S',T'}^{\mathrm{GL}_t(\mathbb{F}_q)}$$

gives the sum indexed by "gluing" surjections $V \oplus V' \twoheadrightarrow W$ for some $\mathbb{F}_q$-vector space W, which are injective on each direct summand, of the quotient maps given by the composition of $f \oplus f'$ with the corresponding gluing surjection.

The unit $1 = \mathrm{Id}_1$ is defined by the isomorphism $\mathbb{F}_q^{\emptyset} \to \mathbb{F}_q^{\emptyset}$ and $\iota = \mathrm{Id}_X$ is defined by the quotient

$$\mathbb{F}_q \oplus \mathbb{F}_q \twoheadrightarrow \mathbb{F}_q$$

which sends the basis elements $(1,0)$ and $(0,1)$ of $\mathbb{F}_q \oplus \mathbb{F}_q$ to $1 \in \mathbb{F}_q$.

The T-algebras corresponding to $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ and $\mathfrak{Rep}(\mathrm{O}_t(\mathbb{F}_q))$ also have combinatorial descriptions of their corresponding T-algebras $\mathcal{T}^{\mathrm{Sp}_{2t}(\mathbb{F}_q)}$ and $\mathcal{T}^{\mathrm{O}_t(\mathbb{F}_q)}$. Specifically, to construct $\mathcal{T}^{\mathrm{Sp}_{2t}(\mathbb{F}_q))}$ (resp. $\mathcal{T}^{\mathrm{O}_t(\mathbb{F}_q)}$), the vector space corresponding to a pair of finite sets S, T can be taken to be the free $\mathbb{C}$-vector space generated by the data of a quotient (5.4.6) and a (not necessarily non-degenerate) antisymmetric (resp. symmetric) form on the target V, up to composition with a form-preserving isomorphism. When taking partial trace, we retrict the forms on the target. For example, in, say, $\mathcal{T}_{S,T}^{\mathrm{Sp}_{2t}(\mathbb{F}_q)}$, consider the partial trace of the element corresponding to a quotient $f : \mathbb{F}_q^{S \amalg T} \twoheadrightarrow V$ and a form

A on V, along a bijection $\phi : S' \to T'$ of subsets $S' \subseteq S$, $T' \subseteq T$. Again, we take the partial trace to be 0 if there exists an $i \in S'$ such that $f(e_i) \neq f(e_{\phi(i)})$, and otherwise, define the partial trace to be the generator corresponding to the restriction $f|_{\mathbb{F}_q^{(S \amalg T) \smallsetminus (S' \amalg T')}}$ and the restriction of A to a form on the image of $f|_{\mathbb{F}_q^{(S \amalg T) \smallsetminus (S' \amalg T')}}$, multiplied by a certain coefficient polynomial in q^t replacing (5.4.7) for the corresponding formula for symplectic groups.

5.5. The interpolated metaplectic group and oscillator representation. Finally, in this subsection, we define the category

$$\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$$

obtained by interpolating the oscillator representation. This category can in fact be considered as the interpolated *metaplectic group* representation category over $\mathbb{F}_q$, capturing an interesting effect where the cocycle representing the oscillator representation as a projective representation splits in every category $\mathrm{Rep}(\mathrm{Sp}_{2N}(\mathbb{F}_q))$, but does *not* in the interpolated setting. We also define the interpolation of the category $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q) \rtimes \mathbb{H}_t(\mathbb{F}_q)$ generated by the Weil-Shale representation in this subsection. We prove the semisimplicity of these categories at generic values of t and the semisimplicity of the semisimplification of these categories at values $t = n + \frac{\pi i k}{\ln(q)}$ for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ (i.e. where $q^t = \pm q^n$).

5.5.1. *Statement of the construction of the T-algebra for the category* $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$. Recall that, for every N, we can express the standard representation as

$$\mathbb{C}V_N = \omega_a[V_N] \otimes \omega_a^\vee[V_N]$$

for any choice of $a \in \mathbb{F}_q^\times$. The description of the algebra structure $\star$ given in the remark at the beginning of this section allows us to see this structure, since we may consider $\overline{\omega}_a \otimes \overline{\omega}_a^\vee \cong \mathrm{End}_{\mathbb{C}}(\overline{\omega}_A)$, and

$$(\mathrm{End}_{\mathbb{C}}(\overline{\omega}_A), \circ) \cong (\mathbb{C}V_N, \star).$$

Fix a finite field $\mathbb{F}_q$ and fix $a, b \in \mathbb{F}_q^\times$ so that $b \neq a \in \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$. We define the vector spaces for the T-algebra $\mathcal{T}^{\mathrm{Osc}}$ by putting, for finite sets S, T,

$$\mathcal{T}^{\mathrm{Osc}}_{S,T} = \mathrm{Hom}_{\mathrm{Sp}(V_N)}((\omega_a[V_N] \oplus \omega_b[V_N])^{\otimes S}, (\omega_a[V_N] \oplus \omega_b[V_N])^{\otimes T}) \tag{5.5.1}$$

for $N >> |S|, |T|$ (the space being again stable as N grows). Recalling the duality of the oscillator representations, we may also express the vector space (5.5.1) as

$$\mathrm{Hom}_{\mathrm{Sp}(V_N)}(1, (\omega_a[V_N] \oplus \omega_b[V_N])^{\otimes T} \otimes (\omega_{-a}[V_N] \oplus \omega_{-b}[V_N])^{\otimes S}).$$

There are clear candidates for the action of product and trace, arising from tensor product and trace on $\mathrm{Rep}(\mathrm{Sp}(V_N))$ which (since they have a combinatorial description as in the construction of $\mathcal{T}^{\mathrm{GL}_t}(\mathbb{F}_q)$ given in Subsection 5.4) are independent of N and can be therefore used to define partial trace and product operations on $\mathcal{T}^{\mathrm{Osc}}_{S,T}$. For example, to describe the trace

$$\boldsymbol{\tau}_{\mathrm{Id}_{\{1\}}} : \mathcal{T}^{\mathrm{Osc}}_{\{1\},\{1\}} \to \mathbb{C},$$

(which we also note should determine all the partial trace operations by the commutation of trace and product axiom of a T-algebra) let us use express the vector space $\mathcal{T}^{\mathrm{Osc}}_{\{1\},\{1\}}$ as the Hom-space in the representations of $\mathrm{Sp}(V_N)$ from the trivial representation to the tensor product of $\omega_a[V_N] \oplus \omega_b[V_N]$ with its dual:

$$\mathcal{T}^{\mathrm{Osc}}_{\{1\},\{1\}} = \mathrm{End}_{\mathrm{Sp}(V_N)}(\omega_a[V_N] \oplus \omega_b[V_N]) = \\ \mathrm{Hom}_{\mathrm{Sp}(V_N)}(1, (\omega_a[V_N] \oplus \omega_b[V_N]) \otimes (\omega_{-a}[V_N] \oplus \omega_{-b}[V_N])).$$

Multiplying out the tensor product, the target of morphisms in this Hom-space can be reduced to

$$(\mathbb{C}V_N)^{\oplus 2} \oplus (\omega_a[V_N] \otimes \omega_{-b}[V_N]) \oplus (\omega_b[V_N] \otimes \omega_{-a}[V_N]).$$

The copies of $\mathbb{C}V_N$ are obtained by

$$\mathbb{C}V_N = \omega_a[V_N] \otimes \omega_{-a}[V_N], \quad \mathbb{C}V_N = \omega_b[V_N] \otimes \omega_{-b}[V_N], \tag{5.5.2}$$

while the latter two two summands have no copies of 1 by our assumption that $a \neq b$ in $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$, since $\omega_a[V_N]$ and $\omega_{-b}[V_N]$ are then not each other's dual (and contain no dual summands). Therefore, we in fact have

$$\mathcal{T}^{\mathrm{Osc}}_{\{1\},\{1\}} = \mathrm{Hom}_{\mathrm{Sp}(V_N)}(1, (\mathbb{C}V_N)^{\oplus 2}) = \\ (\mathbb{C}V_N \oplus \mathbb{C}V_N)^{\mathrm{Sp}(V_N)} \subseteq \mathbb{C}V_N \oplus \mathbb{C}V_N \tag{5.5.3}$$

We define $\boldsymbol{\tau}_{\mathrm{Id}_{\{1\}}}$ by taking it to be defined as

$$\boldsymbol{\tau}_{\mathrm{Id}_{\{1\}}}(0) = q^t, \text{ and } \boldsymbol{\tau}_{\mathrm{Id}_{\{1\}}}(v) = 0 \text{ for } v \neq 0 \in V_N$$

on each summand $\mathbb{C}V_N$ in (5.5.3).

In principle, this and the consistency of partial trace with products and permutations on S and T should fully determine general partial traces. In fact, we shall find the following closed formula for a partial trace $\boldsymbol{\tau}_{i\mapsto j}$ matching the ith and jth coordinates of an element

$$(v_1, \dots, v_k) \in \mathbb{C}V_N^k = \mathrm{End}(\omega_a[V_N]^{\otimes k}) \subset \mathcal{T}^{\mathrm{Osc}}_{[k],[k]}$$

(for $N >> 0$), writing $[k] = \{1, \dots, k\}$:

$$\begin{gathered}\boldsymbol{\tau}_{i\mapsto j}((v_1, \dots, v_k)) = \\ \psi_a(\tfrac{S(v_i,v_j)}{2}) \cdot (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_i + v_j, v_{j+1}, \dots, v_k)\end{gathered} \tag{5.5.4}$$

considered in

$$\mathbb{C}V_N^k = \mathrm{End}(\omega_a[V_N]^{\otimes k-1}) \subset \mathcal{T}^{\mathrm{Osc}}_{[k]\smallsetminus\{i\},[k]\smallsetminus\{i\}} \cong \mathcal{T}^{\mathrm{Osc}}_{[k]\smallsetminus\{i\},[k]\smallsetminus\{j\}},$$

according to the identification of $[k] \smallsetminus \{i\}$ with $[k] \smallsetminus \{j\}$ obtained by swapping i and j. Consistency with permuations requires that (5.5.4) can be obtained as

$$\boldsymbol{\tau}_{i\mapsto j}((v_1, \dots, v_k)) = \boldsymbol{\tau}_{\mathrm{Id}_{\{i\}}}(\sigma \circ (v_1, \dots, v_k)) \tag{5.5.5}$$

where σ denotes the action of the switch of the i and j coordinates on $\mathcal{T}^{\mathrm{Osc}}_{[k]}$.

5.5.2. *Details on the switch operation.* To prove (5.5.4), we need to discuss the switch operation in more detail. The functoriality of the vector spaces $\mathcal{T}^{\mathrm{Osc}}_{S,T}$ under bijections of the sets S and T (i.e. the action of permutations in the product of symmetric groups $\Sigma_S \times \Sigma_T$) is more subtle than in any of the previous constructions.We therefore now explicitly describe the action of permutations on the vector space (5.5.1) and prove consistency (and independence of N) using the $\star$ operation.

Specifically, by taking product with the unit and composing (according to $\star$) it is enough to define the switch

$$\sigma : \mathcal{T}^{\mathrm{Osc}}_{\{1,2\},\{1,2\}} \to \mathcal{T}^{\mathrm{Osc}}_{\{1,2\},\{1,2\}}.$$

Rewrite (for $N > 2$)

$$\begin{gathered}\mathcal{T}^{\mathrm{Osc}}_{\{1,2\},\{1,2\}} = \mathrm{End}_{\mathrm{Sp}(V_N)}((\omega_a[V_N] \oplus \omega_b[V_N])^{\otimes 2}) = \\ = \mathrm{End}_{\mathrm{Sp}(V_N)}((\overline{\omega}_a)^{\otimes 2} \oplus (\overline{\omega}_b)^{\otimes 2} \oplus (\overline{\omega}_b \otimes \overline{\omega}_a) \oplus (\overline{\omega}_a \otimes \overline{\omega}_b)).\end{gathered} \tag{5.5.6}$$

Since the switch operation is $\mathrm{Sp}(V_N)$-equivariant, it is composition $\circ s$ with an element

$$s \in \mathrm{End}_{\mathrm{Sp}(V_N)}((\overline{\omega}_a)^{\otimes 2} \oplus (\overline{\omega}_b)^{\otimes 2} \oplus (\overline{\omega}_b \otimes \overline{\omega}_a) \oplus (\overline{\omega}_a \otimes \overline{\omega}_b)).$$

This element is the sum of components in

$$\mathrm{End}_{\mathrm{Sp}(V_N)}(\overline{\omega}_a^{\otimes 2}),\ \mathrm{End}_{\mathrm{Sp}(V_N)}(\overline{\omega}_b^{\otimes 2}), \tag{5.5.7}$$

and

$$\mathrm{Hom}_{\mathrm{Sp}(V_N)}((\overline{\omega}_b \otimes \overline{\omega}_a) \oplus (\overline{\omega}_a \otimes \overline{\omega}_b)) \tag{5.5.8}$$

all of which are isomorphic to the $\mathrm{Sp}(V_N)$-fixed points of $\mathbb{C}(V_N \oplus V_N)$.

5.5.3. Lemma. *On each of copy of $\mathbb{C}(V_N \oplus V_N)$ in (5.5.6) appearing as one of the summands in (5.5.7) or (5.5.8), the switch σ is defined by $\star$-composition with s, where $s \in \mathbb{C}(V_N \oplus V_N)$ denotes the "switch element"*

$$s = \frac{1}{q^t} \sum_{v \in V_N} (v, -v). \tag{5.5.9}$$

Proof. We must verify that at $t = N$, the element (5.5.9) acts as the switch in $\mathrm{End}_{\mathrm{Sp}(V_N)}(\omega_a[V_N]^{\otimes 2})$. The other components are similar. Explicitly, Lemma 1.2.1 identifies $\mathbb{C}(V_N \oplus V_N)$ with the vector space endomorphisms of the oscillator representation

$$\mathbb{C}(V_N \oplus V_N) \cong \mathrm{End}(\omega_a[V_N] \otimes \omega_a[V_N]). \tag{5.5.10}$$

Now, as representations of

$$\mathrm{Sp}(V_N) \subset \mathrm{Sp}(V_N) \times \mathrm{Sp}(V_N) \subset \mathrm{Sp}(V_N \oplus V_N),$$

$\omega_a[V_N] \otimes \omega_a[V_N]$ is isomorphic to the oscillator representation $\omega_a[V_N \oplus V_N]$. Viewed as an element acting on the oscillator representation $\omega_a[V_N \oplus V_N]$, considering $s \in \mathrm{Sp}(V_N \oplus V_N)$, we have that s is order 2 and commutes with switching the two copies of V_N.

In particular, this implies that s is a scalar multiple of the switch of coordinates

$$g : V_N \oplus V_N \to V_N \oplus V_N$$

i.e. the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

In fact, since s is an involution, we have that

$$s = (\pm 1) \cdot g. \tag{5.5.11}$$

To determine how g acts on the level of $\mathbb{C}(V_N \oplus V_N)$, since we are studying the oscillator representation, restricted along the diagonal embedding

$$V_N \subset V_N \oplus V_N,$$

we need to consider the alternative orthogonal decomposition $V_N \oplus V_N = \Delta_{V_N} \oplus \Delta^-_{V_N}$, into the diagonal and codiagonal

$$\Delta_{V_N} = \{(v, v) \mid v \in V_N\}$$
$$\Delta^-_{V_N} = \{(v, -v) \mid v \in V_N\}.$$

Changing bases, therefore, s is expressed as the matrix

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

It remains to understand the composition action of $-I \in \mathrm{Sp}(V_N)$ (using $\Delta^-_{V_N} \cong V_N$) on $\omega_a[V_N]$ as the element of $\mathbb{C}V_N$. Recalling the Schrödinger model of $\omega_a[V_N]$ as, for a decomposition into Lagrangians $V_N = V_N^+ \oplus V_N^-$, the space of functions

$$f : V_N^+ \to \mathbb{C}$$

(on which $(v_+, v_-) \in V_N^+ \oplus V_N^-$ sends f to

$$((v_+, v_-)[f])(x) = \psi_a(S(v_-, x))f(x - v_+).)$$

Therefore, it is clear that $-I$ acts by

$$-I[f](x) = \epsilon(-1)^N f(-x).$$

Finally, for $\mathbf{1}_x$ the character function of $x \in \Lambda$,

$$\sum_{(\ell,\ell')\in V_N} \mathbf{1}_x = q^n \cdot \mathbf{1}_{-x}$$

Identifying the character function of a vector with the vector itself, and using $\mathbb{C}V_N$, we see that $-I \in \mathrm{Sp}(V_N)$ corresponds to the element

$$\frac{\epsilon(-1)^N}{q^N} \cdot \sum_{v\in V_N} v \in \mathbb{C}V_N.$$

Therefore, as an element of $\mathbb{C}(V_N \oplus V_N)$, g is

$$g = \frac{\epsilon(-1)^N}{q^N} \sum_{v\in V_N} (v, -v). \tag{5.5.12}$$

It remains to determine the sign of (5.5.11) and that it matches (5.5.9) by proving

$$s = \epsilon(-1)^N \cdot g.$$

For clarity, let us write $(V_N)_1$, $(V_N)_2$ for the two copies of V_N appearing in (5.5.10) corresponding to the two factors of $\omega_a[V_N]$. Again, using the Schrödinger model, for decompositions into Lagrangians $(V_N)_i = (V_N)_i^+ \oplus (V_N)_i^-$, we can identify the two copies of the oscillator representation $\omega_a[(V_N)_i]$ with the spaces of functions $f : (V_N)_i^- \to \mathbb{C}$ for $i = 1, 2$. In particular, we identify $\omega_a[V_N] \otimes \omega_a[V_N]$ with the space of functions

$$f : \Lambda_1' \oplus \Lambda_2' \to \mathbb{C}.$$

On this level, the involution $s : (\omega_a[V_N])^{\otimes 2} \to (\omega_a[V_N])^{\otimes 2}$ acts by sending

$$f(x, y) \mapsto f(y, x)$$

for $(x, y) \in \Lambda'_1 \oplus \Lambda'_2$. To check (5.5.9), it suffices to prove that, for a pair $(x, y) \in \Lambda'_1 \oplus \Lambda'_2$, applying (5.5.9) i.e.

$$\frac{1}{q^N} \sum_{\ell \in \Lambda, \ell \in \Lambda'} (\ell + \ell') \otimes (-(\ell + \ell'))) \tag{5.5.13}$$

to $\mathbf{1}_{(x,y)} : \Lambda'_1 \oplus \Lambda'_2 \to \mathbb{C}$ gives $\mathbf{1}_{(y,x)}$. Now as elements of $\mathbb{C}V_N$, we have

$$(\ell + \ell') = \psi_a(-\frac{1}{2}S(\ell', \ell)) \cdot (\ell') \star (\ell).$$

Therefore, applying (5.5.13) to $\mathbf{1}_{(x,y)}$ gives

$$\frac{1}{q^N} \cdot \sum_{\ell \in \Lambda} \sum_{\ell' \in \Lambda'} (\ell') \otimes (-\ell') \left(\psi_a(-S(\ell', \ell)) \cdot \mathbf{1}_{x+\ell, y-\ell}\right). \tag{5.5.14}$$

Now $(\ell') \otimes (-\ell')$ is considered as an element of $\mathbb{C}V_N \otimes \mathbb{C}V_N$ here, and it acts on $\mathbf{1}_{x+\ell,y-\ell}$ by

$$(\ell') \otimes (-\ell') \, (\mathbf{1}_{x+\ell,y-\ell}) = \psi_a(S(\ell', x + \ell) + S(-\ell', y - \ell)) \cdot \mathbf{1}_{x+\ell,y-\ell}.$$

Therefore, at each $\ell \in \Lambda$, the corresponding term of (5.5.14) is

$$\begin{aligned} &\frac{1}{q^N} \sum_{\ell' \in \Lambda'} (\ell') \otimes (-\ell') \left(\psi_a(-S(\ell', \ell)) \cdot \mathbf{1}_{x+\ell,y-\ell}\right) = \\ &\frac{1}{q^N} \sum_{\ell' \in \Lambda'} \psi(S(\ell', x - y + \ell)) \cdot \mathbf{1}_{x+\ell,y-\ell}. \end{aligned} \tag{5.5.15}$$

This sum vanishes for every choice of ℓ with $x - y + \ell \neq 0$. Thus, the only surviving term occurs for $\ell = y - x$, for which (5.5.15) is

$$\frac{1}{q^N} \cdot q^N \cdot \mathbf{1}_{y,x} = \mathbf{1}_{y,x},$$

as required.

□

5.5.4. *The interpolated metaplectic representation category and its semisimplicity.* Now we can check the consistency of (5.5.4) and (5.5.5). It suffices to check, by commutation of trace and product, the case of $k = 2$, i.e. applying $\boldsymbol{\tau}_{2\mapsto 1}$ to $(v_1, v_2) \in \mathrm{End}(\omega_a[V_N]^{\otimes 2}) \subset \mathcal{T}^{\mathrm{Osc}}_{\{1,2\},\{1,2\}}$. By (5.5.9), we find that, writing σ for the switch of the first and second coordinates,

$$\sigma \circ (v_1, v_2) = s \star (v_1, v_2) = \frac{1}{q^t} \sum_{w \in V} (v_1, v_2) \star (w, -w) =$$
$$\frac{1}{q^t} \sum_{w \in V} \psi_a\left(\frac{S(v_1, w)}{2} + \frac{S(v_2, -w)}{2}\right) \cdot (v_1 + w, v_2 - w).$$

Applying $\boldsymbol{\tau}_{\mathrm{Id}_{\{2\}}}$ to this, we see that the only surviving term occurs if the second coordinate of $(v_1 + w, v_2 - w)$ is 0, i.e. $v_2 = w$. Simplifying, therefore, we find that (5.5.5) reduces to

$$\boldsymbol{\tau}_{\mathrm{Id}_{\{2\}}}(\sigma \circ (v_1, v_2)) = \frac{1}{q^t} \psi\left(\frac{S(v_2, v_2)}{2}\right) \cdot \boldsymbol{\tau}_{\mathrm{Id}_{\{2\}}}(v_1 + v_2, 0) =$$
$$\frac{q^t}{q^t} \psi\left(\frac{S(v_2, v_2)}{2}\right) \cdot (v_1 + v_2) = \psi\left(\frac{S(v_2, v_2)}{2}\right) \cdot (v_1 + v_2),$$

matching (5.5.4).

