

CHARACTER RATIO SUMS IN THE STABLE RANGE FOR FINITE SYMPLECTIC GROUPS

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ABSTRACT. We prove the vanishing of certain character ratio sums over representations of a finite symplectic group constructed in the restriction of an oscillator representation to a type I reductive dual pair in the symplectic stable range. This answers a question of S. Gurevich and R. Howe, which was a part of their program for proving the Shalev conjecture on uniform distribution of commutators of symplectic groups. Our approach is a “sieve method” using a suitable stratification of the oscillator representation.

1. INTRODUCTION

Many of the most interesting questions about the structure of a finite group can be expressed as problems in representation theory. A quintessential example of this is the way a group’s commutators can be studied using the character theory of its irreducible representations.

For a finite group G , let us write \widehat{G} for the set of its irreducible representations. For an element $g \in G$ and a G -representation ρ , we write $\chi_\rho(g)$ for the character value of ρ at g (i.e. the trace of g ’s action on ρ). For each $g \in G$, we may consider the *character ratio sum of G at g*

$$(1) \quad \sum_{1 \neq \rho \in \widehat{G}} \frac{\chi_\rho(g)}{\dim(\rho)},$$

obtained by summing the fractions of each non-trivial irreducible representation’s character value at g over its dimension. These character ratio sums were used in M. Liebeck, E. O’Brien, A. Shalev, and P.H. Tiep’s proof [13, 16] of the *Ore conjecture* [17] for commutators in finite non-abelian simple groups. To be concrete, the Ore conjecture asks if for a finite non-abelian simple group G , its commutator map

$$c_G : G \times G \rightarrow G \\ (x, y) \mapsto xyx^{-1}y^{-1}$$

must necessarily be surjective. A classical formula of Frobenius [5] states that the order of the fiber $c_G^{-1}(g)$ of the commutator map at an

element $g \in G$ is expressible in terms of the character ratio sum

$$|c_G^{-1}(g)| = |G| \cdot \sum_{\rho \in \widehat{G}} \frac{\chi_\rho(g)}{\dim(\rho)} = |G| \cdot \left(1 + \sum_{1 \neq \rho \in \widehat{G}} \frac{\chi_\rho(g)}{\dim(\rho)}\right).$$

In particular, estimating certain cases of the character ratio sums (1) played a key role in the proof that $c_G^{-1}(g)$ is always non-empty (and hence c_G is surjective) in [13, 16]. Naturally, the positive solution to the Ore conjecture leads to the question of whether the commutator map for such a non-abelian simple group is uniformly distributed (away, of course, from the identity element). This question was conjectured by Shalev in [18], Conjecture 1.11. For example, one may ask if for a finite non-abelian group of Lie type $G(\mathbb{F}_q)$, the character ratio sum at a non-trivial element $1 \neq g \in G(\mathbb{F}_q)$ vanishes as q goes to infinity

$$(2) \quad \lim_{q \rightarrow \infty} \sum_{1 \neq \rho \in \widehat{G(\mathbb{F}_q)}} \frac{\chi_\rho(g)}{\dim(\rho)} = 0.$$

(We note that in this paper, we will consider $\lim_{q \rightarrow \infty}$ as the limit over powers of the q with $g \in G(\mathbb{F}_q)$ with degree going to infinity, e.g.

$$\lim_{r \rightarrow \infty} \sum_{1 \neq \rho \in \widehat{G(\mathbb{F}_{q^r})}} \frac{\chi_\rho(g)}{\dim(\rho)}.$$

In 2009, Liebeck and Tiep found counterexamples to this conjecture, with the right hand side of (2) tending to 1 and -1 for g a transvection in $G(\mathbb{F}_q) = \mathrm{PSL}_3(\mathbb{F}_q)$ and $G(\mathbb{F}_q) = \mathrm{PSU}_3(\mathbb{F}_q)$, respectively (see [19]).

In this paper, we specifically consider the case of the symplectic groups $G(\mathbb{F}_q) = \mathrm{Sp}_{2N}(\mathbb{F}_q)$ for an odd prime power q . In [7, 8], S. Gurevich and R. Howe introduced a program for proving (2), for example, in this case. Computationally, they observed two effects in the behavior of the sums for the case of g a transvection:

- (A) First, the “small size” representations ρ contribute the largest terms of (2).
- (B) Second, and very importantly in the case of these small sizes, the terms of (2) contributed by representations of “similar size” tend to cancel.