Since the formula (5.5.9) is consistent for every N, we can therefore conclude that $\mathcal{T}^{\mathrm{Osc}}$ is a consistent T-algebra.

5.5.5. Definition. *We write $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ for the category $\mathscr{C}[\mathcal{T}^{Osc}]$ obtained from applying a pseudo-abelian enveloped to the diagrammatic category constructed from the T-algebra $\mathcal{T}^{Osc}$. We call this the* interpolated metaplectic group representation category *or the* interpolated category generated by the oscillator representation.

5.5.6. Proposition. *In every case of t, the semisimplification of the category $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ is a semisimple (and, in particular, abelian) pre-Tannakian category. For values of $t \in \mathbb{C}$ such that $q^t \neq \pm q^n$ for $n \in \mathbb{N}_0$, the category $\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))$ itself is semisimple.*

Proof. First, there is an inclusion of T-algebras

$$\mathcal{T}^{\mathrm{Osc}} \hookrightarrow \mathcal{T}^{\mathrm{GL}_t(\mathbb{F}_q)},$$

since for every N, the restriction of an oscilaltor representation ω_a to $GL_N(\mathbb{F}_q) \subseteq \mathrm{Sp}_{2N}(\mathbb{F}_q)$ is isomorphic to

$$\mathrm{Res}_{GL_N(\mathbb{F}_q)}(\omega_a[V_N]) \cong (\mathbb{C}\mathbb{F}_q^N) \otimes \epsilon(\det).$$

In particular, then for finite sets S, T, restriction gives an inclusion from each Hom-space

$$\mathrm{Hom}_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))}(\omega_a[V_t]^{\otimes S_1} \otimes \omega_b[V_t]^{\otimes S_2}, \omega_a[V_t]^{\otimes T_1} \otimes \omega_b[V_t]^{\otimes T_2}) = \mathrm{Hom}_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(\omega_a^{\otimes S_1} \otimes \omega_b^{\otimes S_2}, \omega_a^{\otimes T_1} \otimes \omega_b^{\otimes S_2})$$

making up $\overline{\mathcal{T}}[\mathrm{Sp}_{2t}(\mathbb{F}_q)]_{S,T}$ for $S_1 \amalg S_2 = S, T_1 \amalg T_2 = T$, into the $GL_N(\mathbb{F}_q)$-equivariant Hom-space on the restrictions

$$\mathrm{Hom}_{GL_N(\mathbb{F}_q)}((\mathbb{C}\mathbb{F}_q^N \otimes \epsilon(\det))^{\otimes S_1 \amalg S_2}, (\mathbb{C}\mathbb{F}_q^N \otimes \epsilon(\det))^{\otimes T_1 \amalg T_2}),$$

which is isomorphic to $\mathcal{T}_{S,T}^{\mathrm{GL}_t(\mathbb{F}_q)}$. Partial trace, tensor product, and functoriality (and therefore composition) are all compatible.

Therefore, since the semisimplification of $\mathfrak{Rep}(GL_t(\mathbb{F}_q))$ is semisimple, we may apply Proposition 5.3.2 to conclude that the semisimplification of $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ must also be semisimple.

The second claim follows, for example, by examining the polynomial order of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, and replacing N by t, to conclude that every indecomposable object is non-vanishing in the semisimplification. □

To summarize: We have constructed a category $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ interpolating the category generated by the oscillator representations over $\mathbb{F}_q$, in that

$$\mathrm{Hom}_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))}(\omega_a[V_t]^{\otimes S} \otimes \omega_b[V_t]^{\otimes T}, \omega_a[V_t]^{\otimes S'} \otimes \omega_b[V_t]^{\otimes T'}) = \mathrm{Hom}_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(\omega_a[V_N]^{\otimes S} \otimes \omega_b[V_N]^{\otimes T}, \omega_a[V_N]^{\otimes S'} \otimes \omega_b[V_N]^{\otimes T'})$$

for $N >> |S|, |T|, |S'|, |T'|$. The trace operation for endomorphisms of each $\omega_a[V_t]$ is given by

$$\mathrm{End}_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))_0}(\omega_a[V_t]) = \mathrm{End}_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(\omega_a[V_N]) = \mathrm{Hom}_{\mathrm{Sp}_{2N}(\mathbb{F}_q)}(1, \mathbb{C}V_N) \cong (\mathbb{C}V_N)^{\mathrm{Sp}_{2N}},$$

which has a basis consisting of (0) and $\sum_{v \neq 0 \in V_N} (v)$, where we put

$$tr((0)) = q^t, \quad tr(\sum_{v \neq 0 \in V_N} (v)) = 0.$$

Recalling Proposition 1.3.3, the categories $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ can in fact be considered a grading of the standard interpolated category

$\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ by the Witt group

$$W(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/2 \times \mathbb{Z}/2 & \text{for } q = 1 \bmod 4 \\ \mathbb{Z}/4 & \text{for } q = 3 \bmod 4. \end{cases}$$

The category's objects are graded by $W(\mathbb{F}_q)$, according to whether they can be constructed in a tensor product of

$$1,\ \omega_a[V_t],\ \omega_{-b}[V_t],\ \text{or } \omega_a[V_t] \otimes \omega_{-b}[V_t]$$

with a tensor power of the "vector representation" $\mathbb{C}V_t := \omega_a[V_t] \otimes \omega_{-a}[V_t]$ (still using our choice of a, b distinct in $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$). In the case of values of t such that $q^t = \pm q^N$ for $N \in \mathbb{N}_0$, the central extension of the symplectic group by the Witt group splits so that the oscillator representation can be considered as a genuine representaiton of the symplectic group $\mathrm{Sp}_{2N}(\mathbb{F}_q)$. For generic values of t, however, the central extension does not fully split. We may still split off the $\mathbb{Z}/2$ factor of $W(\mathbb{F}_q)$ corresponding to even-dimensional quadratic forms in every case. This is because, for a non-split even-dimensional symmetric bilinear form B over $\mathbb{F}_q$, the sum $B \oplus B$ is always split and can therefore be expressed as a sum of two copie of a split symmetric bilinear form. Therefore, the central extension by $W(\mathbb{F}_q)$ must at least split by $\mathbb{Z}/2$, leaving that in the interpolated setting, $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ corresponds to a "central extension" of the standard interpolation by $\mathbb{Z}/2$. For this reason, we consider $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ to be an "interpolation of the metaplectic group," though, of course, there is no genuine metaplectic group over the classical finite symplectic groups.

Comment: *The categories* $\mathfrak{Rep}(S_t)$, $\mathfrak{Rep}(GL_t(\mathbb{F}_q))$, $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q))$ *and* $\mathfrak{Rep}(O_t(\mathbb{F}_q))$ *all also have constructions from the perspective of* oligomorphic groups [**26**]. *More specifically, they can be constructed by interpreting the representation theory of the groups* S_∞, $GL_\infty(\mathbb{F}_q)$, $Sp_\infty(\mathbb{F}_q)$, *and* $O_\infty(\mathbb{F}_q)$ *(respectively), by introducing a certain "measure" associated to the parameter* t *to make sense of trace on the regular representations. Every oligomorphic category* $\mathscr{C}$ *in particular admits a functor from* $\mathfrak{Rep}(S_t)$ *for some value of* t*, the universal property of which implies that the standard object* X *of* $\mathscr{C}$ *forms an algebra object with multiplication morphism*

$$\mu : X \otimes X \to X$$

whose composition with the "algebra trace" morphism

$$Tr_X : X \xrightarrow{Id \otimes coev} X \otimes X \otimes X^\vee \xrightarrow{\mu \otimes Id} X \otimes X^\vee \xrightarrow{ev} 1$$

gives self-duality for X (this structure can be considered a "set quantum field theory"). There is no consistent choice of this for the interpolated metaplectic representation category $\overline{\mathfrak{Rep}}(Sp_{2N}(\mathbb{F}_q))$, meaning that it cannot be constructed in this way.

5.6. The interpolated Weil-Shale representation. In this subsection, we briefly mention another interesting interpolated model which can be obtained by these methods. This model is not directly used in the theory of Howe duality, so a reader primarily interested in that subject can skip this subsection.

The Weil-Shale representations $\boldsymbol{\omega}_a[V_N]$ of the semidirect products $\mathrm{Sp}_{2N}(\mathbb{F}_q) \ltimes \mathbb{H}_N(\mathbb{F}_q)$ can also be interpolated. Let us denote by

$$\mathcal{T}^{\text{W-S}}_{S,T} \tag{5.6.1}$$

the subspace of (5.4.5) of morphisms that preserve the action of

$$\mathrm{GL}_N(\mathbb{F}_q) \subset \mathrm{Sp}(V_N) \ltimes \mathbb{H}_N(\mathbb{F}_q) \tag{5.6.2}$$

(taking V_N to be a symplectic space of dimension $2N$, and embedding $\mathrm{GL}_N(\mathbb{F}_q)$ as the general linear group of $\boldsymbol{\omega}_a[V_N]$'s underlying vector space), for $N >> 0$. We may then see that the spaces $\mathcal{T}^{\text{W-S}}_{S,T}$ forms a sub-T-algebra of $\mathcal{T}^{\mathrm{GL}_t(\mathbb{F}_q)}$: The only non-trivial point to verify is that for finite sets S, T and a bijection $\phi : S' \to T'$ betweeen subsets $S' \subseteq S$, $T' \subseteq T$, the image of $\mathcal{T}^{\text{W-S}}_{S,T}$ under the partial trace

$$\boldsymbol{\tau}_\phi : \mathcal{T}^{\mathrm{GL}_t(\mathbb{F}_q)}_{S,T} \to \mathcal{T}^{\mathrm{GL}_t(\mathbb{F}_q)}_{S\smallsetminus S', T\smallsetminus T'}$$

is contained in $\mathcal{T}^{\text{W-S}}_{S\smallsetminus S', T\smallsetminus T'}$, which follows since the action of (5.6.2) on a tensor power is diagonal and is preserved by partial trace.

Again, the semisimplification of the category constructed from $\mathcal{T}^{\text{W-S}}$ is semisimple by Proposition 5.3.2 and the semisimplicity of (the semisimplification of) $\mathfrak{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ for every $t \in \mathbb{C}$. We write

$$\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q) \ltimes \mathbb{H}_t(\mathbb{F}_q)) := \mathscr{S}(\mathscr{C}[\mathcal{T}^{\text{W-S}}])$$

for this semisimplification.

5.6.1. THEOREM. *For every power q of a prime not equal to 2 or 3 and every $t \in \mathbb{C}$ such that $q^t \neq \pm 1, \pm q$, the category $\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q) \ltimes \mathbb{H}_N(\mathbb{F}_q))$ is a semisimple pre-Tannakian category generated by an object X of dimension q^t such that X, $\Lambda^2(X)$, and $Sym^2(X)$ are simple and*

$$dim(End_{\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q)\ltimes \mathbb{H}_N(\mathbb{F}_q))}(X^{\otimes 3}) = 2q + 2. \tag{5.6.3}$$

If $q^t = \pm \pm q$, the same conditions hold, with

$$dim(End_{\mathfrak{Rep}(Sp_{2t}(\mathbb{F}_q)\ltimes \mathbb{H}_N(\mathbb{F}_q))}(X^{\otimes 3}) = 2q + 1 \tag{5.6.4}$$

instead of (5.6.3).

To prove this, we must recall the relationship between the endomorphism algebras of tensor products of Weil-Shale representations and those of the oscillator representations. One consequence of Lemma 1.2.1 is the following

5.6.2. THEOREM. *Consider a Weil-Shale representation* $\boldsymbol{\omega}[V_N]$ *of* $Sp(V_N) \ltimes \mathbb{H}(V_N)$ *and an oscillator representation* $\omega[V_N]$. *For any degree* k, *there is an isomorphism of vector spaces*

$$End_{Sp(V_N)}(\omega[V_N]^{\otimes k}) \cong End_{Sp(V_N)\ltimes\mathbb{H}(V_N)}(\boldsymbol{\omega}[V_N]^{\otimes k+1}) \tag{5.6.5}$$

PROOF. Fix a Weil-Shale representation $\boldsymbol{\omega}_a[V_N]$. We recall that our notation $\Omega = \boldsymbol{\omega}[V_N] \otimes \boldsymbol{\omega}^\vee[V_N]$. Of course, the restriction to $\mathrm{Sp}(V_N)$ is isomorphic to the vector representation

$$\mathrm{Res}_{\mathrm{Sp}(V_N)}(\Omega) \cong \mathbb{C}V_N$$

(though we use the convention that Ω consists of functions from V_N to $\mathbb{C}$ rather than linear combinations of vectors, so this isomorphism is described by sending an indicator function $\mathbf{1}_v$ to (v)). As usual, by duality, we may consider the endomorphism algebra of a tensor power of the Weil-Shale representation as the space of fixed points in the corresponding tensor power of Ω:

$$\mathrm{End}_{\mathrm{Sp}(V_N)\ltimes\mathbb{H}(V_N)}(\boldsymbol{\omega}_a[V_N]^{\otimes k+1}) \cong \mathrm{Hom}_{\mathrm{Sp}(V_N)\ltimes\mathbb{H}(V_N)}(1, \Omega^{\otimes k+1}) \cong \\ (\Omega^{\otimes k+1})^{\mathrm{Sp}(V_N)\ltimes\mathbb{H}(V_N)}$$

Let us examine the fixed points of Ω more closely, specifically focusing on the fixed points under $(\Omega^{\otimes k+1})^{\mathbb{H}(V_N)}$ with respect to the Heisenberg group action. First, since the center $\mathbb{F}_q$ of the Heisenberg group acts trivially on Ω, we may identify $(\Omega^{\otimes k+1})^{\mathbb{H}(V_N)}$ with the fixed points of V_N acting on the restriction of Ω to $\mathrm{Sp}(V_N) \ltimes V_N \subset \mathrm{Sp}(V_N) \ltimes \mathbb{H}(V_N)$

$$(\mathrm{Res}_{\mathrm{Sp}(V_N)\ltimes V_N}(\Omega^{\otimes k+1}))^{V_N}. \tag{5.6.6}$$

Considering $\Omega^{\otimes k+1}$, as a vector space, to consist of functions from V_N^{k+1} to $\mathbb{C}$, recall that the action of V_N on $\Omega^{\otimes k+1}$ is given by, for $u \in V_N$, $(v_1, \ldots, v_{k+1}) \in V_N^{k+1}$, putting

$$u(\mathbf{1}_{(v_1,\ldots,v_{k+1})}) = \prod_{i=1}^{k+1} \psi(S(u, v_i)) \cdot \mathbf{1}_{(v_1,\ldots,v_{k+1})} = \\ \psi(S(u, v_1 + \cdots + v_{k+1})) \cdot \mathbf{1}_{(v_1,\ldots,v_{k+1})}.$$

Therefore, the fixed point space (5.6.6) is the $\mathbb{C}$-vector space generated by indicator functions $\mathbf{1}_{(v_1,\dots,v_{k+1})}$ such that

$$S(u, v_1 + \cdots + v_{k+1}) = 0 \text{ for every } u \in V_N. \tag{5.6.7}$$

This condition is equivalent to requiring that

$$v_1 + \cdots + v_{k+1} = 0, \tag{5.6.8}$$

and therefore, we find that (5.6.6) is isomorphic to the space of functions on V_N^{k+1} only supported on $(k+1)$-tuples $(v_1, \dots, v_{k+1})$ satisfying (5.6.8). This is precisely isomorphic to the space of $\mathbb{C}$-valued functions on V_N^k (for example, by embedding $V_N^k \subset V_N^{k+1}$ by putting $v_{k+1} = -v_1 - \cdots - v_k$). Again, we may identify the space of $\mathbb{C}$-valued functions on V_N^k with

$$\mathbb{C}V_N^k \cong (\mathbb{C}V_N)^{\otimes k}$$

(by identifying an indicator function with its corresponding vector). Note also that all of the above isomorphisms are $\mathrm{Sp}(V_N)$-equivariant since the symplectic group is take to act diagonally.

Therefore, as $\mathrm{Sp}(V_N)$-representations, the fixed points of a tensor power of Ω satisfy

$$(\Omega^{\otimes k+1})^{\mathbb{H}(V_N)} \cong (\mathrm{Res}_{\mathrm{Sp}(V_N)}(\Omega))^{\otimes k} \cong (\mathbb{C}V_N)^{\otimes k}.$$

Hence, returning to endomorphism algebras, we find that, as vector spaces,

$$\begin{gathered}\mathrm{End}_{\mathrm{Sp}(V_N)\ltimes\mathbb{H}(V_N)}(\boldsymbol{\omega}_a[V_N]^{\otimes k+1}) \cong (\Omega^{\otimes k+1})^{\mathrm{Sp}(V_N)\ltimes\mathbb{H}(V_N)} \cong \\ (\mathbb{C}V_N^{\otimes k})^{\mathrm{Sp}(V_N)} \cong \mathrm{End}_{\mathrm{Sp}(V_N)}(\omega_a[V_N]^{\otimes k}).\end{gathered}$$

□

We also note that Theorem 1.3.1 can then be used to see that the algebra structures of endomorphisms of the oscillator and the Weil-Shale representations are also related. For the algebra structure, however, we must be more careful with the central characters of the Weil-Shale representation factors than for the vector space structure. For example, we have the following

5.6.3. Corollary. *Consider a choice of Weil-Shale representations* $\boldsymbol{\omega}_{a_1}[V_N], \dots, \boldsymbol{\omega}_{a_{k+1}}[V_N]$ *of* $Sp(V_N) \ltimes \mathbb{H}(V_N)$ *such that*

$$b_i = a_{i+1} \cdot (\sum_{j=1}^{i} a_j) \cdot (\sum_{j=1}^{i+1} a_j) \tag{5.6.9}$$

are all non-zero for $i = 1, \dots, k$. Then there is an isomorphism of algebras

$$\begin{gathered} End_{Sp(V_N)\ltimes \mathbb{H}(V_N)}(\boldsymbol{\omega}_{a_1}[V_N] \otimes \cdots \otimes \boldsymbol{\omega}_{a_{k+1}}[V_N]) \cong \\ End_{Sp(V_N)}(\omega_{b_1}[V_N] \otimes \cdots \otimes \omega_{b_k}[V_N]) \end{gathered} \tag{5.6.10}$$

In particular, for any irreducible sub-$Sp(V_N)$-representation ρ of $\omega_{b_1}[V_N] \otimes \cdots \otimes \omega_{b_k}[V_N]$, the tensor product

$$\rho \otimes \boldsymbol{\omega}_{a_1+\cdots+a_k}[V_N]$$

is an irreducible representation of $Sp(V_N) \ltimes \mathbb{H}(V_N)$ (taking $\mathbb{H}(V_N)$ to act trivially on ρ).

Proof. Given the assumptions of this statement, iterating Theorem 1.3.1 implies that there is an isomorphism of $\mathrm{Sp}(V_N) \ltimes \mathbb{H}(V_N)$-representations

$$\begin{gathered} \boldsymbol{\omega}_{a_1}[V_N] \otimes \cdots \otimes \boldsymbol{\omega}_{a_{k+1}}[V_N] \cong \\ \omega_{b_1}[V_N] \otimes \cdots \otimes \omega_{b_k}[V_N] \otimes \boldsymbol{\omega}_{a_1+\cdots+a_{k+1}}[V_N], \end{gathered} \tag{5.6.11}$$

where on the right hand side, each $\omega_{b_i}[V_N]$ is inflated from a $\mathrm{Sp}(V_N)$-representation to be considered as a $\mathrm{Sp}(V_N) \ltimes \mathbb{H}(V_N)$-representation by letting the Heisenberg group act trivially. Considering the decomposition of the tensor product of oscillator representations into irreducible representations of $\mathrm{Sp}(V_N)$

$$\omega_{b_1}[V_N] \otimes \cdots \otimes \omega_{b_k}[V_N] \cong \bigoplus_{\rho \in \widehat{\mathrm{Sp}(V_N)}} \rho^{\oplus m(\rho)}$$

for multiplicities $m(\rho) \in \mathbb{N}_0$, the right hand side of (5.6.10) decomposes into matrix algebras

$$\mathrm{End}_{\mathrm{Sp}(V_N)}(\omega_{b_1}[V_N] \otimes \cdots \otimes \omega_{b_k}[V_N]) = \prod_{\rho \in \widehat{Sp(V_N)}} M_{m(\rho)}(\mathbb{C}). \tag{5.6.12}$$

Substituting in (5.6.11), we then also obtain a decomposition

$$\boldsymbol{\omega}_{a_1}[V_N] \otimes \cdots \otimes \boldsymbol{\omega}_{a_{k+1}}[V_N] \cong \bigoplus_{\rho \in \widehat{\mathrm{Sp}(V_N)}} (\rho \otimes \boldsymbol{\omega}_{a_1+\cdots+a_{k+1}}[V_N])^{\bigoplus m(\rho)},$$

(again inflating ρ to $\mathrm{Sp}(V_N) \ltimes \mathbb{H}(V_N)$-representations by letting the Heisenberg group act trivially), implying that the left hand side of (5.6.10) contains (5.6.12) (with equality precisely when every

$$\rho \otimes \boldsymbol{\omega}_{a_1+\cdots+a_{k+1}}[V_N] \tag{5.6.13}$$

is irreducible).

By Theorem 5.6.2, we know that this inclusion must be equality as claimed, since the dimensions of the two sides of (5.6.10) are equal (since the underlying characters of the oscillator and Weil-Shale representations do not affect the dimension of the endomorphism algebra

$$\dim(\mathrm{End}_{\mathrm{Sp}(V_N)}(\omega_{b_1}[V_N] \otimes \cdots \otimes \omega_{b_k}[V_N])) =$$
$$\dim(\mathrm{End}_{\mathrm{Sp}(V_N)}(\omega[V_N]^{\otimes k})) = \dim(\mathrm{End}_{\mathrm{Sp}(V_N)\ltimes \mathbb{H}(V_N)}(\boldsymbol{\omega}[V_N]^{\otimes k+1})) =$$
$$\dim(\mathrm{End}_{\mathrm{Sp}(V_N)\ltimes \mathbb{H}(V_N)}(\boldsymbol{\omega}_{a_1}[V_N] \otimes \cdots \otimes \boldsymbol{\omega}_{a_{k+1}}[V_N])).$$

(Therefore, in particular, the representations (5.6.13) are in fact irreducible.)

□

These results also hold in the interpolated categories $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q) \ltimes \mathbb{H}_N(\mathbb{F}_q))$ and $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ since they are deduced only from considering endomorphism algebra results, independent of N, and can be applied to prove Theorem 5.6.1:

Proof of Theorem 5.6.1. First let us consider the tensor square and cube of a Weil-Shale representation $\boldsymbol{\omega}_a[V_N]$ of $\mathrm{Sp}_{2N}(\mathbb{F}_q) \ltimes \mathbb{H}_N(\mathbb{F}_q)$. The irreducibility of $\mathrm{Sym}^2(\boldsymbol{\omega}_a[V_N])$ and $\Lambda^2(\boldsymbol{\omega}_a[V_N])$ follow from Theorem 5.6.2, since

$$\dim(\mathrm{End}_{\mathrm{Sp}(V_N)\ltimes \mathbb{H}(V_N)}(\boldsymbol{\omega}_a[V_N]^{\otimes 2})) = \dim(\mathrm{End}_{\mathrm{Sp}(V_N)}(\omega[V_N])) = 2$$

for every natural number N. The objects $\mathrm{Sym}^2(\boldsymbol{\omega}_a[V_t])$ and $\Lambda^2(\boldsymbol{\omega}_a[V_t])$ are also therefore simple in $\mathfrak{Rep}(\mathrm{Sp}_{2t}(\mathbb{F}_q) \ltimes \mathbb{H}_N(\mathbb{F}_q))$ and non-zero (since they are of non-zero dimension $(q^{2t} \pm 1)/2$ by the assumption that $q^t \neq \pm 1$ and therefore survive the semisimplification). Similarly, Theorem 5.6.2 also gives

$$\dim(\mathrm{End}_{\mathrm{Sp}(V_N)\ltimes \mathbb{H}(V_N)}(\boldsymbol{\omega}_a[V_N]^{\otimes 3})) = \dim(\mathrm{End}_{\mathrm{Sp}(V_N)}(\omega[V_N]^{\otimes 2})),$$

which, recalling Lemma 2.2.1, is $2q+2$ for $N > 1$, and $2q+1$ for $N = 1$. It remains to argue that, again, in the interpolated case, all irreducible summands of $\boldsymbol{\omega}_a[V_t]^{\otimes 3}$ survive semisimplification for $q^t \neq \pm 1, \pm q$ and only one is identified to zero for $q^t = \pm q$. According to Corollary 5.6.3, we have

$$\boldsymbol{\omega}_a[V_N]^{\otimes 3} \cong \omega_{6a}[V_N] \otimes \omega_{2a}[V_N] \otimes \boldsymbol{\omega}_{3a}[V_N]$$

for every N. Applying Theorems 2.1.4 and 4.1.1, we know the dimension of the $\mathrm{Sp}(V_N)$ decomposition of $\omega_{6a}[V_N] \otimes \omega_{2a}[V_N]$:

If $q = 1 \bmod 3$ (in which case -3 is a square in $\mathbb{F}_q$), then $\omega_{6a}[V_N] \otimes \omega_{2a}[V_N]$ is isomorphic to the restriction of $\omega[V_N \otimes (\mathbb{F}_q^2, +)]$, writing $(\mathbb{F}_q^2, +)$ for the 2-dimensional orthogonal space with split orthogonal form, to $\mathrm{Sp}(V_N)$. In this case, the dimensions of the occuring irreducible $\mathrm{Sp}(V_N)$-representations are 1 for the trivial representation copies and

$$\frac{q^{2N}-1}{q-1}, \frac{q^{2N}-1}{2(q-1)}, \text{ and } \frac{(q^N \mp 1)(q^N \pm q)}{2(q-1)} \tag{5.6.14}$$

(obtained from applying $\eta_{V_N}^{(\mathbb{F}_q^2,+)}$ to the $O_2^+(\mathbb{F}_q)$-representations with semisimple data with eigenvalues not equal to ± 1, the two representation with semisimple data $-I$, and the two representations with semisimple data I, respectively). Specifically, note that in the case of the trivial and sign representations (i.e. the two $O_2^+(\mathbb{F}_q)$-representations with semisimple data I), $\eta_{V_N}^{(\mathbb{F}_q^2,+)}(1) = \binom{0<N}{1}$ and $\eta_{V_N}^{(\mathbb{F}_q^2,+)}(-1) = \binom{1<N}{0}$, giving the dimensions

$$\dim(\eta_{V_N}^{(\mathbb{F}_q^2,+)}(\pm 1)) = \frac{(q^N \mp 1)(q^N \pm q)}{2(q-1)}.$$

The irreducible summands of $\boldsymbol{\omega}_a[V_t]^{\otimes 3}$, by Corollary 5.6.3, then have dimension equal to (5.6.14), with N replaced by t, multiplied by $q^t = \dim(\boldsymbol{\omega}_{3a}[V_t])$. In particular, we see that all the dimensions are nonzero if $q^t \neq \pm 1$, while precisely one of the summands obtained by interpolating $\eta_{V_N}^{(\mathbb{F}_q^2,+)}(\pm 1)$ and tensoring with $\boldsymbol{\omega}_{3a}[V_t]$ has 0 dimension (and is therefore identified with the 0 object during semisimplification) if $q^t = \pm q$.

Similarly, if $q = 2 \bmod 3$ (in which case -3 is not a square in $\mathbb{F}_q$), then $\omega_{6a}[V_N] \otimes \omega_{2a}[V_N]$ is isomorphic to the restriction of $\omega[V_N \otimes (\mathbb{F}_q^2, -)]$, writing $(\mathbb{F}_q^2, -)$ for the 2-dimensional orthogonal space with non-split orthogonal form, to $\mathrm{Sp}(V_N)$. In this case, the dimensions of the occuring irreducible $\mathrm{Sp}(V_N)$-representations are

$$\frac{q^{2N}-1}{q+1}, \frac{q^{2N}-1}{2(q+1)}, \text{ and } \frac{(q^N \pm 1)(q^N \pm q)}{2(q+1)} \tag{5.6.15}$$

(obtained from applying $\eta_{V_N}^{(\mathbb{F}_q^2,-)}$ to the $O_2^-(\mathbb{F}_q)$-representations with semisimple data with eigenvalues not equal to ± 1, the two representation with semisimple data $-I$, and the two representations with semisimple data I, respectively). Specifically, note that in the case of the trivial and sign representations (i.e. the two $O_2^-(\mathbb{F}_q)$-representations with semisimple data I), $\eta_{V_N}^{(\mathbb{F}_q^2,-)}(1) = \binom{0<1}{N}$ and $\eta_{V_N}^{(\mathbb{F}_q^2,-)}(-1) = \binom{0<1<N}{\emptyset}$, giving the dimensions

$$\dim(\eta_{V_N}^{(\mathbb{F}_q^2,+)}(\pm 1)) = \frac{(q^N \pm 1)(q^N \pm q)}{2(q+1)}.$$

The irreducible summands of $\boldsymbol{\omega}_a[V_t]^{\otimes 3}$, by Corollary 5.6.3, then have dimension equal to (5.6.15), with N replaced by t, multiplied by $q^t = \dim(\boldsymbol{\omega}_{3a}[V_t])$. In particular, we again see that all the dimensions are non-zero if $q^t \neq \pm 1$, while precisely one of the summands obtained by interpolating $\eta_{V_N}^{(\mathbb{F}_q^2,-)}(\pm 1)$ and tensoring with $\boldsymbol{\omega}_{3a}[V_t]$ has 0 dimension (and is therefore identified with the 0 object during semisimplification) if $q^t = \pm q$.

$\square$

CHAPTER 6

The full explicit decomposition statement

In this chapter, we use the background introduced in Chapter 5 to finally formulate and prove the full statement of finite field Howe duality, in particular focusing on the cases outside of the stable ranges. A naive guess would be that fixing one of the groups in the pair, i.e. $\mathrm{O}(W, B)$ resp. $\mathrm{Sp}(V)$, we may simply interpolate the representations of its partner (by varying $\dim(V)$ resp. $\dim(W)$) to explain their behavior outside of the stable range, with inter- (or extra?)-polated representations of predicted dimension 0 disappearing from the classification. While this does happen, it is not the only effect that occurs. With an integer parameter outside of the stable range, one may also encounter interpolated representations of predicted *negative* dimensions. As it turns out, these correspond to "illegal Lusztig symbols" related to true Lusztig symbols by a simple permutation of entries. There is always one "legal" Lusztig symbol in the group, corresponding to representations which actually occur in the "real world" Howe correspondence, in a kind of an analogue of certain parabolic BGG resolutions for reductive groups over finite fields. Identifying the alternating sums corresponding to these resolutions is the purpose of this chapter; we will further study them in the next.

To be more precise, this behavior occurs in certain ranges which fill out regiones from a certain "middle line" to the corresponding stable range. We call this a *metastable range*. Outside of the stable and metastable ranges, even more profound singularities occur, corresponding to "Lusztig symbols" with actual negative entries. We do not study these effects here. However, the stable and metastable ranges fill out all the possible choices of pairs $(\mathrm{Sp}(V), \mathrm{O}(W, B))$.

To start a more precise discussion, recall again our setting. Consider a type I reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$. Then in the symplectic resp. orthogonal metastable ranges, there are correspondences

$$\eta^V_{W,B} : \widehat{\mathrm{O}(W, B)} \to \widehat{\mathrm{Sp}(V)} \cup \{0\}$$

$$\zeta^{W,B}_V : \widehat{\mathrm{Sp}(V)} \to \widehat{\mathrm{O}(W, B)} \cup \{0\},$$

explicitly described in terms of Lusztig's classification, such that the restriction of the oscillator representation of $\mathrm{Sp}(V \otimes W)$ to $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$ is a direct sum of terms of the form $\eta_V^{W,B}(\pi) \otimes \pi$ resp. $\rho \otimes \zeta_{W,B}^V(\rho)$ in the "top part" and tensor products of the images of eta resp. zeta correspondences from smaller orthogonal resp. symplectic groups with certain alternating sums of parabolic Harish-Chandra induction modules for smaller choices of V, W, each of which adds up to a linear combination of irreducible representations with positive integral coefficients. The alternating sums are explicitly resolved as sums of irreducible representations in terms of Lusztig's classification.

The precise statement takes some preliminaries, and will be made in Theorem 6.2.8 below.

In Subsection 6.3, we define certain subcategories of $\mathscr{C}_B^{int}(t) \subseteq \overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ isolating the ranges of objects appearing in a certain tensor power of oscillator representations (a "partial pseudo-abelian envelope") corresponding to a specific choice of orthogonal space (W, B). For a fixed irreducible representation $\rho \in \widehat{\mathrm{O}(W, B)}$, we further consider a subcategory $\mathscr{C}_\rho^{int}(t)$ detecting objects that interpolate representations of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ symplectic stable with $\mathrm{O}(W, B)$ whose tensor product with ρ appears in the restriction of $\omega[\mathbb{F}_q^{2N} \otimes W]$. We denote their images under semisimplification by $\widetilde{\mathscr{C}_B^{int}}(t)$, $\widetilde{\mathscr{C}_\rho^{int}}(t)$. We enumerate the objects of this category in terms of "formal Lusztig symbols" and write down their dimensions. An interpolated decomposition of the restricted oscillator representation holds in these categories (Theorem 6.3.1 below).

In Subsection 6.4, we find that, at $t = N$ with $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ in the symplectic metastable range with $\mathrm{O}(W, B)$, the relationship between an interpolated category $\widetilde{\mathscr{C}_\rho^{int}}(N)$ and the subcategory of genuine $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representations π such that $\rho \otimes \pi$ appears in the restricted oscillator representation, gives that the decomposition of the restricted oscillator representation as a genuine representation can be derived from the interpolated statement after "cancelling terms." Then we check that simplifying the cancelled terms gives the claimed decomposition as a genuine representation of the symplectic group.

6.1. The metastable ranges. First, for a general unstable choice of symplectic and orthogonal spaces V and (W, B), we must still choose whether it is "closer" to the symplectic or orthogonal stable range.

We separate the pairs $(V, (W, B))$ which do not lie in the symplectic stable range or the orthogonal stable range into "metastable ranges" to indicate whether we intend to approach the decomposition of the restriction of $\omega[V \otimes W]$ by extending the eta or zeta correspondence. We consider the different cases of $\mathrm{O}(W, B)$ individually.

6.1.1. DEFINITION. *Consider a choice of symplectic and orthogonal spaces V and (W, B). Write $dim(V) = 2N$.*

- *If W is of odd dimension $dim(W) = 2m + 1$, then we say $(V, (W, B))$ is in the* symplectic metastable range *if*
$$m < N < 2m + 1.$$
Say $(Sp(V), O(W, B))$ is in the orthogonal metastable range *if*
$$m < 2N \leq 2m$$
- *If W is of even dimension $dim(W) = 2m$ and B is not completely split, then we say $(Sp(V), O(W, B))$ is in the* symplectic metastable range *if*
$$m \leq N < 2m.$$
Say $(Sp(V), O(W, B))$ is in the orthogonal metastable range *if*
$$m - 1 < 2N < 2m.$$
- *If W is of even dimension $dim(W) = 2m$ and B is completely split, then we say $(Sp(V), O(W, B))$ is in the* symplectic metastable range *if*
$$m \leq N < 2m.$$
Say $(Sp(V), O(W, B))$ is in the orthogonal metastable range *if*
$$m < 2N < 2m.$$

We see that under this definition, every unstable choice of symplectic and orthogonal spaces V and (W, B) is contained in precisely one metastable range. More specifically, the disjoint union of the symplectic stable and metastable ranges consists of all choices of symplectic spaces V and orthogonal spaces (W, B) such that

$$\frac{dim(V)}{2} \geq \lfloor \frac{dim(W)}{2} \rfloor, \tag{6.1.1}$$

while the disjoint union of the orthogonal stable and metastable ranges consist of V and (W, B) satisfying the complimentary condition

$$\frac{dim(V)}{2} < \lfloor \frac{dim(W)}{2} \rfloor. \tag{6.1.2}$$

Broadly, the conditions (6.1.1) and (6.1.2) should be thought of as detecting whether it is more computationally viable to decompose the oscillator representation in terms of the eta correspondence (i.e. as a sum of distinct irreducible $\mathrm{Sp}(V)$-representations with potentially non-irreducible $\mathrm{O}(W,B)$-coefficients), or in terms of the zeta correspondence (i.e. as a sum of distinct irreducible $\mathrm{O}(W,B)$-representations with potentially non-irreducible $\mathrm{Sp}(V)$-coefficients). Concretely, in our constructions of the eta and zeta correspondence, the conditions (6.1.1) and (6.1.2) ensure that, in either case, we never attempt to concatenate a negative coordinate to a Lusztig symbol. It is possible to further extend the eta and zeta correspondences to all ranges by interpreting them to output 0 when this occurs (Lusztig's dimension formula for symbols indeed suggests that symbols with negative coordinates are 0-dimensional). However, approaching the decomposition of the oscillator representation from the "wrong side" is in general less computationally efficient, and we do not consider it for the purposes of this paper.

6.2. The precise statement of the metastable decomposition statement. We now concretely describe our proposed decomposition of the restriction of an oscillator representation to a reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$.

6.2.1. *The extended and virtual eta and zeta correspondences.* In this paragraph, we describe our proposed construction of the extended eta and zeta correspondences. We also describe certain *virtual eta* and *zeta correspondences.* For the purpose of decomposing the restricted oscillator representation, we want a statement only involving genuine representations of the symplectic and orthogonal groups, so we only use the virtual eta and zeta correspondences as an intermediate step. Still, for applications such as character computation, the virtual correspondences are useful in their own right (Subsection 7.6).

For a reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic (resp. orthogonal) metastable range, we in broad terms define the extended eta (resp. zeta) correspondence of an irreducible representation of $\mathrm{O}(W,B)$ (resp. $\mathrm{Sp}(V)$) by the same process we used in Subsection 4.2 to define $\phi_V^{W,B} = \eta_V^{W,B}$ (resp. $\psi_{W,B}^V = \zeta_{W,B}^V$) when possible, and when the construction is not possible, we set the extended correspondence to output 0. In more concrete terms, recalling the terminology of Definition 4.2.5, we make the following

6.2.2. Definition. *Consider a reductive dual pair $(Sp(V), O(W,B))$. Write N for the rank of $Sp(V)$ and m for the rank of $O(W,B)$.*

(1) *Say the reductive dual pair is in the symplectic metastable range. Consider an irreducible representation* π *of* $O(W,B)$*. Consider the alterable part* $u_{alt} = \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ *of its unipotent data. Consider* $m'_\pi = m - N + \lfloor \frac{a+b}{2} \rfloor$ *(which the metastable range condition ensures is non-negative). Say the extended sign data of* π*'s classification data corresponds to concatenating* m'_π *to the end of the* λ_i *(resp.* μ_j*) row of* u_{alt}*.*
   (a) *If* $m'_\pi > \lambda_a$ *(resp. if* $m'_\pi > \mu_b$*), then we put*
$$\eta_V^{W,B}(\pi) = \phi_V^{W,B}(\pi).$$
   (b) *Otherwise, we put*
$$\eta_V^{W,B}(\pi) = 0.$$
(2) *Say the reductive dual pair is in the orthogonal metastable range. Consider an irreducible representation* ρ *of* $Sp(V)$*. Consider the alterable part* $u_{alt} = \binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ *of its unipotent data. Consider* $N'_\rho = N - m + \lfloor \frac{a+b}{2} \rfloor$ *(which the metastable range condition ensures is non-negative). Say the central sign data of* ρ*'s classification data corresponds to concatenating* N'_ρ *to the end of the* λ_i *(resp.* μ_j*) row of* u_{alt}*.*
   (a) *If* $N'_\rho > \lambda_a$ *(resp. if* $N'_\rho > \mu_b$*), then we put*
$$\zeta_{W,B}^V(\rho) = \psi_{W,B}^V(\rho).$$
   (b) *Otherwise, we put*
$$\zeta_{W,B}^V(\rho) = 0.$$

Now we define the virtual eta correspondence, mapping irreducible representations of the orthogonal group into the Grothendeick group of the representations of the symplectic group
$$\boldsymbol{\eta}_V^{W,B} : \widehat{\mathrm{O}(W,B)} \to K(\mathrm{Rep}(\mathrm{Sp}(V)))$$
for $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic metastable range, and the virtual eta correspondence, mapping irreducible representations of the symplectic group into the Grothendeick group of the representations of the orthogonal group
$$\boldsymbol{\zeta}_{W,B}^V : \widehat{\mathrm{Sp}(V)} \to K(\mathrm{Rep}(\mathrm{O}(W,B)))$$
for $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the orthogonal metastable range.

Again, for an input representation ρ of $\boldsymbol{\eta}_V^{W,B}$ or π of $\boldsymbol{\zeta}_{W,B}^V$, we compute the output by attempting the appropriate case of the construction described in Subsection 4.2. However, instead of taking the output to be 0 if the we cannot concatentate N'_π or m'_ρ to the end of the row of

u_{alt} specified by the extended/central sign data of π or ρ, we shuffle N'_ρ or m'_π into a place where it concatentatable, and use the signature of the shuffle as the coefficient of the resulting representation in the Grothendeick group. If even this is not possible (i.e. if N'_π resp. m'_ρ is equal to one of the entries of the row of u_{alt} to which we are meant to concatenate it), then we put the output to be 0.

6.2.3. Definition. *Consider a reductive dual pair $(Sp(V), O(W,B))$. Write N for the rank of $Sp(V)$ and m for the rank of $O(W,B)$.*

(1) *Say the reductive dual pair is in the symplectic metastable range. Consider an irreducible representation π of $O(W,B)$. Consider the alterable part $u_{alt} = \binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ of its unipotent data. Consider $m'_\pi = m - N + \lfloor \frac{a+b}{2} \rfloor$ (which the metastable range condition ensures is non-negative). Say the extended sign data of π's classification data corresponds to concatenating m'_π to the end of the λ_i (resp. μ_j) row of u_{alt}.*
   (a) *Suppose the extended sign data of π corresponds to adding N'_ρ to the first row of the alterable symbol for some $i \leq a+1$, we have $\lambda_{i-1} < N'_\rho < \lambda_i$ (setting $\lambda_0 = 0$ and $\lambda_{a+1} = \infty$). Then we set $\boldsymbol{\eta}_V^{W,B}(\pi)$ to be the genuine representation obtained from (4.2.9) or (4.2.17) with the factor*

$$\boldsymbol{\phi}^{(\pm)}(u) = \begin{pmatrix} \lambda_1 < \cdots < \lambda_{i-1} < N'_\rho < \lambda_i < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

   instead of $\phi^{(\pm)}(u)$, multiplied by the coefficient $(-1)^{a-i}$ in $K(Rep(Sp(V)))$.
   (b) *Otherwise, there exists a $1 \leq i \leq a$ so that $N'_\rho = \lambda_i$, in which case we put*

$$\boldsymbol{\eta}_V^{W,B}(\pi) = 0 \in K(Rep(Sp(V)))$$

(2) *Say the reductive dual pair is in the orthogonal metastable range. Consider an irreducible representation ρ of $Sp(V)$. Consider the alterable part $u_{alt} = \binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ of its unipotent data. Consider $N'_\rho = N - m + \lfloor \frac{a+b}{2} \rfloor$ (which the metastable range condition ensures is non-negative). For simplicity, switch rows so that the central sign data of ρ's classification data corresponds to concatenating N'_ρ to the end of the λ_i row of u_{alt}.*
   (a) *Suppose the central sign data of ρ corresponds to adding m'_ρ to the first row of the alterable symbol for some $i \leq a+1$, we have $\lambda_{i-1} < m'_\pi < \lambda_i$ (setting $\lambda_0 = 0$ and $\lambda_{a+1} = \infty$). Then we set $\boldsymbol{\zeta}_{W,B}^V(\rho)$ to be the genuine representation*

obtained from (4.2.26) or (4.2.32) with the factor

$$\boldsymbol{\psi}^{(\pm)}(u) = \begin{pmatrix} \lambda_1 < \dots < \lambda_{i-1} < N'_\rho < \lambda_i < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}$$

instead of $\psi^{(\pm)}(u)$*, multiplied by the coefficient* $(-1)^{a-i}$ *in* $K(Rep(O(W,B)))$.