The definition of “size” Gurevich and Howe concretely use is the notion of U -rank, which is defined for a $\rho \in \widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)}$ as the maximal rank of a symmetric matrix corresponding in the standard way to a character of the Siegel unipotent subgroup $U \subseteq \mathrm{Sp}_{2N}(\mathbb{F}_q)$ appearing in

the restriction of ρ . Representations are called *small* if their U -rank is strictly less than N , and are called *large* otherwise.

The main result of the present paper proves the cancellation effect Gurevich and Howe observed for small representations (and for a certain class of large representations) for all $g \neq \pm I \in \mathrm{Sp}_{2N}(\mathbb{F}_q)$.

Our approach is a sieve method using the *oscillator representations* $\omega_\psi[V]$ of a symplectic group $\mathrm{Sp}(V)$ for V a $2N$ -dimensional symplectic space over \mathbb{F}_q . In [7, 8], Gurevich and Howe considered the restriction of an $\mathrm{Sp}(V \otimes W)$ -oscillator representation $\omega[V \otimes (W, B)]$ to a type I reductive dual pair of the form

$$\mathrm{Sp}(V) \times \mathrm{O}(W, B) \hookrightarrow \mathrm{Sp}(V \otimes W)$$

for orthogonal spaces (W, B) satisfying $N \geq \dim(W)$ (this is called the *symplectic stable range condition*). They observed that in such a restriction, for each irreducible representation $\rho \in \widehat{\mathrm{O}(W, B)}$, there is a corresponding “top” partner $\eta_{W,B}^V(\rho) \in \widehat{\mathrm{Sp}(V)}$ of U -rank $\dim(W)$ such that

$$\rho \otimes \eta_{W,B}^V(\rho) \subseteq \mathrm{Res}_{\mathrm{Sp}(V) \times \mathrm{O}(W,B)}^{\mathrm{Sp}(V \otimes W)}(\omega[V \otimes (W, B)]).$$

They called the resulting system of correspondences

$$\eta_{W,B}^V : \widehat{\mathrm{O}(W, B)} \hookrightarrow \widehat{\mathrm{Sp}(V)}$$

the *eta correspondences*.

In [9], [10], and [11], we proved a full decomposition statement for the restriction of an oscillator representation to a type I reductive dual pair in terms of certain “layers” made up of an extended version of the eta correspondence, for reductive dual pairs $(\mathrm{Sp}(V), \mathrm{O}(W, B))$ in a certain *symplectic metastable range* (in the complementary *orthogonal metastable range*, we also prove an analogous result involving a “zeta correspondence” in the opposite direction). By calculating the eta correspondence (and the relevant extension of it) explicitly, we were able to prove Gurevich and Howe’s *rank and exhaustion conjecture* for symplectic groups, meaning that all small representations in $\widehat{\mathrm{Sp}(V)}$ of U -rank $n < N$ appear in the image of an eta correspondence $\eta_{W,B}^V$ for an orthogonal space (W, B) of dimension n . We note that appropriate statements of the rank and exhaustion conjectures were proved in the type A case by R.M. Guralnick, M. Larsen, and P.H. Tiep [6], and in the type B and D cases by Larsen and Tiep in [12].

In the present paper, we apply the decomposition results of [9, 10] in the stable range $\dim(W) \leq N$ to prove a vanishing statement for

the sums of character ratios over the images of the stable eta correspondences, which, in particular, then proves Gurevich and Howe's cancellation observation for all small representations. We note that in fact, we prove that the cancellation effect also works for the first level of "large" representations arising from eta correspondences from $\dim(W) = N$.

To be more specific, let us establish some additional notation. Consider a sign $\alpha = \pm$. Let us write $(\widehat{\mathbb{F}_q^{2m+1}}, \alpha)$ for the $(2m+1)$ -dimensional orthogonal \mathbb{F}_q -space with a form of discriminant α . Let us write $(\widehat{\mathbb{F}_q^{2m}}, \alpha)$ for the $2m$ -dimensional orthogonal \mathbb{F}_q -space such that $O(\widehat{\mathbb{F}_q^{2m}}, \alpha) = O_{2m}^\alpha(\mathbb{F}_q)$ (using the convention that $O_{2m}^+(\mathbb{F}_q)$ denotes the orthogonal group of a sum of m copies of the 2-dimensional hyperbolic over \mathbb{F}_q , and that $O_{2m}^-(\mathbb{F}_q)$ denotes the orthogonal group of a sum of $(m-1)$ copies of the 2-dimensional hyperbolic and one copy of the 2-dimensional anisotropic form over \mathbb{F}_q). For a sign α , we then may consider the eta correspondences