(b) *Otherwise, there exists a* $1 \leq i \leq a$ *so that* $m'_\rho = \lambda_i$*, in which case we put*

$$\boldsymbol{\zeta}^V_{W,B}(\rho) = 0 \in K(Rep(O(W,B)))$$

In particular, note that the virtual eta and zeta correspondences always output a sign times an irreducible representation. We note that for a representation which is sent to zero by the original eta or zeta correspondence, the virtual eta or zeta correspondence outputs a sign of an irreducible representation already in the image of the genuine eta or zeta correspondence for a smaller orthogonal or symplectic group (respectively). Also note that if $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ is in the symplectic (resp. orthogonal) stable range, we take the virtual eta and zeta correspondence $\boldsymbol{\eta}^{W,B}_V$ (resp. $\boldsymbol{\zeta}^V_{W,B}$) to be equal to $\eta^{W,B}_V$ (resp. $\zeta^V_{W,B}$).

Using the virtual eta and zeta correspondences, we can make a first version of the decomposition of the restricted oscillator representation to a reductive dual pair in the symplectic or orthogonal metastable range.

6.2.4. THEOREM. *Fix a reductive dual pair* $(Sp(V), O(W,B))$.

(1) *If* $(Sp(V), O(W,B))$ *is in the symplectic metastable range, then the restriction of* $\omega[V \otimes W]$ *to* $Sp(V) \times O(W,B)$ *can be expressed as*

$$\bigoplus_{k=0}^{h_W} \bigoplus_{\pi \in \widehat{O(W[-k], B[-k])}} \boldsymbol{\eta}^{W[-k],B[-k]}_V(\pi) \otimes Ind_{P^k_{O(W,B)}}(\pi^-) \tag{6.2.1}$$

(using the same notation for the maximal parabolic $P^k_{O(W,B)}$ *and the signed representations* π^- *as in Theorem 2.1.4).*

(2) *If* $(Sp(V), O(W,B))$ *is in the orthogonal metastable range, then the restriction of* $\omega[V \otimes W]$ *to* $Sp(V) \times O(W,B)$ *can be expressed as*

$$\bigoplus_{k=0}^{dim(V)/2} \bigoplus_{\rho \in \widehat{Sp(V[-k])}} Ind_{P^k_{Sp(V)}}(\rho^-) \otimes \boldsymbol{\zeta}^{V[-k]}_{W,B}(\rho) \tag{6.2.2}$$

(again, using the same notation as in Theorem 2.1.4).

To process the virtual sums (6.2.1) and (6.2.2) further into a genuine decomposition of the restriction of $\omega[V \otimes W]$, we must replace the parabolic induction coefficient of the virtual eta and zeta correspondence terms by alternating sums of parabolics. We describe these new coefficients now.

6.2.5. *Alternating sums of parabolic inductions.* Now we define the alternating sums of parabolic inductions which we replace the parabolic induction coefficient by in (6.2.1) and (6.2.2) to cancel the terms corresponding to $\boldsymbol{\eta}_V^{W,B}(\pi) \neq 0$ when $\eta_V^{W,B}(\pi) = 0$ and terms corresponding to $\boldsymbol{\zeta}_{W,B}^V(\rho) \neq 0$ when $\zeta_{W,B}^V(\rho) = 0$. The description given in this paragraph is somewhat technical, but in each case, the principle is to preserve all of the classification data, except for the symbol factor of the unipotent part that is altered in the construction of the extended eta or zeta correspondence. We take the sum of the representations obtained by replacing that symbol by those appearing in the alternating sum of parabolic inductions of the symbols obtained by concentenating a final coordinate to one of the rows as in the construction of $\eta_{W,B}^V$ or $\zeta_V^{W,B}$, and removing another coordinate in the same row to recover the original symbol's row lengths (see (6.2.5) and (6.2.8) below).

Suppose we want to consider the decomposition of the restricted oscillator representation $\mathrm{Res}_{\mathrm{Sp}(V)\times \mathrm{O}(W,B)}(\omega[V \otimes W])$. Fix our choice of reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$.

Consider an input representation which is mapped to a non-zero representation by the extended eta correspondence $\eta_V^{W,B}$ (if the pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ is in the symplectic metastable range) or the extended zeta correspondence $\zeta_{W,B}^V$ (if $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ is in the orthogonal metastable range). Consider the symbol corresponding to the unipotent alterable data

$$u_{\mathrm{alt}} = \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \tag{6.2.3}$$

forming a unipotent irreducible representation of the group $K[-k]$, where K is the factor of the centralizer of the semisimple data (s) of an input representation (of $\mathrm{O}(W,B)$ in the case of a pair in the symplectic metastable range or of $\mathrm{Sp}(V)$ in case of a pair in the orthogonal metastable range) corresponding to the alterable semisimple data (s_{alt}). We use the notation $[-k]$ to refer to the group of the same type and subtracting k from the rank (e.g. for $K = \mathrm{SO}(W,B)$, we write $K[-k] = \mathrm{SO}(W[-k], B[-k])$). Let us switch rows in (6.2.3) so that in the construction of the extended eta or zeta correspondence, ϕ^+ or ψ^+ concatenates a new coordinate to the top row $\lambda_1 < \cdots < \lambda_a$ of (6.2.3) (this corresponds to a condition on the row lengths a, b, which varies depending on the case of the extended eta or zeta correspondence we consider). We do this in order to consolidate the notation and treat every case at once.

Fix a non-negative integer k and a natural number N' satisfying

$$\lambda_1 < \lambda_2 < \cdots < \lambda_a < N'. \tag{6.2.4}$$

Consider the index $1 \leq c \leq a$ such that

$$N' - \lambda_c \leq k < N' - \lambda_{c+1}.$$

Then, for every choice of i such that $c \leq i \leq a+1$, let us consider the symbol ${u_{\mathrm{alt}}}^+(N')_i$ with top row given by (6.2.4) with the ith coordinate removed, and with the same bottom row as u_{alt} in (6.2.3). For each $c \leq i \leq a$ this gives

$$u^+_{\mathrm{alt}}(N')_i := \begin{pmatrix} \lambda_1 < \cdots < \widehat{\lambda}_i < \cdots < \lambda_a < N' \\ \mu_1 < \cdots < \mu_b \end{pmatrix}. \tag{6.2.5}$$

In the case of $q = a+1$, we remove the $(a+1)$th coordinate of (6.2.4), which is N' itself, and thus we get

$$u^+_{\mathrm{alt}}(N')_{a+1} = u_{\mathrm{alt}}.$$

Each of the symbols (6.2.5) has the same row lengths as the original symbol u_{alt}. Therefore, $u^+_{\mathrm{alt}}(N')_i$ describes a unipotent representation of $K[-(k - N' + \lambda_i))]$. We may hence consider the alternating sum

$$A^+_k(u_{\mathrm{alt}}, N') := \bigoplus_{i=c}^{a+1} (-1)^{a-i+1} \cdot \mathrm{Ind}_{P^K_{k-N'+\lambda_i}}(u^+_{\mathrm{alt}}(N')_i) \tag{6.2.6}$$

where we write P^K_j for the standard maximal parabolic in the group K with Levi factor $K[-j] \times GL_j(\mathbb{F}_q)$, and we take trivial $GL_j(\mathbb{F}_q)$-action in each induction term (and, as usual, inflate from the Levi factor to the full parabolic by taking the unipotent factor to act trivially).

Entirely symmetrically, for a choice of natural number N' such that

$$\mu_1 < \mu_2 < \cdots < \mu_b < N', \tag{6.2.7}$$

we may consider the index of the bottom row $1 \leq c \leq b$ such that

$$N' - \mu_c \leq k < N' - \mu_{c+1}.$$

Then for every choice of i satsifying $c \leq i \leq b+1$, we may construct symbols with the same top row as u_{alt} in (6.2.3) and bottom row obtained by removing the ith entry in the sequence (6.2.7). For $c \leq i \leq b$

this gives

$$u^{-}_{\mathrm{alt}}(N')_i := \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \widehat{\mu_i} < \cdots < \mu_b < N' \end{pmatrix}. \tag{6.2.8}$$

Again, in the case of $i = b+1$, we have

$$u^{-}_{\mathrm{alt}}(N')_{b+1} = u_{\mathrm{alt}}.$$

We can then define the alternating sum

$$A^{-}_k(u_{\mathrm{alt}}, N') := \bigoplus_{i=c}^{b+1} (-1)^{b-i+1} \cdot \mathrm{Ind}_{P_{k-N'+\mu_i}}(u^{-}_{\mathrm{alt}}(N')_i). \tag{6.2.9}$$

(using the same notation as in (6.2.6)).

We find that (6.2.6) and (6.2.9), in every case we consider, define genuine representations. We give a concrete description of which symbols appear in their decompositions in Chapter 7.

Again, we approach the case of the extended eta correspondence first.

6.2.6. Definition. *Consider a choice of V and (W, B) in the symplectic (stable or) metastable range. Suppose we are given $0 \leq k \leq h_W$ and $N' > 0$. Consider an irreducible representation ρ of the group $O(W[-k], B[-k])$.*

(1) *Suppose $dim(W) = 2m+1$ is odd. Consider the classifcation data so that $\pi = r^{O(W[-k],B[-k])}[(s), u]^{\alpha}$, and write u_{alt} for the alterable part of π's unipotent data with respect to the reductive dual pair $(Sp(V), O(W[-k], B[-k]))$, and write $u = u_{H^*} \otimes u_{alt}$. Then define $\mathcal{A}_k(\rho, N')$ by*

$$\mathcal{A}_k(\rho, N') = \bigoplus_{\chi \subset A^{\pm}_k(u_{alt}, N')} r^{O(W,B)}[(s \oplus -I_{2k}), u_{H^*} \otimes \chi]^{\alpha}, \tag{6.2.10}$$

with the sum running over every distinct irreducible unipotent summand χ appearing in the decomposition of $A^{\alpha}_k(u_{alt}, N')$.

(2) *Suppose $dim(W) = 2m$ is even. Consider the classification data so that $\pi = r^{O(W[-k],B[-k])}[(s), u]^{\gamma}$, and write u_{alt} for the alterable part of π's unipotent data with respect to the reductive dual pair $(Sp(V), O(W[-k], B[-k]))$, and write $u = u_{H^*} \otimes u_{alt}$. Then define $\mathcal{A}_k(\rho, N')$ by*

$$\mathcal{A}_k(\rho, N') = \bigoplus_{\chi \subset A^{\gamma\alpha}_k(u_{alt}, N')} r^{O(W,B)}[(s \oplus I_{2k}), u_{H^*} \otimes \chi]^{\gamma} \tag{6.2.11}$$

with the sum running over every distinct irreducible unipotent χ *appearing in* $A_k^{\gamma_\alpha}(u_{alt}, N')$.

Similarly, consider the orthogonal metastable range. We note that we still need the separate the cases of the parity of the dimension of the orthogonal space W, though the role of W is somewhat hidden in the notation.

6.2.7. Definition. *Consider a choice of* V *and* (W, B) *in the orthogonal (stable or) metastable range, and write* $dim(V) = 2N$. *Suppose we are given* $0 \leq k \leq N$ *and* $m' > 0$. *Consider an irreducible representation* $\rho \in \widehat{Sp(V[-k])}$.

(1) *Suppose* $dim(W) = 2m + 1$ *is odd. Consider the classification data so that* $\rho = r^{Sp(V[-k])}[(s), u, \alpha]$, *and write* u_{alt} *for the alterable part of* ρ*'s unipotent data with respect to the reductive dual pair* $(Sp(V[-k]), O(W, B))$, *and write* $u = u_{H^*} \otimes u_{alt}$. *Then define* $\mathcal{A}_k(\rho, m')$ *to be the* $Sp(V)$*-representation*

$$\mathcal{A}_k(\rho, m') = \bigoplus_{\chi \subset A_k^{\alpha}(u_{alt}, m')} r^{Sp(V)}[(s \oplus -I_{2k}), u_{H^*} \otimes \chi, \alpha] \tag{6.2.12}$$

with the sum running over every distinct irreducible unipotent χ *appearing in* $A_k^{\alpha}(u_{alt}, m')$.

(2) *Suppose* $dim(W) = 2m$ *is even. Write the classification data of* ρ *as* $\rho = r^{Sp(V[-k])}[(s), u, \alpha]$, *and write* u_{alt} *for the alterable part of* ρ*'s unipotent data with respect to the reductive dual pair* $(Sp(V[-k]), O(W, B))$, *and write* $u = u_{H^*} \otimes u_{alt}$. *Then define* $\mathcal{A}_k(\rho, m')$ *to be the* $Sp(V)$*-representation*

$$\mathcal{A}_k(\rho, m') = \bigoplus_{\chi \subset A_k^{\alpha}(u_{alt}, m')} r^{Sp(V)}[(s \oplus I_{2k}), u_{H^*} \otimes \chi, \alpha] \tag{6.2.13}$$

with the sum running over every distinct irreducible unipotent χ *appearing in* $A_k^{\pm}(u_{alt}, m')$.

Now that we have established the necessary notation to describe the terms of the decomposition of a restricted oscillator representation claimed in the introduction to this chapter, we may restate it concretely.

6.2.8. Theorem. *Fix a reductive dual pair* $(Sp(V), O(W, B))$.

(1) *If* $(Sp(V), O(W,B))$ *is in the symplectic metastable range, then*

$$Res_{Sp(V)\times O(W,B)}(\omega[V\otimes W]) =$$

$$\bigoplus_{k=0}^{h_B} \bigoplus_{\pi\in \widehat{O(W[-k],B[-k])}} \mathcal{A}_k(\pi, N'_\pi)\otimes \eta^V_{W,B}(\pi). \tag{6.2.14}$$

(2) *If* $(Sp(V), O(W,B))$ *is in the orthogonal metastable range, then*

$$Res_{Sp(V)\times O(W,B)}(\omega[V\otimes W]) =$$

$$\bigoplus_{k=0}^{N} \bigoplus_{\rho\in \widehat{Sp(V[-k])}} \zeta^{W,B}_V(\rho)\otimes \mathcal{A}_k(\rho, m'_\rho) \tag{6.2.15}$$

First, we note that given Theorem 6.2.4, Theorem 6.2.8 follows elementarily by reconciling the cancelled terms:

Proof of Theorem 6.2.8. We assume Theorem 6.2.4. It remains to reconcile this cancellation with the claimed answer. We do this in the symplectic metastable case for odd-dimensional orthogonal groups (every other case is similar). In this case, the restriction of the oscillator representation $\omega[V\otimes W]$ to $\mathrm{Sp}(V)\times \mathrm{O}(W,B)$ can be expressed as (6.2.1). Consider a term arising from $\pi\in \widehat{\mathrm{O}(W[-k],B[-k])}$ such that

$$\eta^{W,B}_V(\pi)\neq 0. \tag{6.2.16}$$

We must collect all terms of (6.2.1) whose $\mathrm{Sp}(V)$-representation factor is equal to a sign multiple of $\eta^{W,B}_V(\pi)$. Similarly as in the stable case, we find that the images of the extended eta correspondences are always disjoint when $\phi^{W,B}_V$ is constructible, meaning

$$\bigcap_{k=0}^{h_B} \mathrm{Im}(\eta_V^{W[-k],B[-k]}) = \{0\}.$$

Therefore the only terms of (6.2.1) with $\mathrm{Sp}(V)$-representation factor equal to a sign multiple of $\eta^{W,B}_V(\pi)$ are those of the form

$$\boldsymbol{\eta}_V^{W[-j],B[-j]}(\pi')\otimes \mathrm{Ind}_{P^j_{\mathrm{O}(W,B)}}((\pi')^-)$$

for $0\leq j\leq h_B$ and $\pi'\in \widehat{\mathrm{O}(W[-j],B[-j])}$ satisfying

$$\eta^{W,B}_V(\pi')=0,\quad \boldsymbol{\eta}^{W,B}_V(\pi')=\pm\eta^{W,B}_V(\pi). \tag{6.2.17}$$

Consider the classification data so that

$$\pi = r^{\mathrm{O}(W[-k],B[-k])}[(s),u]^\alpha.$$

To satisfy (6.2.17), an irreducible representation π' must first have classification data $\pi' = r^{O(W[-j],B[-j])}[(s'),u']^{\alpha'}$ so that the non-alterable parts of s' and s (i.e. the sum of blocks corresponding to eigenvalues not equal to -1) match, and writing $u = u_{H^*} \otimes u_{\text{alt}}$, $u' = u'_{(H')^*} \otimes u'_{\text{alt}}$, the unipotent data satisfies

$$u'_{(H')^*} = u_{H^*}$$

(note that the factors H, H' of the centralizers of s and s' are equal by the condition on the semisimple data), and the extension sign data α, α' match. Writing $Z_{\mathrm{Sp}_{2(m-k)}(\mathbb{F}_q)}(s) = H \times \mathrm{Sp}_{2\ell}(\mathbb{F}_q)$, we then have

$$Z_{\mathrm{Sp}_{2(m-j)}(\mathbb{F}_q)}(s') = H \times \mathrm{Sp}_{2(\ell+k-j)}(\mathbb{F}_q).$$

Let us denote the symbol corresponding to the alterable part of π's unipotent data by $u_{\text{alt}} = \binom{\lambda_1<\dots<\lambda_a}{\mu_1<\dots<\mu_b}$, switching rows to assume without loss of generality that in the construction of $\eta_V^{W,B}(\pi)$, the number $N'_\pi = N-(m-k)+(a+b-1)/2$ is concatenated to the top row (meaning $N'_\pi > \lambda_a$, by our assumption (6.2.16)). The alterable part of (π')'s unipotent data must correspond to a symbol of $\mathrm{Sp}_{2(\ell+k-j)}(\mathbb{F}_q)$ such that the true symbol appearing in $\eta_V^{W,B}(\pi)$

$$\begin{pmatrix} \lambda_1 < \dots < \lambda_a < N'_\pi \\ \mu_1 < \dots < \mu_b \end{pmatrix}$$

has the same bottom row as u'_{alt} and the top row is a shuffle of the sequence obtained from concatenating

$$N'_{\pi'} = N-(m-j)+\frac{a+b-1}{2} = N'_\pi + j - k. \tag{6.2.18}$$

to the end of the top row of u'_{alt}. In other words, the (π')'s alterable unipotent part must be of the form

$$\begin{pmatrix} \lambda_1 < \dots < \widehat{\lambda_i} < \dots < \lambda_a < N'_\pi \\ \mu_1 < \dots < \mu_b \end{pmatrix} \tag{6.2.19}$$

with (6.2.18) equal to λ_i for some $1 \leq i \leq a$, meaning

$$j = k - N'_\pi + \lambda_i. \tag{6.2.20}$$

In this case, we have

$$\boldsymbol{\eta}_V^{W,B}(\pi') = (-1)^{a-i+1} \cdot \eta_V^{W,B}(\pi) \in K(\mathrm{Rep}(O(W,B))).$$

We hence have found that the alterable factors of the unipotent parts of the π' satisfying (6.2.17) are precisely those appearing in $A_k^\alpha(u_{\text{alt}}, N')$ (corresponding to indices $c \leq i \leq a$, since we can only have π' where (6.2.20) is non-negative), with the same induction steps and coinciding signs. The $(a+1)$th term of $A_k^\alpha(u_{\text{alt}}, N')$ corresponds to the alterable factor of the unipotent part of π itself. The non-alterable pieces of data all are carried through the induction without change, and therefore collecting terms gives precisely the summand

$$\mathcal{A}_k(\pi, N'_\pi) \otimes \eta_V^{W,B}(\pi),$$

as claimed in (6.2.14).

□

6.3. Interpolating the stable results. The purpose of this subsection is to describe how the stable decomposition result proved in Chapter 2 can be interpreted in the interpolated category $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$. We find that the objects of $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ appearing in a certain tensor power of oscillator representations can be written down according to an "formally interpolated classification," and can therefore also interpret the results of Chapter 4 in the interpolated representation category.

First let us define subcategories of $\mathscr{C}_B^{int}(t) \subseteq \overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$ isolating the ranges of objects appearing in a certain tensor power of oscillator representations (a "partial pseudo-abelian envelope") corresponding to a specific choice of orthogonal space (W, B). For a fixed irreducible representation $\pi \in \widehat{\mathrm{O}(W,B)}$, we further consider an even smaller subcategory $\mathscr{C}_\pi^{int}(t)$ consisting only of objects that interpolate representations of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ symplectic stable with $\mathrm{O}(W, B)$ whose tensor product with π appears in the restriction of $\omega[\mathbb{F}_q^{2N} \otimes W]$. We denote their images under semisimplification by $\mathscr{S}(\mathscr{C}_B^{int}(t))$, $\mathscr{S}(\mathscr{C}_\rho^{int}(t))$ (we note that here $\mathscr{S}$ denotes the semisimplification quotient functor for the whole category $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2t}(\mathbb{F}_q))$, not the categories $\mathscr{C}_B^{int}(t)$, $\mathscr{C}_\rho^{int}(t)$ which do not even have their own tensor product structure).

More specifically, an object of $\mathscr{C}_B^{int}(t)$ is of the form

$$\eta^t_{W,B}(\rho),$$

for W of dimension $2m+1$ for $m \in \mathbb{N}$, with a form B of discriminant $\mathrm{disc}(B) = \alpha$, and an irreducible representation ρ of $\mathrm{O}(W, B)$. Say that as a representation of $\mathrm{O}(W, B) = \mathrm{SO}_{2m+1}(\mathbb{F}_q) \times \mathbb{Z}/2$, ρ is of the form $r[(s), u] \otimes (\pm 1)$ where $r[(s), u]$ is an irreducible representation of $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$ corresponding to classification data with semisimple part $(s) \in \mathrm{Sp}_{2m}(\mathbb{F}_q) = \mathrm{SO}^*_{2m+1}(\mathbb{F}_q)$ and unipotent part $u \in \widehat{Z_{\mathrm{Sp}^*_{2m}(\mathbb{F}_q)}(s)}_u$. More concretely, further say that s has -1 as an eigenvalue of multiplicity 2ℓ, writing $Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s) = H \times \mathrm{Sp}_{2\ell}(\mathbb{F}_q)$, and

$$u = u_{H^*} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

for $u_{H^*} \in \widehat{H^*}_u$ and $\binom{\lambda_a<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ (switching rows so that $a-b$ is 1 mod 4) specifying a unipotent representation of $SO_{2\ell+1}(\mathbb{F}_q) = Sp^*_{2\ell}(\mathbb{F}_q)$. Then, for the sign $+$, we say that $\eta^t_{W,B}(r[(s),u]\otimes(+1))$ corresponds to "interpolated classification data"

$$[\phi^+(s), \widetilde{u_{H^*}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a & \\ \mu_1 < \cdots < \mu_b & t'_\rho \end{pmatrix}], \tag{6.3.1}$$

writing $t'_\rho = t - m + (a+b-1)/2$. This is exactly the classification data of a stable range eta correspondence $\eta^t_{W,B}(\rho)$ for $dim(V) = 2N$, with N replaced by t (we omit the final $<$ sign in the second row of the symbol notation, since at an interpolated value of t, writing $\mu_b < t'_\rho$ may be false or incomparable). Again, $Sp_{2t}(\mathbb{F}_q)$ is not a genuine group, and writing $\phi^+(s)$ indicates an element with finitely many eigenvalues not equal to -1 (which would contribute genuine factors in its "centralizer") and has -1 as an eigenvalue of "multiplicity $2(t-m+\ell)$." Interpolating the stable formula one would obtain for $Sp_{2N}(\mathbb{F}_q)$, replacing N by t, its dimension is

$$\dim(\eta_t^{W,B}(r^{O(W,B)}[(s),u,+1])) = \dim(r^{O(W,B)}[(s),u,+1])\cdot \frac{\prod_{i=t'+1}^{t}(q^{2i}-1)\cdot\prod_{i=1}^{a}(q^{t'_\rho}+q^{\lambda_i})\cdot\prod_{i=1}^{b}(q^{t'_\rho}-q^{\mu_i})}{2\cdot q^{(a+b-1)(a+b+1)/4}\cdot|SO_{2m+1}(\mathbb{F}_q)|_{q'}}. \tag{6.3.2}$$

Similarly, at the sign -1, we say that $\eta^t_{W,B}(r[(s),u]\otimes(-1))$ corresponds to "interpolated classification data"

$$[\phi^-(s), \widetilde{u_{H^*}} \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a & t'_\rho \\ \mu_1 < \cdots < \mu_b & \end{pmatrix}], \tag{6.3.3}$$

writing $t'_\rho = t-m+(a+b-1)/2$. Interpolating the stable formula one would obtain for $Sp_{2N}(\mathbb{F}_q)$, replacing N by t, its dimension is

$$\dim(\eta_t^{W,B}(r^{O(W,B)}[(s),u,-1])) = \dim(r^{O(W,B)}[(s),u,-1])\cdot \frac{dim(\rho)\cdot\prod_{i=t'+1}^{t}(q^{2i}-1)\cdot\prod_{i=1}^{a}(q^{t'_\rho}-q^{\lambda_i})\cdot\prod_{i=1}^{b}(q^{t'_\rho}+q^{\mu_i})}{2\cdot q^{(a+b-1)(a+b+1)/4}\cdot|SO_{2m+1}(\mathbb{F}_q)|_{q'}}. \tag{6.3.4}$$

Considering the "restriction" functors

$$\mathbf{Res} : \mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2nt}(\mathbb{F}_q))) \to \mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))) \boxtimes Rep(O(W,B))$$

$$\mathbf{Res} : \mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2nt}(\mathbb{F}_q))) \to Rep(Sp(V)) \boxtimes \mathscr{S}(\mathfrak{Rep}(O_t(\mathbb{F}_q))),$$

the interpolation of Theorem 2.1.4 can then be stated as follows:

6.3.1. Theorem. *Fix a choice of $t \in \mathbb{C}$.*

(1) *Fix an orthogonal space (W, B). In the semisimplification $\mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q)))$, the original decomposition of $\mathbf{Res}(\boldsymbol{\omega}_1)$ as*

$$\bigoplus_{k=0}^{h_B} \bigoplus_{\pi \in \widehat{O(W[-k], B[-k])}} \eta_t^{W,B}(\pi) \boxtimes Ind_{P_k^{O(W,B)}}(\pi^-) \tag{6.3.5}$$

holds, as objects of $\mathscr{S}(\overline{\mathfrak{Rep}}(Sp_{2t}(\mathbb{F}_q))) \boxtimes Rep(O(W, B))$.

(2) *In the semisimplification $\mathscr{S}(\overline{\mathfrak{Rep}}(O_t(\mathbb{F}_q)))$, the original decomposition of $\mathbf{Res}(\boldsymbol{\omega}_1)$ as*

$$\bigoplus_{k=0}^{dim(V)/2} \bigoplus_{\rho \in \widehat{Sp(V[-k])}} Ind_{P_k^{Sp(V)}}(\rho^-) \boxtimes \zeta_t^V(\rho) \tag{6.3.6}$$

holds, as objects of $Rep(Sp(V)) \boxtimes \mathscr{S}(\mathfrak{Rep}(O_t(\mathbb{F}_q)))$.

In summary, the objects of $\mathscr{C}_B^{int}(t)$ are precisely direct sums of all

$$\eta^t_{W[-k],B[-k]}(\rho) \tag{6.3.7}$$

for irreducible representations $\rho \in \widehat{\mathrm{O}(W[-k], B[-k])}$. For a fixed irreducible representation ρ of $\mathrm{O}(W, B)$, the objects of $\mathscr{C}_\rho^{int}(t)$ consist of direct sums of objects $\eta^t_{W[-k],B[-k]}(\rho')$ corresponding to irreducible representations $\rho' \in \widehat{\mathrm{O}(W[-k], B[-k])}$ such that ρ is a summand of the parabolic induction

$$\rho \subseteq \mathrm{Ind}_{P_k}(\rho' \otimes \epsilon(\det))$$

writing $P_k \subseteq \mathrm{O}(W, B)$ for the maximal parabolic subgroup with Levi factor $\mathrm{O}(W[-k], B[-k]) \times \mathrm{GL}_k(\mathbb{F}_q)$, considering $\epsilon(\det)$ as a representation of the factor $\mathrm{GL}_k(\mathbb{F}_q)$.

At $t = N$ corresponding to a reductive dual pair $(\mathrm{Sp}_{2N}(\mathbb{F}_q), \mathrm{O}(W, B))$ in the symplectic metastable range, the semisimplification images $\widetilde{\mathscr{C}}_B(N)$, $\widetilde{\mathscr{C}}_\rho(N)$ are semisimple categories with objects consisting of direct sums of simple objects corresponding to all formal interpolated classification, eliminating 0-dimensional objects, which occur precisely when $N'_\rho = \lambda_i$ or μ_i for some i in (6.3.1) or (6.3.3). Note that the remaining formal interpolated classification objects, say where $\lambda_i < N'_\rho < \lambda_{i+1}$ in (6.3.1) or $\mu_i < N'_\rho < \mu_{i+1}$ in (6.3.3), have dimension equal to a genuine irreducible $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representation where N'_ρ is inserted in the appropriate place, multiplied by $(-1)^{a-i}$ or $(-1)^{b-i}$. Call this the *true permutation* of the formal interpolated Lusztig data as $t = N$.

6.4. The proof of the full explicit decomposition theorem. It remains to relate the results of the previous subsection to true representations of a symplectic or orthogonal group for integer values of t.

It then suffices to prove that at $t = N$ with $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ in the symplectic metastable range with $\mathrm{O}(W, B)$, the relationship between an interpolated category $\widetilde{\mathscr{C}}_\rho^{int}(N)$ and the subcategory of genuine $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representations π such that $\rho \otimes \pi$ appears in the restricted oscillator representation.

Fix a choice of reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the symplectic metastable range. Applying Theorem 6.3.1 to $t = N$ gives a decomposition in the semisimplification $\mathscr{S}(\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2(t=N)}(\mathbb{F}_q)))$ in terms of objects of $\widetilde{\mathscr{C}}_B(N)$. We must relate this category to genuine $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representations, so that the corresponding decomposition holds of $\omega[V \otimes W]$ in the category of virtual $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$-representations holds, replacing the formal classification data of an object in the category $\mathscr{S}(\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2(t=N)}(\mathbb{F}_q)))$ by its true permutation, with the corresponding sign and interpreting the result as an object in the Grothendieck group $K(\mathrm{Rep}(\mathrm{Sp}_{2N}(\mathbb{F}_q)))$ (precisely matching the construction of the virtual eta correspondence).

For each $\pi \in \widehat{\mathrm{O}(W, B)}$, consider functors

$$\Phi_\pi : \mathscr{C}_\pi^{\mathrm{gen}}(N) \to \widetilde{\mathscr{C}}_\pi^{\mathrm{int}}(N)$$

defined as follows: Consider a simple object π of the source, such that

$$\pi \otimes \rho \subseteq \mathrm{Res}_{\mathrm{Sp}(V) \times \mathrm{O}(W,B)}(\omega[V \otimes W]).$$

We may consider an idempotent ι_π in the $\mathrm{Sp}(V)$-equviariant endomorphism algebra of $\mathrm{Res}_{\mathrm{Sp}(V)}(\omega[V \otimes W]) \cong \omega^{\otimes B}$ whose image is one of these copies of π. By duality, since each oscillator representation has $\omega_a \otimes (\omega_a)^\vee \cong \mathbb{C}V)$, we may conisder

$$\iota_\pi \in \mathrm{End}_{\mathrm{Sp}(V)}(\omega^{\otimes B}) \cong \mathrm{Hom}_{\mathrm{Sp}(V)}(1, (\mathbb{C}V)^{\otimes B}) = (\mathbb{C}(V \otimes W))^{\mathrm{Sp}(V)}$$

as a linear combination of $\mathrm{Sp}(V)$-orbits on $V \otimes W = V^{\oplus n}$ (recall that an $\mathrm{Sp}(V)$-orbit consits of a set of n-tuples of vectors $(v_1, \dots, v_n)$ satisfying some linear (in)dependence conditions, and equations for the values of the symplectic form on pairs of them). Theses orbits can also be considered as orbits of any $\mathrm{Sp}_{2M}(\mathbb{F}_q)$ acting on $(\mathbb{F}_q^{2M})^{\oplus n}$ for any higher

M, and therefore ι_π corresponds to an interpolated endomorphism

$$\mathrm{End}_{\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2N}(\mathbb{F}_q))}(\boldsymbol{\omega}^{\otimes B}) \cong ((\mathbb{C}(\mathbb{F}_q^{2M} \otimes W))^{\mathrm{Sp}_{2M}(\mathbb{F}_q)}$$

for $M >> n$ (by the definition of $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2N}(\mathbb{F}_q))$). Since partial trace operations (and therefore compositions) are computed the same in $Rep(\mathrm{Sp}_{2N}(\mathbb{F}_q))$ and $\overline{\mathfrak{Rep}}(\mathrm{Sp}_{2N}(\mathbb{F}_q))$ (the difference between them arising instead from certain morphisms in the interpolated category not existing in the genuine category), this new endomorphism is still an idempotent, with image equal to an object in $\widetilde{\mathscr{C}_\rho^{int}}(N)$ of the same dimension as π. We put $\Phi(\pi)$ to be this object.

On the other hand, define

$$\Psi : K(\widetilde{\mathscr{C}_\rho^{int}}(N)) \to K(\mathscr{C}_\rho^{gen}(N)) \tag{6.4.1}$$

by sending a simple object in $\widetilde{\mathscr{C}_\rho^{int}}(N)$ which must be of the form (6.3.1) or (6.3.3) at $t = N$ for some choice of s, u, $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ to $(-1)^{b-i}$, where i is the index so that $\mu_i < t'_\rho < \mu_{i+1}$ or $\lambda_i < t'_\rho < \lambda_{i+1}$, times the genuine irreducible representation of $\mathrm{Sp}(V)$ whose Lusztig calssification data is the same as (6.3.1) or (6.3.3) with the formal interpolated symbol replaced by

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_i < t'_\rho < \mu_{i+1} < \cdots < \mu_b \end{pmatrix}$$

or

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_i < t'_\rho < \lambda_{i+1} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix},$$

respectively. In other words, we them precisely to their signed true permutations.

6.4.1. Proposition. *The composition of*

$$K(\mathscr{C}_\rho^{gen}(N)) \xrightarrow{K(\Phi)} K(\widetilde{\mathscr{C}_\rho^{int}}(N)) \xrightarrow{\Psi} K(\mathscr{C}_\rho^{gen}(N))$$

is $Id_{K(\mathscr{C}_\rho^{gen}(N))}$.

Proof. Note that this holds immediately when formal classification data is actually genuine classification data defining a $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representation, since both $K(\Phi)$ and Ψ act as the identity on these objects, considered in either categories. The general statement follows since dimensions are preserved by Ψ and Φ, and it is not possible for $\Psi \circ K(\Phi)$ when applied to an irreducible representation of $\mathscr{C}_\rho^{gen}(N)$ to output a linear combination of multiple different irreducible representations in $Rep(\mathrm{Sp}_{2N}(\mathbb{F}_q))$ with integer coefficients, both by dimension and the fact that it would violate the decomposition (6.3.5).

□

If $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the symplectic metastable range, we then find that the virtual eta correspondence satisfies

$$\boldsymbol{\eta}_V^{W,B} \circ \Phi = \Psi \circ \eta_{t=N}^{W,B},$$

where $\eta_{t=N}^{W,B}$ denotes the interpolated eta correspondence applied at $t = N$, and Ψ is defined as in (6.4.1) Similarly, if $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ is in the orthogonal metastable range, we then find the virtual zeta correspondence

$$\boldsymbol{\zeta}_{W,B}^V \circ \Phi = \Psi \circ \zeta_{t=N}^V,$$

where $\zeta_{t=N}^V$ denotes the interpolated eta correspondence applied at $t = N$.

Therefore, the decomposition of the restricted oscillator representation as a genuine representation of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ can be obtained from (6.3.5) by applying Ψ, and cancelling terms as in $K(\mathscr{C}_B(N))$.

We also note that in concrete cases, Theorems 1, 2 can also be checked elementarily, using observations about how the dimensions of endomorphism algebras of tensor powers of oscillator representations degenerate in the degrees corresponding to metastable reductive dual pairs:

6.4.2. Example. *Consider the reductive dual pair* $(SL_2(\mathbb{F}_q), O(\mathbb{F}_q^3, B))$, *for a* 3*-dimensional orthogonal form with discriminant a.*

CHAPTER 7

The complete picture

In this chapter, we finally complete the picture of Howe duality over finite fields. In particular, we identiffy precisely where the pairs of representations identified by S. Y. Pan [**50, 51**] occur in this correspondence, proving that their multiplicity is always 1.

We will also show in Subsection 7.5 below how, inductively the eta and zeta correspondence can be used to construct explicitly the cuspidal representations of $O(W, B)$, $Sp(V)$, and also give an inductive method for computing their characters.

The work left in practical terms is, of course, to resolve concretely the alternating sums identified in the last chapter. In Theorem 6.2.8, we decompose the restriction $Res_{Sp(V)\times O(W,B)}(\omega[V \otimes W])$ of the oscillator representation in terms of the eta correspondence (in the case of the symplectic stable or metastable range, corresponding to the condition (6.1.1)) or the zeta correspondence (in the orthogonal metastable or stable ranges, corresponding to the complementary condition (6.1.2)). Further, we described the eta and zeta correspondences in terms of classification data, allowing us to directly compute the $Sp(V)$-representation and $O(W, B)$-representation summands occuring in the restriction of $\omega[V \otimes W]$.

However, in (6.2.14) and (6.2.15), the eta and zeta correspondence terms appear with coefficients $\mathcal{A}_k(\rho, N'_\rho)$ (resulting in an $O(W, B)$-representation, for ρ an irreducible $O(W[-k], B[-k])$-representation) or $\mathcal{A}_k(\rho, m'_\rho)$ (resulting in a $Sp(V)$-representations, for ρ an irreducible $Sp(V[-k])$, respectively. Our description of these terms for the purpose of proving the Theorem was as certain alternating sums of parabolic inductions. The purpose of this section is to simplify these sums $\mathcal{A}_k(\rho, N'_\rho)$ (resp. $\mathcal{A}_i(\rho, m'_\rho)$) and describe their irreducible $O(W, B)$- (resp. $Sp(V)$-) representation summands in a way that can be used for concrete computations.

7.1. More about Lusztig symbols. The main statement of this chapter is the following

7.1.1. THEOREM. *Fix a reductive dual pair $(Sp(V), O(W,B))$ in the symplectic stable or metastable range. For an irreducible representation ρ of $O(W[-k], B[-k])$, consider the factor of the unipotent part of its classification data writable as a symbol*

$$\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}, \tag{7.1.1}$$

such that it is replaced by a symbol

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

in the construction of $\eta^V_{W[-k],B[-k]}(\rho)$ (switch rows if necessary). Then the $O(W,B)$-representation $\mathcal{A}_k(\rho, N'_\rho)$ consists of the irreducible summands appearing in the parabolic induction $Ind_{P_k}(\rho \otimes \epsilon(det))$ such that, when performing the Pieri rule (see Proposition 7.2.1) on the row $\lambda_1 < \cdots < \lambda_a$ of (7.3.1), the highest coordinate $\lambda'_{a'}$ of the corresponding row of the new symbol satisfies

$$\lambda'_{a'} < N'_\rho + (a' - a).$$

There is a similar description in the case of $(Sp(V), O(W,B))$ in the orthogonal stable or metastable range, of the $Sp(V)$-representations $\mathcal{A}_i(\rho, m'_\rho)$ for $Sp(V[-i])$-representations ρ.

First, let us briefly recall the role of symbols as representations. In (4.6.2), (4.7.1) of [**44**], Lusztig described how the combinatorial data of a pair of increasing sequences

$$\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b} \tag{7.1.2}$$

corresponds to an irreducible representation of a certain Hecke algebra $\mathscr{H}_n(q, q^d)$ defined according to certain relations (see Subection 4.1 of [**44**]) which are equivalent to the classical Iwahori relations and recover the group algebra of the Weyl group (see [**16**], §68A). In Subsection 4.8 of [**44**], Lusztig also describes how induction is preserved by these correspondences. For the Weyl groups of the groups we consider here, the irreducible representations in each case are classified by pairs of Young diagrams whose total numbers of boxes add up to the rank. Therefore, the induction $Ind_{P_k}(\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b})$ can be computed by applying the Pieri rule to these Young diagrams i.e., by considering all choices of $k_1 + k_2 = k$, and adding a row of length k_1 to the top row's corresponding Young diagram and a row of length k_2 to the bottom row's corresponding Young diagram.

To find the Weyl group representation corresponding to a symbol (7.1.2), without loss of generality, switch rows so that $a \geq b$, and denote the defect by $d = a - b$.

First suppose d is strictly positive. The symbol notation then indicates that the unipotent representation $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ is constructed in an induction of the cuspidal unipotent representation corresponding to the symbol

$$\begin{pmatrix} 0 < 1 < 2 < \cdots < d-1 \\ \emptyset \end{pmatrix}$$

(the minimal rank symbol of defect d). The first step of Lusztig's procedure is to "remove" this cuspidal representation from the symbol (i.e. by undoing the bijection j of Proposition 3.2 of [**44**]), to obtain a defect one symbol

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ 0 < 1 < \cdots < d-2 < \mu_1 + (d-1) < \cdots < \mu_b + (d-1) \end{pmatrix}, \tag{7.1.3}$$

(using the convention of [**44**] describing how to reduce a symbol if the coordinate of its first two rows is 0).

The next step is to undo the procedure described in Subsection 4.6 of [**44**] to obtain Young diagrams. In the case of (7.1.3), we obtain a Young diagram

$$(\lambda_1 \leq \lambda_2 - 1 \leq \cdots \leq \lambda_a - (a-1)) \tag{7.1.4}$$

where the ith row has length $\lambda_{a-i+1} - (a-i)$ corresponding to the top row, and a Young diagram

$$(\mu_1 \leq \mu_2 - 1 \leq \cdots \leq \mu_b - (b-1)) \tag{7.1.5}$$

where the ith row has length $\mu_{b-i+1} - (b-i)$ corresponding to the bottom row (not writing the rows with length 0 corresponding to the coordinates $0 < 1 < \cdots < d-2$ concatenated onto the bottom row in (7.1.3)).

In the case of defect $d = 0$, we undo the procedure in Subsection 4.7 of [**44**] to obtain this same answer, of a Weyl group representation corresponding to Young diagrams (7.1.4), (7.1.5).

We will denote the Young diagrams (7.1.4), (7.1.5), by α, β, denoting the ith row lengths by

$$\begin{aligned} \alpha_i &:= \lambda_{a-i+1} - (a-i) \\ \beta_i &:= \mu_{b-i+1} - (b-i). \end{aligned} \tag{7.1.6}$$

(We use the convention that the first row of the Young diagram is the longest.)