$$\eta_{2m+1, \alpha}^{2N} : O_{2m+1}(\widehat{\mathbb{F}_q}, \alpha) \hookrightarrow Sp_{2N}(\widehat{\mathbb{F}_q})$$

for ranks satisfying $N \geq 2m+1$ (the *odd stable range condition*) and

$$\eta_{2m, \alpha}^{2N} : O_{2m}^\alpha(\widehat{\mathbb{F}_q}) \hookrightarrow Sp_{2N}(\widehat{\mathbb{F}_q})$$

for ranks satisfying $N \geq 2m$ (the *even stable range condition*). We note that for these stable ranges, for a fixed N , the images of each eta correspondence for a choice of orthogonal group dimension and sign α are all disjoint.

Our main result is the following:

Theorem 1. *Fix an element $g \neq \pm I \in Sp_{2N}(\mathbb{F}_q)$. Fix a natural number n such that $n \leq N$.*

- (1) *Say $n = 2m+1$ is odd. Then, for each sign α , the character ratio sum over the image of the corresponding eta correspondence vanishes*

$$(3) \quad \lim_{q \rightarrow \infty} \sum_{\rho \in O_{2m+1}(\widehat{\mathbb{F}_q})} \frac{\chi_{\eta_{2m+1, \alpha}^{2N}(\rho)}(g)}{\dim(\eta_{2m+1, \alpha}^{2N}(\rho))} = 0.$$

- (2) *Say $n = 2m$ is even. Then the character ratio sum over the union of the images of both eta correspondences from orthogonal*

spaces of $2m$ vanishes

$$(4) \quad \lim_{q \rightarrow \infty} \sum_{\rho \in \widehat{O_{2m}^+(\mathbb{F}_q)}} \frac{\chi_{\eta_{2m,+}^{2N}(\rho)}(g)}{\dim(\eta_{2m,+}^{2N}(\rho))} + \sum_{\rho \in \widehat{O_{2m}^-(\mathbb{F}_q)}} \frac{\chi_{\eta_{2m,-}^{2N}(\rho)}(g)}{\dim(\eta_{2m,-}^{2N}(\rho))} = 0.$$

Remark: In part (2) of the theorem, we note that the claim (4) is genuinely false for the case of $g = -I$. This can be seen even in the case of $m = 1$, since each character ratio summand is a sign ± 1 , and the sum of the number of irreducible representations of $O_2^+(\mathbb{F}_q)$ with the number of irreducible representations of $O_2^-(\mathbb{F}_q)$ is odd.

Our broad approach will be to rewrite the character ratio sums

$$(5) \quad \sum_{\rho \in \widehat{O_{2m+1}(\mathbb{F}_q)}} \frac{\chi_{\eta_{2m+1,\alpha}^{2N}(\rho)}(g)}{\dim(\eta_{2m+1,\alpha}^{2N}(\rho))}$$

and

$$(6) \quad \sum_{\rho \in \widehat{O_{2m}^+(\mathbb{F}_q)}} \frac{\chi_{\eta_{2m,+}^{2N}(\rho)}(g)}{\dim(\eta_{2m,+}^{2N}(\rho))} + \sum_{\rho \in \widehat{O_{2m}^-(\mathbb{F}_q)}} \frac{\chi_{\eta_{2m,-}^{2N}(\rho)}(g)}{\dim(\eta_{2m,-}^{2N}(\rho))}$$

by first simplifying the denominator by estimating the dimension of the outputs of a stable eta correspondence (Section 2) and then using orthogonality of characters for each $O(W, B)$ to reassemble the terms as characters of oscillator representations (Section 3). The characters of the oscillator representation can, in turn, be calculated and bounded using the Schrödinger model, allowing us to control the q -degree of each term. This is done in Section 4. Finally, in Sections 5 and 6, we conclude the odd and even cases of Theorem 1, respectively.

2. THE DIMENSION OF THE OUTPUTS OF AN ETA CORRESPONDENCE

The purpose of this section is to process the denominators in the character ratio sums (5) and (6). In other words, this section is focused on considering the dimension of a representation's image under the eta correspondence.

2.1. Lusztig classification data for symplectic and orthogonal groups. In general, for an algebraic group G over \mathbb{F}_q , we may construct G -representations according to *classification data* consisting of

- (1) A conjugacy class of a semisimple element (s) in the dual group G^*

- (2) An irreducible unipotent representation u of the (dual of the) centralizer $Z_{G^*}(s)$.