7.2. The Pieri rule. We use the terminology that, for a Young diagram $(\gamma_n \leq \cdots \leq \gamma_1)$ and for a natural number k, we say Young diagrams of the form

$$(k_{n+1} \leq \gamma_n + k_n \leq \gamma_n + k_{n-1} \leq \cdots \leq \gamma_1 + k_1)$$

where k_i are natural numbers satisfying $k_1 + \cdots + k_{n+1} = k$ and, for every $i = 1, \ldots, n$, we have $k_{i+1} \leq \gamma_i - \gamma_{i+1}$ (putting $\gamma_{n+1} = 0$ are the Young diagrams *obtained by adding a row of length* k *to* γ.

The Pieri rule for symbols can then be stated as follows:

7.2.1. Proposition. *Fix an orthogonal space* (W, B) *(resp. a symplectic space* V*) and consider a symbol* $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ *defining a unipotent representation of* $O(W[-k], B[-k])$ *(resp.* $Sp(V[-k])$*). Recall that we denote by* P_k *the standard maximal parabolic with Levi subgroup* $O(W[-k], B[-k]) \times GL_k(\mathbb{F}_q)$ *(resp.* $Sp(V[-k]) \times GL_k(\mathbb{F}_q)$*), and consider the symbol as a* P_k*-representation by letting* $GL_k(\mathbb{F}_q)$ *act trivially and inflating by letting the unipotent part of the parabolic act trivially. Then its parabolic induction to an* $O(W, B)$*-representation decomposes as a sum of symbols*

$$Ind_{P_k}(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}) = \bigoplus_{\mathcal{S}_k[\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}]} \begin{pmatrix} \lambda'_1 < \cdots < \lambda'_{a'} \\ \mu'_1 < \cdots < \mu'_{b'} \end{pmatrix}$$

where the sum runs over the set of symbols $\mathcal{S}_k[\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}]$ *consisting of* $\binom{\lambda'_1 < \cdots < \lambda'_{a'}}{\mu'_1 < \cdots < \mu'_{b'}}$ *where, for some* $k_1 + k_2 = k$*, the Young diagram*

$$(\lambda'_1 \leq \lambda'_2 - 1 \leq \cdots \leq \lambda'_{a'} - (a' - 1))$$

is obtained by adding a row of length k_1 *to*

$$(\lambda_1 \leq \lambda_2 - 1 \leq \cdots \leq \lambda_a - (a - 1)),$$

and the Young diagram

$$(\mu'_1 \leq \mu'_2 - 1 \leq \cdots \leq \mu'_{b'} - (b' - 1))$$

is obtained by adding a row of length k_2 *to*

$$(\mu_1 \leq \mu_2 - 1 \leq \cdots \leq \mu_b - (b - 1)).$$

(The awkwardness of this statement is in order to accomodate all cases of input symbols, including those where one of the rows of the original symbol begins with a 0 coordinate, and to properly address the case when a' and b' are larger than a and b.)

7.3. Resolving the alternating sums. In Theorem 1 and 2, we decompose the restricted oscillator representation $\mathrm{Res}_{\mathrm{Sp}(V)\times \mathrm{O}(W,B)}(\omega[V\otimes W])$ in terms of the eta correspondence (in the case of the symplectic stable or metastable range, corresponding to the condition (6.1.1)) or the zeta correspondence (in the orthogonal metastable or stable ranges, corresponding to the complementary condition (6.1.2)). Further, we described the eta and zeta correspondences in terms of classification data, allowing us to directly compute the $\mathrm{Sp}(V)$-representation and $\mathrm{O}(W,B)$-representation summands occuring in the restriction of $\omega[V\otimes W]$.

However, in (6.2.14) and (6.2.15), the eta and zeta correspondence terms appear with coefficients $\mathcal{A}_k(\rho, N'_\rho)$ (which we recall form representation $\mathrm{O}(W,B)$, for ρ irreducible $\mathrm{O}(W[-k],B[-k])$-representations) or $\mathcal{A}_k(\rho, m'_\rho)$ (giving a $\mathrm{Sp}(V)$-representations, for ρ an irreducible representattion of $\mathrm{Sp}(V[-k])$), respectively. Our description of these terms for the purpose of proving the Theorem was as certain alternating sums of parabolic inductions. The purpose of this section is to simplify these sums $\mathcal{A}_k(\rho, N'_\rho)$ (resp. $\mathcal{A}_i(\rho, m'_\rho)$) and describe their irreducible $\mathrm{O}(W,B)$- (resp. $\mathrm{Sp}(V)$-) representation summands in a way that can be used for concrete computations.

We find

7.3.1. THEOREM. *Fix a reductive dual pair $(Sp(V), O(W,B))$ in the symplectic stable or metastable range. For an irreducible representation ρ of $O(W[-k],B[-k])$, consider the factor of the unipotent part of its classification data writable as a symbol*

$$\begin{pmatrix}\lambda_1<\cdots<\lambda_a\\ \mu_1<\cdots<\mu_b\end{pmatrix}, \tag{7.3.1}$$

such that it is replaced by a symbol

$$\begin{pmatrix}\lambda_1<\cdots<\lambda_a<N'_\rho\\ \mu_1<\cdots<\mu_b\end{pmatrix}$$

in the construction of $\eta^V_{W[-k],B[-k]}(\rho)$ (switch rows if necessary). Then the $O(W,B)$-representation $\mathcal{A}_k(\rho,N'_\rho)$ consists of the irreducible summands appearing in the parabolic induction $Ind_{P_k}(\rho\otimes\epsilon(det))$ such that, when performing the Pieri rule (see Proposition 7.2.1) on the row $\lambda_1<\cdots<\lambda_a$ of (7.3.1), the highest coordinate $\lambda'_{a'}$ of the corresponding row of the new symbol satisfies

$$\lambda'_{a'} < N'_\rho + (a'-a).$$

There is a similar description in the case of $(Sp(V), O(W,B))$ in the orthogonal stable or metastable range, of the $Sp(V)$-representations $\mathcal{A}_i(\rho, m'_\rho)$ for $Sp(V[-i])$-representations ρ.

7.4. A dictionary on notations. The purpose of this subsection is to provide a dictionary between the notation used in this paper, and the notation used by S.-Y. Pan in [**50, 51**].

First note that Pan uses a modified description of the unipotent irreducible representations of the symplectic and orthogonal groups. We consider the classical description of unipotent irreducible representations of a connected group G, say $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$, $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, or $\mathrm{SO}^{\pm}_{2m}(\mathbb{F}_q)$ in terms of Lusztig symbols with unordered rows, so that

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} = \begin{pmatrix} \mu_1 < \cdots < \mu_b \\ \lambda_1 < \cdots < \lambda_a \end{pmatrix} \tag{7.4.1}$$

(requiring, in the case when G is of B- or C-type, simply that the defect is odd). This is sufficient for our discussion of the irreducible unipotent representations of symplectic groups and the odd orthogonal groups (where the center splits off $\mathrm{O}_{2m+1}(\mathbb{F}_q) = \mathbb{Z}/2 \times \mathrm{SO}_{2m+1}(\mathbb{F}_q)$). In the case of D-type, the combinatorial data of symbols (7.4.1), requiring defect to be 0 or 2 mod 4, correspond to the irreducible unipotent representations of $\mathrm{SO}^{+}_{2m}(\mathbb{F}_q)$ and $\mathrm{SO}^{-}_{2m}(\mathbb{F}_q)$.

However, performing an induction from $\mathrm{SO}^{\pm}_{2m}(\mathbb{F}_q)$ to $\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)$ (for $m > 0$), the resulting unipotent representation of $\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)$ of twice the dimension of $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ splits into two non-isomorphic equidimensional pieces. In our notation, we identify these pieces by labelling these unipotent irreducible $\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)$-representations according to their underlying symbol (7.4.1) and the data of a central sign $\pm$ indicating the action of $\mathbb{Z}/2 = \mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)/\mathrm{SO}^{\pm}_{2m}(\mathbb{F}_q)$, determining the relevant piece of the underlying symbol's induction:

$$\mathrm{Ind}^{\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)}_{\mathrm{SO}^{\pm}_{2m}(\mathbb{F}_q)} \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} = r[1, \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}, (+1)] \oplus r[1, \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix}, (-1)].$$

On the other hand, Pan labels these two unipotent irreducible $\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)$-representations by enforcing an ordering on the rows of a symbol (in this paper, we distinguish this notation by using square brackets), so that

$$\mathrm{Ind}^{\mathrm{O}^{\pm}_{2m}(\mathbb{F}_q)}_{\mathrm{SO}^{\pm}_{2m}(\mathbb{F}_q)} \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} = \begin{bmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{bmatrix} \oplus \begin{bmatrix} \mu_1 < \cdots < \mu_b \\ \lambda_1 < \cdots < \lambda_a \end{bmatrix}$$

for non-isomorphic

$$\begin{bmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{bmatrix} \neq \begin{bmatrix} \mu_1 < \cdots < \mu_b \\ \lambda_1 < \cdots < \lambda_a \end{bmatrix}.$$

(We note Pan also changes the order of a symbol's entries in his notation in [**50**], writing each row in strictly decreasing order). Pan also imposes an ordering on the symbols ccorresponding to the unipotent irreducible representations of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ or $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$, demanding then that

$$\begin{bmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{bmatrix}$$

has defect $a-b$ exactly 1 mod 4 (so that there is exactly one ordered symbol corresponding to an odd defect unordered symbol (7.4.1)).

Now, Pan describes in [**51**] the decomposition of the unipotent part of

$$\mathrm{Res}_{\mathrm{Sp}(V)\times \mathrm{O}(W,B)}(\omega[V\otimes W]) \tag{7.4.2}$$

for even-dimensional W as a direct sum of tensor products $\rho\otimes\pi$ for certain pairs of irreducible unipotent representations $\rho\in\widehat{\mathrm{Sp}(V)}$ and $\pi\in\widehat{\mathrm{O}(W,B)}$. Specfically, Pan describes that in the split case $\mathrm{O}(W,B)=\mathrm{O}^+_{2m}(\mathbb{F}_q)$, a tensor product of an ordered symbol of $\mathrm{Sp}(V)$ with an ordered symbol of $\mathrm{O}(W,B)$

$$\begin{bmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{bmatrix} \otimes \begin{bmatrix} \lambda'_1 < \cdots < \lambda'_{a'} \\ \mu'_1 < \cdots < \mu'_{b'} \end{bmatrix} \tag{7.4.3}$$

is a summand of the unipotent part of (7.4.2) precisely when

- The Young diagram with row lengths
$$(\lambda'_{a'}-(a'-1), \lambda_{a'-1}-(a'-2), \dots, \lambda'_1)$$
can be obtained by adding a row (recalling the terminology in Subsection 7.2) to the Young diagram with row lengths
$$(\mu_b-(b-1), \mu_{b-1}-(b-2), \dots, \mu_1).$$
- The Young diagram with row lengths
$$(\lambda_a-(a-1), \lambda_{a-1}-(a-2), \dots, \lambda_1)$$
can be obtained by adding a row to the Young diagram with row lengths
$$(\mu'_{b'}-(b'-1), \mu'_{b'-1}-(b'-2), \dots, \mu'_1).$$
- The defects precisely satisfy $a'-b'=-(a-b)+1$.

Similarly, Pan proves that in the non-split case $\mathrm{O}(W,B)=\mathrm{O}^-_{2m}(\mathbb{F}_q)$, a tensor product (7.4.3) of an ordered symbol of $\mathrm{Sp}(V)$ with an ordered symbol of $\mathrm{O}(W,B)$ is a summand of the unipotent part of (7.4.2) precisely when

- The Young diagram with row lengths
$$(\mu'_{b'} - (b'-1), \mu_{b'-1} - (b'-2), \dots, \mu'_1)$$
can be obtained by adding a row to the Young diagram with row lengths
$$(\lambda_a - (a-1), \lambda_{a-1} - (a-2), \dots, \lambda_1).$$
- The Young diagram with row lengths
$$(\mu_b - (b-1), \mu_{b-1} - (b-2), \dots, \mu_1)$$
can be obtained by adding a row to the Young diagram with row lengths
$$(\lambda'_{a'} - (a'-1), \lambda'_{a'-1} - (a'-2), \dots, \lambda'_1).$$
- The defects precisely satisfy $a' - b' = -(a-b) - 1$.

Therefore, Pan decomposes the unipotent part of the restriction of an oscillator representation to $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$, for even-dimensional W. (Of course, in the case of odd-dimensional W, there is no unipotent part of the restriction of $\omega[V \otimes W]$.)

In [**51**], Pan approaches the question of which pairs $\rho \otimes \pi$ appear with non-zero multiplicity in the restricted oscillator representation for general irreducible representations $\rho \in \widehat{\mathrm{Sp}(V)}$ and $\pi \in \widehat{\mathrm{O}(W, B)}$ by claiming a "commutation with the Lusztig correspondence." In essence, this means that a pair of irreducible representations appears with non-zero multiplicity precisely when a pair of factors (corresponding to a certain eigenvalue of the semisimple part of the classification data) of the unipotent part of their classification data appears in the unipotent part of the appropriate restricted oscillator representation.

We now explain how our decomposition recovers Pan's classification of the occurring summands.

First, we note that our constructions of the pieces of the restricted oscillator representation always only involve "altering" the Lusztig data of an input representation associated to a certain specific eigenvalue of the semisimple data (the eigenvalue 1 for even orthogonal groups and the eigenvalue -1 for odd orthogonal groups). In particular, we can see the effect of "commuting with the Lusztig correspondence" in the sense which Pan uses to pass from his decomposition of the unipotent part of the restricted oscillator representation to information about the irreducible pairs which could appear in the full representation. This reduces us to needing to see that the unipotent part of our decomposition in each case matches the description of Pan's unipotent summands.

We begin with considering $(V,(W,B)$ in the symplectic stable or metastable range. We use the eta correspondence and its extension in this case. From the decomposition given in Theorem 1, we find the unipotent part of the restriction of $\omega[V\otimes W]$ to $\mathrm{Sp}(V)\times \mathrm{O}(W,B)$ is the sum over $k=0,\dots,h_B$ and over every unipotent irreducible representation π of $\mathrm{O}(W[-k],B[-k])$ of summands of the form

$$\eta^V_{W[-k],B[-k]}(\pi)\otimes \mathcal{A}_k(\pi,N'_\pi).$$

Say the restriction of π to $\mathrm{SO}(W[-k],B[-k])$ corresponds to the symbol $\binom{\alpha_1<\dots<\alpha_a}{\beta_1<\dots<\beta_b}$ (we use different notation for the entries here to avoid confusion with Pan's notation), so that $N'_\pi=N-m+\frac{a+b}{2}$. Say $\eta^V_{W,B}(\pi)$ is constructible. Then, depending on the central sign, $\eta^V_{W,B}(\pi)$ is the unipotent irreducible representation of $\mathrm{Sp}(V)$ corresponding to one of the symbols

$$\begin{pmatrix}\alpha_1<\dots<\alpha_a\\ \beta_1<\dots<\beta_b<N'_\pi\end{pmatrix}\text{ or }\begin{pmatrix}\alpha_1<\dots<\alpha_a<N'_\pi\\ \beta_1<\dots<\beta_b\end{pmatrix}. \tag{7.4.4}$$

Now, for the orthogonal groups representation factor, the Harish-Chandra induction of this symbol is a sum of symbols

$$\begin{pmatrix}\alpha'_1<\dots<\alpha'_{a'}\\ \beta'_1<\dots<\beta'_{b'}\end{pmatrix} \tag{7.4.5}$$

such that the Young diagram with row lengths

$$(\alpha'_{a'}-(a'-1),\alpha'_{a'-1}-(a'-2),\dots,\alpha'_1)$$

can be obtained by adding a row to the Young diagram with row lengths

$$(\alpha_a-(a-1),\alpha_{a-1}-(a-2),\dots,\alpha_1),$$

and similarly, the Young diagram with row lengths

$$(\beta'_{b'}-(b'-1),\beta'_{b'-1}-(b'-2),\dots,\beta'_1)$$

can be obtained by adding a row to the Young diagram with row lengths

$$(\beta_b-(b-1),\beta_{b-1}-(b-2),\dots,\beta_1), \tag{7.4.6}$$

according to the Pieri rule. The summands which contribute and survive in the alternating sum $\mathcal{A}_k(\pi,N'_\pi)$ are these symbols (7.4.5) such that

$$\beta'_{b'}-(b'-1)\le N'_\pi-b\text{ or }\alpha'_{a'}-(a'-1)\le N'_\pi-a,$$

respectively (corresponding to (7.4.4)). To understand the relation with Pan's description, we can rephrase the conditions (7.4.6) as demanding that the Young diagram

$$(N'_\pi-a,\alpha_a-(a-1),\alpha_{a-1}-(a-2),\dots,\alpha_1)$$

can be obtained by adding a row to $(\alpha'_{a'}-(a'-1),\alpha'_{a'-1}-(a'-2),\dots,\alpha'_1)$ or, respectively, that the Young diagram

$$(N'_\pi - b, \beta_b - (b-1), \beta_{b-1} - (b-2), \dots, \beta_1)$$

can be obtained by adding a row to $(\beta'_{b'}-(b'-1),\beta'_{b'-1}-(b'-2),\dots,\beta'_1)$.

Let us now suppose we are specifically working in the split case, i.e. $\mathrm{SO}(W,B) = \mathrm{SO}^+_{2m}(\mathbb{F}_q)$, so that $a-b$ is 2 mod 4. The summand we have identified, in the ordered symbol notation, is

$$\begin{bmatrix} \beta_1 < \dots < \beta_b < N'_\pi \\ \alpha_1 < \dots < \alpha_a \end{bmatrix} \otimes \begin{bmatrix} \alpha'_1 < \dots < \alpha'_{a'} \\ \beta'_1 < \dots < \beta'_{b'} \end{bmatrix}$$

or

$$\begin{bmatrix} \alpha_1 < \dots < \alpha_a < N'_\pi \\ \beta_1 < \dots < \beta_b \end{bmatrix} \otimes \begin{bmatrix} \beta'_1 < \dots < \beta'_{b'} \\ \alpha'_1 < \dots < \alpha'_{a'} \end{bmatrix}$$

(with the above corresponding restrictions on α'_i and β'_j), respectively. We can therefore see that by re-labelling the top rows of the symbols as λ and λ' and the bottom rows as μ and μ', we exactly recover the conditions Pan described. The non-split case of $\mathrm{O}^-_{2m}(\mathbb{F}_q)$ proceeds similarly.

Now let us consider $(V,(W,B)$ in the orthogonal stable or metastable range. We use the zeta correspondence and its extension in this case. From the decomposition given in Theorem 2, we find the unipotent part of the restriction of $\omega[V\otimes W]$ to $\mathrm{Sp}(V)\times \mathrm{O}(W,B)$ is the sum over $k=0,\dots,N$ and over every unipotent irreducible representation ρ of $\mathrm{Sp}(V[-k])$ of summands of the form

$$\mathcal{A}_k(\rho, N'_\rho) \otimes \zeta^{W,B}_{V[-k]}(\rho).$$

Say ρ corresponds to a symbol $\binom{\alpha_1<\dots<\alpha_a}{\mu_1<\dots<\mu_b}$, and switch rows so that $a-b$ is 1 mod 4. Then $N'_\rho = N-m+\frac{a+b-1}{2}$. Say $\zeta^V_{W,B}(\rho)$ is constructible. Then, in the split case $\mathrm{O}(W,B) = \mathrm{O}^+_{2m}(\mathbb{F}_q)$, we get that the underlying $\mathrm{SO}^\pm_{2m}(\mathbb{F}_q)$ symbol is

$$\begin{pmatrix} \alpha_1 < \dots < \alpha_a < N'_\rho \\ \mu_1 < \dots < \mu_b \end{pmatrix},$$

with determined central sign data. Recalling the above description of the Pieri rule, the summands surviving in $\mathcal{A}_k(\rho, N'_\rho)$ consist of symbols $\binom{\alpha'_1<\dots<\alpha'_{a'}}{\mu_1<\dots<\mu'_{b'}}$ such that the Young diagram with row lengths

$$(\alpha'_{a'} - (a'-1), \alpha'_{a'-1} - (a'-2), \dots, \alpha'_1)$$

can be obtained from adding a row to $(\alpha_a-(a-1), \alpha_{a-1}-(a-2), \dots, \alpha_1)$ such that $\alpha_{a'} - (a'-1) \leq N'_\rho - a$ and such that the Young diagram with row lengths

$$(\beta'_{b'} - (b'-1), \beta'_{b'-1} - (b'-2), \dots, \beta'_1)$$

can be obtained from adding a row to $(\beta_b-(b-1), \beta_{b-1}-(b-2), \dots, \beta_1)$. Again, the condition on the top row is equivalent to requiring that the Young diagram

$$(N'_\rho - a, \alpha_a - (a-1), \alpha_{a-1} - (a-2), \dots, \alpha_1)$$

can be obtained by adding a row to $(\alpha'_{a'}-(a'-1), \alpha'_{a'-1}-(a'-2), \dots, \alpha'_1)$.

In the ordered Lusztig symbol notation, these term we have identified are of the form

$$\begin{bmatrix} \alpha'_1 < \cdots < \alpha'_{a'} \\ \beta'_1 < \cdots < \beta'_{b'} \end{bmatrix} \otimes \begin{bmatrix} \beta_1 < \cdots < \beta_b \\ \alpha_1 < \cdots < \alpha_a < N'_\rho \end{bmatrix}$$

and we can see that by re-labelling the entries of the top row as λ and λ' and the bottom row as μ and μ', our conditions for α'_i and β'_j are precisely those stated by Pan. The non-split case of $\mathrm{O}(W,B) = \mathrm{O}^-_{2m}(\mathbb{F}_q)$ proceeds similarly.

7.5. An "inductive" construction. The purpose of this subsection is to describe how the eta and zeta correspondence can be used to construct families of irreducible representations of the finite symplectic and orthogonal groups recursively. This process re-expresses any irreducible representation to a chain of eta and (possibly signed) zeta correspondences, applied to a certain "terminal" representations which can not be expressed as the eta or zeta correspondence applied to a representation of a group of lower rank. A terminal representations is precisely one such that the semisimple component (s) of its classification data has a centralizer only involving factors of type A or ${}^2A$.

For a fixed choice of symplectic space V, let us consider the eta correspondences

$$\eta_V^{W,B} : \widehat{\mathrm{O}(W,B)} \to \widehat{\mathrm{Sp}(V)} \cup \{0\},$$

for every choice of orthogonal space (W,B) such that $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ forms a reductive dual pair in the symplectic metastable range. Recalling that every irreducible representation $\rho \in \mathrm{Sp}_{2N}(\mathbb{F}_q)$ is contained as a summand of a tensor product of oscillator representations

$$\omega_{a_1}[V] \otimes \omega_{a_2}[V] \otimes \cdots \otimes \omega_{a_n}[V], \tag{7.5.1}$$

of degree $n \leq 2N$, which we can further interpret as the restriction of an oscillator representation $\omega[V \otimes W]$ for an n-dimensional orthogonal space W with respect to a form B given by the diagonal matrix with entries $a_1, \dots, a_n$, along the inclusion

$$\mathrm{Sp}(V) \hookrightarrow \mathrm{Sp}(V) \times \mathrm{O}(W,B) \hookrightarrow \mathrm{Sp}(V \otimes W).$$

In particular, we find that the union of the non-zero images of the eta correspondences makes up the whole set of irreducible representation

$$\widehat{\mathrm{Sp}(V)} = \coprod_{\substack{(W,B),\\ \dim(W) \leq 2N}} \mathrm{Im}(\eta_V^{W,B}) \setminus \{0\}.$$

Therefore, knowing $\pi \in \widehat{\mathrm{O}(W,B)}$ satisfying $\rho = \eta^V_{W,B}(\pi)$, we may consider the idempotent

$$\frac{\dim(\pi)}{|\mathrm{O}(W,B)|} \sum_{g \in \mathrm{O}(W,B)} \chi_\pi(g)^{-1} \cdot g \in \mathbb{C}\mathrm{O}(W,B) \tag{7.5.2}$$

in

$$\mathrm{End}_{\mathrm{Sp}(V)}(\omega[V \otimes W]) = \mathrm{End}_{\mathrm{Sp}(V)}(\bigotimes_{i=1}^{n} \omega[V]_{a_i}),$$

where

$$\chi_\pi : \mathrm{O}(W,B) \to \mathbb{C}^\times$$

denotes the character corresponding to π. The image of (7.5.2) in (7.5.1) recovers ρ. Outisde of the stable range, the constructions of the reflection elements are readily interpolated. In the metastable range, if $\rho \otimes \pi$ is in the top part of the restricted oscillator representation, then the corresponding idempotent (7.5.2) survives and is unaltered by semisimplification.

On the other hand, we note that for every irreducible representation π of $\mathrm{SO}(W,B)$, at least one irreducible representation obtained as a summand of the induction of π from $\mathrm{SO}(W,B)$ to $\mathrm{O}(W,B)$ occurs in a tensor product

$$(\epsilon(det) \otimes \mathbb{C}W)^{\otimes N}$$

of degree $N \leq h_W$. Again, for any irreducible representation $\rho \in \widehat{\mathrm{Sp}(V)}$, the summand $\zeta^V_{W,B}(\rho)$ can be obtained as the image of the idempotent

$$\frac{\dim(\rho)}{|\mathrm{Sp}(V)|} \sum_{g \in \mathrm{Sp}(V)} \chi_\rho(g)^{-1} \cdot g \in \mathbb{C}\mathrm{Sp}(V) \tag{7.5.3}$$

in

$$\mathrm{End}_{\mathrm{O}(W,B)}(\omega[V \otimes W]) = \mathrm{End}_{\mathrm{O}(W,B)}((\epsilon(\det) \otimes \mathbb{C}W)^{\otimes N}),$$

again where

$$\chi_\rho : \mathrm{Sp}(V) \to \mathbb{C}^\times$$

denotes to character corresponding to ρ. All representations of $\mathrm{O}(W, B)$ can then be obtained by tensoring the representations appearing in the image of the zeta correspondences by a sign representation.

In particular, one can attempt to attempt to produce a given representation of a symplectic group $\mathrm{Sp}(V)$ or an orthogonal group $\mathrm{O}(W, B)$ as a chain of alternating eta and zeta correspondences (and tensors with sign representations) applied to a simpler representation of a smaller symplectic or orthogonal group. However, we note that both the eta and zeta correspondences can never alter any input classification to introduce eigenvalues other than 1 or -1 to the semisimple component of the data. Therefore, we find that any irreducible representation

7.5.1. DEFINITION. *Consider an irreducible representation ρ of a symplectic group or orthogonal group.*

(1) *For $\rho \in \widehat{Sp_{2N}(\mathbb{F}_q)}$, we call ρ* terminal *if in its classification data, the semisimple compotent $(s) \in SO_{2N+1}(\mathbb{F}_q) = Sp^*_{2N}(\mathbb{F}_q)$ has 1 as an eigenvalue of multiplicity one and no -1 eigenvalues.*
(2) *For $\rho \in \widehat{O_{2m+1}(\mathbb{F}_q)}$, we call ρ* terminal *if in the classification data of the restriction of ρ to the special orthogonal group $SO_{2m+1}(\mathbb{F}_q)$, the semisimple compotent of the data $(s) \in Sp_{2m}(\mathbb{F}_q) = SO^*_{2m+1}(\mathbb{F}_q)$ has no 1 or -1 eigenvalues.*
(3) *For $\rho \in \widehat{O^\pm_{2m}(\mathbb{F}_q)}$, we call ρ* terminal *if in the classification data of the restriction of ρ to the special orthogonal group $SO^\pm_{2m}(\mathbb{F}_q)$, the semisimple compotent $(s) \in SO^\pm_{2m}(\mathbb{F}_q) = (SO^\pm_{2m})^*(\mathbb{F}_q)$ no 1 or -1 eigenvalues.*

We may then produce any irreducible representation of a symplectic or orthogonal group by applying alternating eta and zeta correspondences and possibly tensor products with sign representations during steps of the construction in $\widehat{\mathrm{O}(W, B)}$. Suppose we are given a representation whose Lusztig data has semisimple component involving 1 as an eigenvalue contributing a factor of the centralizer of rank p and -1 as an eigenvalue contributing a factor of the centralizer of rank ℓ. Say the corresponding factors of the unipotent component of the classification data are symbols of the form

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \otimes \begin{pmatrix} \lambda'_1 < \cdots < \lambda'_{a'} \\ \mu'_1 < \cdots < \mu'_{b'} \end{pmatrix}.$$

Then (disregarding steps where we tensor with sign representations to obtain $\mathrm{O}(W,B)$-representations which are only obtained up to sign in the zeta correspondence), $a+b+a'+b'$ steps of applying eta and zeta correspondences are needed.

7.6. Character computations. In this subsection, we use the "inductive" construction to describe an algorithm for computing characters of an arbitrary given irreducible representation. There are three major steps to be considered: The calculation of the character of the top part of an oscillator representation, how to use this top part's character to obtain the character of the representation obtained from applying the eta or zeta correspondence to a given input representation, and the calculation of the characters of the terminal representations.

The action of the symplectic group generators on our model of the oscillator rerpesentation are as follows:

$$\begin{aligned}\omega_a\begin{pmatrix}1 & A\\ 0 & 1\end{pmatrix}(v) &= \psi(\frac{a}{2}A(v,v))\cdot(v)\\ \omega_a\begin{pmatrix}0 & B\\ -B^{-1} & 0\end{pmatrix}(v) &= \sum_{w\in V_-}\frac{\psi(aB(v,w))}{\displaystyle\sum_{u\in V_-}\psi(\frac{a}{2}B(u,u))}\cdot(w)\\ \omega_a\begin{pmatrix}(C^T)^{-1} & 0\\ 0 & C\end{pmatrix}(v) &= \epsilon_{\mathbb{F}_q}(-1)\cdot(Cv).\end{aligned}\tag{7.6.1}$$

The action of any symplectic group element can then be deduced by expressing it as a product of these standard generators. In particular, in principle, we can form a closed expression for the character of the oscillator representation at any symplectic group element. For example, in the case of the standard generators, we find character values

$$\chi_{\omega_a}\begin{pmatrix}1 & A\\ 0 & 1\end{pmatrix} = (-1)^{n(\ell+1)}\mathrm{disc}(\frac{A}{2})(\epsilon_{\mathbb{F}_q}(-1)q)^{n/2}$$

$$\chi_{\omega_a}\begin{pmatrix}0 & B\\ -B^{-1} & 0\end{pmatrix} = (\epsilon_{\mathbb{F}_q}(-2))^n$$

$$\chi_{\omega_a}\begin{pmatrix}(C^T)^{-1} & 0\\ 0 & C\end{pmatrix} = \epsilon_{\mathbb{F}_q}(\det(C))\cdot q^{\dim(\ker(C-I))}.$$

Now, while the expression for the restricted oscillator representation in terms of the extended eta correspondence and the alternating sums given in Theorems 1 and 2 are more ideal from the point of view of a genuine decomposition statement, it is inefficient to organize the characters of the alternating sums when doing concrete computations.

Instead, for the purpose of computing characters, let us recall the *virtual eta* and *virtual zeta correspondences* (see Subsubsection 6.2.1): For a reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the symplectic metastable range, we define the virtual eta correspondence as the function from the set of irreducible $\mathrm{O}(W,B)$-representations to the Grothendieck group on the category of representations of $\mathrm{Sp}(V)$

$$\boldsymbol{\eta}_V^{W,B} : \widehat{\mathrm{O}(W,B)} \to K(\mathrm{Rep}(\mathrm{Sp}(V)))$$

so that the restriction of the oscillator representation to $\mathrm{Sp}(V)\times\mathrm{O}(W,B)$ decomposes as (6.2.1)

Similarly, for any reductive dual pair $(\mathrm{Sp}(V), \mathrm{O}(W,B))$ in the orthogonal metastable range, we define the virtual zera correspondence as the function from the set of irreducible $\mathrm{Sp}(V)$-representations to the Grothendieck group on the category of $\mathrm{O}(W,B)$-representations

$$\boldsymbol{\zeta}_{W,B}^V : \widehat{\mathrm{Sp}(V)} \to K(\mathrm{Rep}(\mathrm{O}(W,B)))$$

so that the restriction of the oscillator representation to $\mathrm{Sp}(V)\times\mathrm{O}(W,B)$ decomposes as (6.2.2)

We consider the character of a virtual representation to be the linear combination (with possible negative coefficients) of the characters of its genuine terms.

Now let us consider the virtual representations

$$\widetilde{\omega^{\mathrm{top}}}[V\otimes W] = \bigoplus_{\pi\in\widehat{\mathrm{O}(W,B)}} \boldsymbol{\eta}_V^{W,B}(\pi)\otimes\pi \tag{7.6.2}$$

of $\mathrm{Sp}(V)\times\mathrm{O}(W,B)$ for every reductive dual pair in the symplectic metastable range, and the virtual representations

$$\widetilde{\omega^{\mathrm{top}'}}[V\otimes W] = \bigoplus_{\rho\in\widehat{\mathrm{Sp}(V)}} \rho\otimes\boldsymbol{\zeta}_{W,B}^V(\rho) \tag{7.6.3}$$

of $\mathrm{Sp}(V)\times\mathrm{O}(W,B)$ for every reductive dual pair in the orthogonal metastable range. These virtual "top parts" are key to understanding the relationship of an irreducibe representation's character and the character of the representation obtained by applying the (virtual) eta or zeta correspondence.

First, we must compute the character values of (7.6.2) and (7.6.3). This can be done recursively by comparing the top parts to the full oscillator representations at each level, similarly to how the dimensions of the stable top parts were computed in Subsection 4.3. In the symplectic metastable case, we may re-write the decomposition of the restriction of $\omega[V\otimes W]$ to $\mathrm{Sp}(V)\times \mathrm{O}(W,B)$ described in (6.2.1) as

$$\bigoplus_{k=0}^{h_W} \mathrm{Ind}_{\mathrm{Sp}(V)\times P_k^{\mathrm{O}(W,B)}}^{\mathrm{Sp}(V)\times \mathrm{O}(W,B)}(\widetilde{\omega^{\mathrm{top}}}[V\otimes W[-k]]^-)$$

and similarly, in the orthogonal metastable case, we may re-write the decomposition restriction of $\omega[V\otimes W]$ to $\mathrm{Sp}(V)\times \mathrm{O}(W,B)$ described in (6.2.2) as

$$\bigoplus_{k=0}^{dim(V)/2} \mathrm{Ind}_{P_k^{\mathrm{Sp}(V)}\times \mathrm{O}(W,B)}^{\mathrm{Sp}(V)\times \mathrm{O}(W,B)}(\widetilde{\omega^{\mathrm{top}\prime}}[V[-k]\otimes W]^-).$$

Now we recall that to compute the character of an induction $\mathrm{Ind}_H^G(\rho)$ for a subgroup $H\subseteq G$ and an H-representation, for a given $g\in G$, we have

$$\chi_{\mathrm{Ind}_H^G(\rho)}(s)=\frac{1}{|H|}\sum_{\substack{x\in G\\ xgx^{-1}\in H}}\chi_\rho(xgx^{-1}).$$

Hence, we obtain that for every choice of $g\in \mathrm{Sp}(V)$, $h\in \mathrm{O}(W,B)$ for $(\mathrm{Sp}(V),\mathrm{O}(W,B))$ in the symplectic metastable range, the sum

$$\sum_{k=0}^{h_W}\frac{1}{|\mathrm{Sp}(V)\times P_k^{\mathrm{O}(W,B)}|}\sum_{\substack{x\in \mathrm{Sp}(V)\times \mathrm{O}(W,B)\\ x(g\otimes h)x^{-1}\in \mathrm{Sp}(V)\times P_k^{\mathrm{O}(W,B)}}}\chi_{\widetilde{\omega^{\mathrm{top}}}[V\otimes W[-k]]^-}(x(g\otimes h)x^{-1})$$

is equal to the character value $\chi_{\omega[V\otimes W]}(g\otimes h)$, which for any explicit choice of g and h could be calculable using (7.6.1). Similarly, for every $g\in \mathrm{Sp}(V)$, $h\in \mathrm{O}(W,B)$, for $(\mathrm{Sp}(V),\mathrm{O}(W,B))$ in the orthogonal metastable range, the sum

$$\sum_{k=0}^{dim(V)/2}\frac{1}{|\mathrm{O}(W,B)\times P_k^{\mathrm{Sp}(V)}|}\sum_{\substack{x\in \mathrm{Sp}(V)\times \mathrm{O}(W,B)\\ x(g\otimes h)x^{-1}\in P_k^{\mathrm{Sp}(V)}\times \mathrm{O}(W,B)}}\chi_{\widetilde{\omega^{\mathrm{top}\prime}}[V[-k]\otimes W]^-}(x(g\otimes h)x^{-1})$$

is equal to the character value $\chi_{\omega[V\otimes W]}(g\otimes h)$. This gives a linear system of equations that can then be used to recursively calculate the virtual characters of the representations $\widetilde{\omega^{\mathrm{top}}}[V\otimes W]$ and $\widetilde{\omega^{\mathrm{top}\prime}}[V\otimes W]$.

Suppose we have calculated the virtual characters of $\widetilde{\omega^{\mathrm{top}}}[V\otimes W]$ (resp. $\widetilde{\omega^{\mathrm{top}\prime}}[V\otimes W]$) for pairs $(\mathrm{Sp}(V),\mathrm{O}(W,B))$ in the symplectic (resp. orthogonal) metastable range. We then use their values to deduce the effect of the extended eta (resp. zeta) correspondence on irreducible representations:

7.6.1. PROPOSITION. *Fix a reductive dual pair $(Sp(V), O(W,B))$.*

(1) *Suppose the pair is in the symplectic metastable range, and fix an irreducible representation π of $O(W,B)$ such that $\eta_V^{W,B}(\pi)$ is non-zero. Then, for any $g \in Sp(V)$, the value of the character associate to $\eta_V^{W,B}(\pi)$ as g can be computed as*

$$\chi_{\eta_V^{W,B}(\pi)}(g) = \sum_{(h)\in O(W,B)} \frac{|(h)|}{|O(W,B)|} \cdot \overline{\chi_\pi(h)} \cdot \chi_{\widetilde{\omega^{top}}[V\otimes W]}(g\otimes h). \tag{7.6.4}$$

(2) *Suppose the pair is in the orthogonal metastabl range, and fix an irreducible representation ρ of $Sp(V)$ such that $\zeta_{W,B}^V(\rho)$ is non-zero. Then, for any $h \in O(W,B)$, the value of the character associate to $\zeta_{W,B}^V(\pi)$ as h can be computed as*

$$\chi_{\zeta_{W,B}^V(\rho)}(h) = \sum_{(g)\in Sp(V)} \frac{|(g)|}{|Sp(V)|} \cdot \overline{\chi_\rho(g)} \cdot \chi_{\widetilde{\omega^{top'}}[V\otimes W]}(g\otimes h). \tag{7.6.5}$$

7.6.2. REMARK. *We note that our proof can also be applied to show that, in fact, (7.6.4) is also equal to*

$$\sum_{(h)\in O(W,B)} \frac{|(h)|}{|O(W,B)|} \cdot \overline{\chi_\pi(h)} \cdot \chi_{\omega^{top}[V\otimes W]}(g\otimes h) \tag{7.6.6}$$

for choice of π which survive the extended eta correspondence. The expressions (7.6.4) and (7.6.6) are different only for irreducible representations $\pi \in O(W,B)$ for which $\eta_{W,B}^V(\pi) = 0$, in which case (7.6.4) gives the virtual character $\chi_{\boldsymbol{\eta}_V^{W,B}}(g)$ (which will be a sign times a genuine irreducible charcter), while the sum (7.6.6) gives 0.

PROOF OF PROPOSITION 7.6.1. This statement follows from elementary manipulations of character theory. Let us fix an order of the conjugacy classes of $\mathrm{O}(W,B)$ and an order of its irreducible characters. We may then consider the square matrix obtained from the character table of $\mathrm{O}(W,B)$ written according to these orderings. Denote this matrix by $\mathrm{ct}(\mathrm{O}(W,B))$. By orthogonality of characters, recall that the inverse of $\mathrm{ct}(\mathrm{O}(W,B))$ can be expressed as a diagonal matrix consisting of the fractions $|(h)|/|\mathrm{O}(W,B)|$ of the order of a conjugacy class (h) of $\mathrm{O}(W,B)$ divided by the group order, multiplied on the right by the conjugate of the transpose of $\mathrm{ctO}(W,B)$

$$\mathrm{ct}(\mathrm{O}(W,B))^{-1} = \begin{pmatrix} \frac{|(h_1)|}{|\mathrm{O}(W,B)|} & 0 & \cdots \\ 0 & \frac{|(h_2)|}{|\mathrm{O}(W,B)|} & \\ \vdots & & \ddots \end{pmatrix} \cdot \overline{\mathrm{ct}(\mathrm{O}(W,B))}^T. \tag{7.6.7}$$

Now fix a group element $g \in \mathrm{Sp}(V)$. Then, for any $h \in \mathrm{O}(W, B)$, by definition (7.6.2), we find that the virtual character of the representation $\widetilde{\omega^{\mathrm{top}}}$ applied to $g \otimes h \in \mathrm{Sp}(V) \times \mathrm{O}(W, B)$ is

$$\chi_{\widetilde{\omega^{\mathrm{top}}}[V\otimes W]}(g \otimes h) = \sum_{\pi \in \widehat{\mathrm{O}(W,B)}} \chi_{\boldsymbol{\eta}_V^{W,B}(\pi)}(g) \cdot \chi_\pi(h).$$

In terms of matrices, this can be interpreted as the statement that the column of character values $\chi_{\widetilde{\omega^{\mathrm{top}}}[V\otimes W]}(g \otimes h)$ (varying conjugacy classes (h)) is obtained by multiplying the transpose of the character table $\mathrm{ctO}(W, B)$ by the column of character values $\chi_{\boldsymbol{\eta}_V^{W,B}(\pi)}(g)$ (varying $\pi \in \widehat{\mathrm{O}(W, B)}$):

$$\begin{pmatrix} \chi_{\widetilde{\omega^{\mathrm{top}}}[V\otimes W]}(g \otimes h_1) \\ \chi_{\widetilde{\omega^{\mathrm{top}}}[V\otimes W]}(g \otimes h_2) \\ \vdots \end{pmatrix} = \mathrm{ct}(\mathrm{O}(W, B))^T \cdot \begin{pmatrix} \chi_{\boldsymbol{\eta}_V^{W,B}(\pi_1)}(g) \\ \chi_{\boldsymbol{\eta}_V^{W,B}(\pi_2)}(g) \\ \vdots \end{pmatrix}.$$

Hence, by applying (7.6.7), we can calculate the column of character values $\chi_{\boldsymbol{\eta}_V^{W,B}(\pi)}(g)$ as

$$\overline{\mathrm{ct}(\mathrm{O}(W, B))} \cdot \begin{pmatrix} \frac{|(h_1)|}{|\mathrm{O}(W,B)|} & 0 & \cdots \\ 0 & \frac{|(h_2)|}{|\mathrm{O}(W,B)|} & \\ \vdots & & \ddots \end{pmatrix} \cdot \begin{pmatrix} \chi_{\widetilde{\omega^{\mathrm{top}}}[V\otimes W]}(g \otimes h_1) \\ \chi_{\widetilde{\omega^{\mathrm{top}}}[V\otimes W]}(g \otimes h_2) \\ \vdots \end{pmatrix}$$

Multiplying this out, we find that each character value $\chi_{\boldsymbol{\eta}_V^{W,B}(\pi)}(g)$ can be calculated as the sum

$$\chi_{\boldsymbol{\eta}_V^{W,B}(\pi)}(g) = \sum_{(h) \in \mathrm{O}(W,B)} \frac{|(h)|}{|\mathrm{O}(W, B)|} \cdot \overline{\chi_\pi(h)} \cdot \chi_{\widetilde{\omega^{\mathrm{top}}}[V\otimes W]}(g \otimes h),$$

matching the right hand side of (7.6.4) The claim then follows, since for $\pi \in \widehat{\mathrm{O}(W, B)}$ such that $\eta_V^{W,B}(\pi)$ is non-zero, we have

$$\boldsymbol{\eta}_V^{W,B}(\pi) = \eta_V^{W,B}(\pi).$$

□

Finally, we consider characters of the terminal representations of the symplectic or orthogonal groups. Terminal representations cannot be reduced as the images of eta or zeta correspondences of irreducible representations of groups of lower rank, so we must compute their characters using some other method.

In summary, then, for any given irreducible representation of a symplectic or orthogonal group, we can in theory obtain a closed formula for the character by recursion using Proposition 7.6.1. Of course, this is difficult to calculate and even write down in practice. We now give a few examples in low rank to show this process in action.

7.6.3. Example. *We begin with the simplest non-trivial example of this: the computation of the characters of the irreducible summands of an oscillator representation*

$$\omega_a^+[V] \oplus \omega_a^-[V] \cong \omega_a[V].$$

We can consider this decomposition by considering the reductive dual pair $(Sp(V), O(\mathbb{F}_q, a)$*, where we consider* $O(\mathbb{F}_q, a)$ *to be the orthogonal group on the* 1*-dimensional space* $\mathbb{F}_q$ *with respect to the symmetric bilinear form corresponding to the* 1×1*-matrix* (a)*. In particular, as a group,*

$$O(\mathbb{F}_q, a) \cong \mu_2,$$

consisting of ± 1*. It has two irreducible representations: the trivial representation* 1 *and the sign representation we denote by* σ*. Considering the reductive dual pair* $(Sp(V), O(\mathbb{F}_q, a))$*, we then find that the pullback of the oscillator representation* $\omega[V]$ *along the Kronecker product* $Sp(V) \times O(\mathbb{F}_q, a) \to Sp(V)$ *decomposes as*

$$\omega_a^+[V] \otimes 1 \oplus \omega_a^-[V] \otimes \sigma.$$

On the level of characters, this gives

$$\chi_{\omega_a[V]}(g \otimes (\pm 1)) = \chi_{\omega_a^+[V]}(g) \pm \chi_{\omega_a^-[V]}(g),$$

noting that $g \otimes (\pm 1) = \pm g$*. Applying Proposition 7.6.1, we find*

$$\chi_{\omega_a^+[V]}(g) = \frac{\chi_{\omega_a[V]}(g) + \chi_{\omega_a[V]}(-g)}{2}$$
$$\chi_{\omega_a^-[V]}(g) = \frac{\chi_{\omega_a[V]}(g) - \chi_{\omega_a[V]}(-g)}{2}.$$

7.6.4. Example. *The next simplest non-trivial example consists of the representations obtained from the eta correspondence for reductive dual pairs* $(Sp(V), O_2^-(\mathbb{F}_q))$*. In this case, the oscillator representation is still the entire top part. We recall*

Our goal in this example will be to compute the character of the unipotent representations obtained from applying the eta correspondence to the trivial and sign representations of $O_2^-(\mathbb{F}_q)$

$$\eta_V^{(\mathbb{F}_q^2, -)}(1) = \binom{0 < 1}{N},$$

$$\eta_V^{(\mathbb{F}_q^2,-)}(\epsilon(det)) = \binom{0<1<N}{\emptyset}.$$

Applying (7.6.4), we get

$$\chi_{\binom{0<1}{N}}(g) = \sum_{(h)\in O_2^-(\mathbb{F}_q)} \frac{|(h)|}{2(q+1)} \chi_{\omega[V\otimes(\mathbb{F}_q^2,-)]}(g\otimes h).$$

CHAPTER 8

The Gurevich-Howe rank conjecture

In this chapter, we discuss an application of our finite field Howe duality statement to the representation theory of finite groups, namely a proof of the Gurevich-Howe rank conjecture for finite groups of Lie type C (the cases of A having been previously resolved in [**22**] and the cases of B, D having been resolved in [**41**]).

8.1. U-rank and the eta correspondence. One striking property of the oscillator representation of a symplectic group $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ is its smallness. To be more precise, the smallest and second-smallest possible dimensions of non-trivial irreducible representations of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ are $(q^N-1)/2$ and $(q^N+1)/2$ (every larger possible dimension is of degree at least 2, as a polynomial in q^N). The irreducible representations attaining these small dimensions are precisely the summands ω_a^+, ω_a^- of the two oscillator representations ω_a. We in fact notice that, for $N >> n$, the irreducible representations appearing for the first time as summands of a degree n tensor product of oscillator representations are roughly of dimension q^{nN}. This suggests some grading of the $\widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)}$ according to . However, the dimension effect notably breaks down in cases when N and n are comparable (see [**25**]). This suggests a finer notion of the "smallness" of representations is needed. In [**24**], S. Gurevich and R. Howe make the concept of "smallness" precise, in the concept of *U-rank*. We review their definition of U-rank, and the *Gurevich-Howe rank conjecture* on its relationship with the oscillator representation in this subsection. We will prove the conjecture for symplectic groups in Subsection 8.2 using our explicit calculation of the eta correspondence, below.

To define U-rank, let us first fix a symplectic group $\mathrm{Sp}_{2N}(\mathbb{F}_q)$. We recall its Siegel unipotent subgroup is defined as

$$U_N = \{\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \mid A \in M_{N\times N}(\mathbb{F}_q) \text{ symmetric}\} \subseteq \mathrm{Sp}_{2N}(\mathbb{F}_q).$$

We note that U_N is isomorphic to the group of symmetric $N \times N$ matrices, with respect to addition. In particular, since U_N is abelian, we may fix an identification it with with its Pontrjagin dual: Fix a non-trivial additive character $\chi_0 : \mathbb{F}_q \to \mathbb{C}^\times$ and define the identification

$$U_N \xrightarrow{\cong} U_N^* \tag{8.1.1}$$

by

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \mapsto \left(\begin{array}{c} U_N \to \mathbb{C}^\times \\ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \mapsto \chi_0(tr(AB)) \end{array} \right).$$

In particular, for a character $\chi \in U_N^*$, we may consider its rank $0 \leq rk(\chi) \leq N$, by putting $rk(\chi) = rk(A_\chi)$, where A_χ is the symmetric $N \times N$ matrix such that its corresponding element of U_N is identified with χ by (8.1.1)

$$\begin{pmatrix} I & A_\chi \\ 0 & I \end{pmatrix} \mapsto \chi.$$

Given a representation of the Seigel unipotent U_N, we may then consider the ranks of the characters defined by trace on each irreducible U_N-representation summand. This is what is used to define U-rank:

8.1.1. Definition (S. Gurevich - R. Howe, [**25**]). *Define the* U-rank *of an* $Sp_{2N}(\mathbb{F}_q)$*-representation* ρ *as the maximal rank of a character appearing in its restriction to* U_N*:*

$$\begin{aligned} rk_U(\rho) := max\{rk(\chi) \mid \chi \in U_N^* \ \textit{irreducible} \\ \textit{and } \chi \subseteq Res_{U_N}(\rho)\}. \end{aligned} \tag{8.1.2}$$

On the other hand, since as described in Subsection 1.3, the oscillator representations of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ tensor generate every other representation. Therefore, the oscillator representations themselves can be used to define a notion of rank grading $\mathrm{Sp}_{2N}(\mathbb{F}_q)$-representations. In [**25**], Gurevich and Howe call this the *tensor rank*.

8.1.2. Definition (S. Gurevich - R. Howe, [**25**]). *The* tensor rank *of a representation* ρ *is then defined as the minimal degree* n *such that every irreducible component of* ρ *appears in a tensor product of less than or equal to* n *oscillator representations:*

$$\begin{gathered} rk_\otimes(\rho) := min\{n \mid \textit{for all } \pi \in \widehat{Sp(V)},\ \pi \subseteq \rho,\ \textit{there exists an} \\ m \leq n,\ a_1, \ldots, a_m \in \mathbb{F}_q^\times \ \textit{with } \pi \subseteq \omega_{a_1}[V] \otimes \cdots \otimes \omega_{a_m}[V]\} \end{gathered}$$

Now recalling again that a degree n tensor product

$$\omega_{a_1}[V] \otimes \cdots \otimes \omega_{a_n}[V]$$

can be considered as the restirction of the oscillator representation of a larger symplectic group $\mathrm{Sp}(V \otimes W)$ along the inclusion

$$\mathrm{Sp}(V) \hookrightarrow \mathrm{Sp}(V) \times \mathrm{O}(W, B) \hookrightarrow \mathrm{Sp}(V \otimes W),$$

where (W, B) denotes an n-dimensional $\mathbb{F}_q$-space with a non-degenerate symplectic form B corresponding to the diagonal matrix with entries $a_1, \dots, a_n$, we see that understanding the restricted oscillator representation $\mathrm{Res}_{\mathrm{Sp}(V)\times \mathrm{O}(W,B)}(\omega[V \otimes W])$ is key to. In particular, the results of this paper for pairs $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in the symplectic stable or metastable range explicitly classify the irreducible representations of each tensor rank $0 \leq rk_{\otimes} \leq 2N$.

The Gurevich-Howe rank conjecture for $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ can then be stated as follows:

8.1.3. Theorem. *For every representation ρ of $Sp_{2N}(\mathbb{F}_q)$, we have*

$$rk_U(\rho) = min(N, rk_{\otimes}(\rho)).$$

More generally, U-rank can also be defined for other finite algebraic groups, and a similar statement can be conjectured about its relationship with the natural ranking given by the oscillator representations. The corresponding statements were proved for split groups of type A and (appropriately modified) and proved for non-split groups of type ${}^2A$ by R. Guralnick, M. Larsen, and P. H. Tiep [**22**], The corresponding statements for orthogonal groups of types B and D were proved by M. Larsen and P. H. Tiep in [**41**].

This connection between U-rank and tensor rank obtained from tensor products of oscillator representations (i.e. the symplectic representation structure of restrictions of the form $\mathrm{Res}_{\mathrm{Sp}(V)\times \mathrm{O}(W,B)}(\omega[V \otimes W])$) was one of the original motivations for Gurevich and Howe's original definition of the eta correspondence in [**24, 25**]:

The original result of Gurevich and Howe describing the eta correspondence given in, say, Theorem 4.3.3 of [**25**], states that

8.1.4. Theorem (S. Gurevich - R. Howe). *For choices of symplectic space V and orthogonal space (W, B) such that the reductive dual pair $(Sp(V), O(W, B))$ is in the symplectic stable range (still defined to mean $dim(W) \leq dim(V)/2$), there is a system of injections*

$$\eta_V^{W,B} : \widehat{O(W, B)} \hookrightarrow \widehat{Sp(V)}$$

(we omit the subscript when the source is determined) such that for every irreducible representations $\rho \in \widehat{O(W, B)}$, the tensor product $\rho \otimes$

$\eta_V^{W,B}(\rho)$ *is a summand of* $Res_{Sp(V)\times O(W,B)}(\omega[V\otimes W])$, *and*

$$rk_U(\eta^V_{W,B}(\rho)) = dim(W). \tag{8.1.3}$$

Further, every other $\pi \in \widehat{Sp(V)}$ *such that* $\rho\otimes\pi$ *appears in the restricted oscillator representation has strictly lower* U*-rank.*

(Note that though Theorem 4.3.3 of [**25**] does not include the case of $dim(W) = dim(V)/2$, the result still applies to this case as described in Remark 4.3.6 of [**25**].)

First, we note that the results of [**24, 25**] immediately imply the agreement of tensor- and U-rank in cases covered by the symplectic stable range:

8.1.5. Corollary. *For irreducible* $Sp_{2N}(\mathbb{F}_q)$*-representations* ρ *of tensor rank* $\leq N$*, the notions of rank coincide:*

$$rk_\otimes(\rho) = rk_U(\rho).$$

Therefore, it only remains to prove the following

8.1.6. Proposition. *Consider an irreducible representation* ρ *of a symplectic group* $Sp_{2N}(\mathbb{F}_q)$ *obtained first in the restriction of an oscillator representation to an unstable reductive dual pair, meaning*

$$N < rk_\otimes(\rho) \leq 2N.$$

Then ρ *attains top* U*-rank*

$$rk_U(\rho) = N.$$

8.2. The Gurevich-Howe rank conjecture. In this subsection, we use Theorem 1 to prove Proposition 8.1.6 and therefore conclude the Gurevich-Howe rank conjecture for symplectic groups $\mathrm{Sp}_{2N}(\mathbb{F}_q)$. In particular, in the decomposition (6.2.14), we may further restrict to $\mathrm{Sp}(V)$-represenations by treating the coefficient $\mathrm{O}(W,B)$-representations as multiplicity spaces, obtaining a classification of the irreducible $\mathrm{Sp}(V)$-represntations of tensor rank $rk_\otimes = r$ for each $0\leq r\leq 2N$ as precisely those constructed in the image of an eta correspondence

$$\eta_V^{W,B} : \widehat{\mathrm{O}(W,B)} \to \widehat{\mathrm{Sp}(V)} \cup \{0\},$$

for one of the two non-equivalent choices of (W,B) with dimension r.

The key step we use to conclude Propostion 8.1.6 and Theorem 8.1.3 is the following result, which gives a relationship between the different rank layers of the eta correspondence, according to parabolic induction:

8.2.1. PROPOSITION. *Fix a $\mathbb{F}_q$-vector space W with symmetric bilinear form B. Consider symplectic spaces V, U of dimension $2N \leq 2M$ respectively, such that both reductive dual pairs $(Sp(V), O(W,B))$ and $(Sp(U), O(W,B))$ are in the symplectic stable or metastable ranges. Then we may consider the eta correspondences*

$$\eta^V : \widehat{O(W,B)} \to \widehat{Sp(V)} \cup \{0\}$$
$$\eta^U : \widehat{O(W,B)} \to \widehat{Sp(U)} \cup \{0\}.$$

For an irreducible representation $\pi \in \widehat{O(W,B)}$ such that $\eta^V(W) \neq 0$, we have

$$\eta^U(\pi) \subseteq Ind_{P^U_{M-N}}(\eta^V(\pi)^{\pm}), \tag{8.2.1}$$

where the $\pm$ denotes whether we consider a sign character on the factor $GL_{M-N}(\mathbb{F}_q)$ of the Levi subgroup before inflating to P^U_{M-N} and applying the induction. The sign is $+$ when W is even dimensional and is $-$ when W is odd dimensional.

To prove this, we now recall briefly the analogue of the Pieri rule for symbols (recall Subsection 7.2). For the purposes of proving Theorem 8.1.3, we only need one case of it, so we do not give the full general statement in this subsection.

Consider a unipotent representation of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ corresponding to a symbol

$$\begin{pmatrix}\lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b\end{pmatrix}.$$

Write P_1 for the maximal parabolic subgroup of $\mathrm{Sp}_{2(N+1)}(\mathbb{F}_q)$ with Levi factor $\mathrm{Sp}_{2N}(\mathbb{F}_q) \times GL_1(\mathbb{F}_q)$, and consider $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ as a its representation by letting the $GL_1(\mathbb{F}_q)$ factor of the Levi subgroup act trivially and inflating on the unipotent radical trivially. Then its parabolic induction to a $\mathrm{Sp}_{2(N+1)}(\mathbb{F}_q)$-representation

$$\mathrm{Ind}^{\mathrm{Sp}_{2(N+1)}(\mathbb{F}_q)}_{P_1^{\mathrm{Sp}_{2(N+1)}(\mathbb{F}_q)}}(\begin{pmatrix}\lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b\end{pmatrix})$$

is a direct sum of unipotent representations corresponding to symbols

$$\begin{pmatrix}\lambda_1 < \cdots < \lambda_{i-1} < \lambda_i + 1 < \lambda_{i+1} < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b\end{pmatrix}, \tag{8.2.2}$$
$$\begin{pmatrix}\lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_{i-1} < \mu_i + 1 < \mu_{j+1} < \cdots < \mu_a\end{pmatrix}$$

when possible, i.e. for $1 \leq i \leq a$ or $1 \leq j \leq b$ where $\lambda_i + 1 < \lambda_{i+1}$ or $\mu_j + 1 < \mu_{j+1}$, respectively, and the unipotent representations

$$\begin{pmatrix} 1 < \lambda_1 + 1 < \lambda_2 + 1 < \cdots < \lambda_a + 1 \\ 0 < \mu_1 + 1 < \cdots < \mu_b + 1 \end{pmatrix}$$

(8.2.3)

$$\begin{pmatrix} 0 < \lambda_1 + 1 < \cdots < \lambda_a + 1 \\ 1 < \mu_1 + 1 < \mu_2 + 1 < \cdots < \mu_b + 1 \end{pmatrix}$$

when possible, i.e. when $\lambda_1 > 0$ or $\mu_1 > 0$, respectively. This is a full description of the "one step" Pieri rule.

More generally, for the "r step" Pieri rules, describing the parabolic induction from a maximal parabolic P_r with Levi subgroup $\mathrm{Sp}_{2N}(\mathbb{F}_q) \times GL_r(\mathbb{F}_q)$ to $\mathrm{Sp}_{2(N+r)}(\mathbb{F}_q)$ (still taking $GL_r(\mathbb{F}_q)$ and the unipotent radical to act trivially on the input representation), instead of adding a single "box" to the underlying Young diagrams corresponding to a symbol $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$, we must add a "row of r boxes." More specifically, one must undo the procedures described in Proposition 3.2 and Subsection 4.6 of [**44**], then apply the classical Pieri rule adding a "row of r boxes" as a Weyl group representation, before re-applying the procedures of [**44**] to recover the original defect and the new rank $N + r$. This rule can be derived directly from the definition of the symbols (see [**44**], Subsection 4.8).

In particular, summands that always appear in $\mathrm{Ind}_{P_r}(\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b})$ are symbols obtained by adding r to the final coordinate in a row:

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_{a-1} < \lambda_a + r \\ \mu_1 < \cdots < \mu_b \end{pmatrix}$$

(8.2.4)

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_{b-1} < \mu_b + r \end{pmatrix}.$$

To apply such a parabolic induction Ind_{P_r} to a general representation ρ of $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, the resulting $\mathrm{Sp}_{2(N+r)}(\mathbb{F}_q)$ representation consists of summands which add 1's to the semisimple part of ρ's classification data and have unipotent part consisting of the input unipotent part with the factor corresponding to the centralizer of 1 eigenvalues replaced by the possible pieces of its r step parabolic induction. We also consider the "signed parabolic induction" $\mathrm{Ind}_{P_r}(\rho^-)$, by which we denote the $\mathrm{Sp}_{2(N+r)}(\mathbb{F}_q)$-representation obtained by tensoring ρ with the sign character of the $GL_r(\mathbb{F}_q)$ factor of the Levi subgroup of P_r before inflating and inducing. The procedure on classification data giving the signed parabolic induction is completely similar to the unsigned case, except that -1's are added to the semisimple part of the data corresponding to the input representation (instead of 1's) and the symbol corresponding to this factor of its centralizer is altered, instead.

In particular, by combining the symbol Pieri rule with our description of the eta correspondence given in Subsubsections 4.2.1 and 4.2.2 we are able to conclude Proposition 8.2.1:

Proof of Proposition 8.2.1. The choice of sign in (8.2.1) precisely specifies whether the induction operation will add 1's or -1's to the classification data of the input representation. Since it is chosen according to the partiy of $dim(W)$, the semisimple part of the classification data of $\eta^U(\pi)$ agrees with that of the irreducible summands of $\mathrm{Ind}_{P^U_{M-n}}(\eta^V(\pi)^{\pm})$, reducing the claim to the fact that in the "altered factor" of the unipotent parts of $\eta^U(\rho)$ and $\eta^V(\rho)$,

$$\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < M'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \subseteq \mathrm{Ind}_{P_{M-N}}(\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\rho \\ \mu_1 < \cdots < \mu_b \end{pmatrix}),$$

which follows from the symbol Pieri rule. □

Considering the effect of parabolic induction on U-rank then allows us to reduce Proposition 8.1.6 to Corollary 8.1.5, and conclude Theorem 8.1.3:

Proof of Proposition 8.1.6 and Theorem 8.1.3. Consider a representation ρ of $\mathrm{Sp}(V)$ of tensor rank

$$N < rk_{\otimes}(\rho) \leq 2N.$$

Then by Theorem 1, there exsits a choice of (W, B) with $dim(W) = rk_{\otimes}(\rho) > N$, and an irreducible representation $\pi \in \widehat{\mathrm{O}(W, B)}$ such that

$$\eta^V_{W,B}(\pi) = \rho.$$

Let us denote by V' the symplectic space of dimension $dim(V') = 2 \cdot dim(W)$, i.e. the maximal dimensional symplectic space such that $(\mathrm{Sp}(V'), \mathrm{O}(W, B))$ is a reducive dual pair in the symplectic stable range. Consider the eta correpsondence

$$\eta^{W,B}_{V'} : \widehat{\mathrm{O}(W, B)} \hookrightarrow \widehat{\mathrm{Sp}(V')},$$

and its image of π. Let us write

$$\rho' = \eta^{W,B}_{V'}(\pi) \in \widehat{\mathrm{Sp}(V')}.$$

Applying Corollary 8.1.5, we know that as a representation of $\mathrm{Sp}(V') = \mathrm{Sp}_{2\cdot dim(W)}(\mathbb{F}_q)$, its U-rank is

$$rk_U(\rho') = rk_{\otimes}(\rho') = dim(W).$$

Now applying Proposition 8.2.1 gives that ρ' appears as a summand of a (possibly signed) parabolic induction of ρ from a parabolic subgroup with Levi factor

$$\mathrm{Sp}(V) \times \mathrm{GL}_{dim(W)-N}(\mathbb{F}_q) \subseteq \mathrm{Sp}(V'),$$

which is an operation that can only increase U-rank by at most the difference $dim(W) - N$. In other words, the U-rank of ρ is at least

$$rk_U(\rho) \geq rk_U(\rho') - (dim(W) - N) =$$
$$rk_{\otimes}(\rho') - (dim(W) - N) = N,$$

and therefore we must have equality $rk_U(\rho) = N$.

□

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