This construction is according to G. Lusztig's parametrization of irreducible characters through Deligne-Lusztig theory. In this section, this may be treated as a black box. We refer to [1, 2, 3] for detailed discussion covering all cases of G we consider here. For each choice of such $[(s), u]$, we may construct a G -representation $r^G[(s), u]$ of dimension

$$\dim(r^G[(s), u]) = \frac{|G|_{q'}}{|Z_{G^*}(s)|_{q'}} \dim(u).$$

Every irreducible G -representation is contained as a summand of one of these representations. If G is connected and has connected center, these representations are all distinct and themselves irreducible, meaning that the classification data exactly classifies the set of irreducible representations of G . We would like to apply this to the cases of symplectic and (special) orthogonal groups which may not satisfy these properties. In this case, some additional data we call *central data* and *extension data* may be needed. We discuss these cases in detail now:

For a symplectic group $\mathrm{Sp}_{2N}(\mathbb{F}_q)$, we consider the classification data of

- (1) A conjugacy class of a semisimple element $(s) \in \mathrm{SO}_{2N+1}(\mathbb{F}_q) = (\mathrm{Sp}_{2N}(\mathbb{F}_q))^*$.
(2) An irreducible unipotent representation u of the (dual of the) centralizer $Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(s)$ of s .

The $\mathrm{Sp}_{2N}(\mathbb{F}_q)$ -representation $r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u]$ associated to this data is irreducible unless s has -1 as an eigenvalue and the factor of u corresponding to s is non-degenerate (see Definition 2), in which case it splits into two non-isomorphic, equidimensional pieces differentiated by the action of the center $Z(\mathrm{Sp}_{2N}(\mathbb{F}_q)) = \mathbb{Z}/2$:

$$r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u] = r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, +1] \oplus r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(s), u, -1]$$

In this case, we then call the extra data of the sign ± 1 indicating the action of $-I \in Z(\mathrm{Sp}_{2N}(\mathbb{F}_q))$ the *central data* specifying an irreducible representation.

Next, consider an odd special orthogonal group $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$. In this case, the classification data defining a representation of $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$ consists of

- (1) A conjugacy class of a semisimple element $(s) \in \mathrm{Sp}_{2m}(\mathbb{F}_q) = (\mathrm{SO}_{2m+1}(\mathbb{F}_q))^*$.

- (2) An irreducible unipotent representation u of the (dual of the) centralizer $Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s)$ of s .

Now, since the center of an odd orthogonal group splits off as

$$\mathrm{O}_{2m+1}(\mathbb{F}_q) = \mathrm{SO}_{2m+1}(\mathbb{F}_q) \times \mathbb{Z}/2,$$

for each choice of $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$ -classification data $[(s), u]$, there are two corresponding $\mathrm{O}_{2m+1}(\mathbb{F}_q)$ -representations

$$r^{\mathrm{O}_{2m+1}(\mathbb{F}_q)}[(s), u]^\pm := r^{\mathrm{SO}_{2m+1}(\mathbb{F}_q)}[(s), u] \otimes (\pm 1)$$

obtained by tensoring the corresponding $\mathrm{SO}_{2m+1}(\mathbb{F}_q)$ with either the trivial (1) or sign (-1) representation of $\mathbb{Z}/2$. These representations $r^{\mathrm{O}_{2m+1}(\mathbb{F}_q)}[(s), u]^\pm$ are exactly the irreducible $\mathrm{O}_{2m+1}(\mathbb{F}_q)$ -representations. Hence, we may classify the irreducible representations of an odd orthogonal group $\mathrm{O}_{2m+1}(\mathbb{F}_q)$ by the classification data of a semisimple conjugacy class $(s) \in \mathrm{Sp}_{2m}(\mathbb{F}_q)$, an irreducible unipotent representation u of the dual of the centralizer of s , and the *extension data* of a choice of sign \pm specifying the action of $\mathbb{Z}/2 = \mathrm{O}_{2m+1}(\mathbb{F}_q)/\mathrm{SO}_{2m+1}(\mathbb{F}_q)$.

Finally, consider an even special orthogonal group $\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)$. In this case, the classification data defining an $\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)$ consists of

- (1) A conjugacy class of a semisimple element $(s) \in \mathrm{SO}_{2m}^\pm(\mathbb{F}_q) = (\mathrm{SO}_{2m}^\pm(\mathbb{F}_q))^*$.
- (2) An irreducible unipotent representation u of the (dual of the) centralizer $Z_{\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)}(s)$ of s .

While, like in the odd case, we still have $\mathrm{O}_{2m}^\pm(\mathbb{F}_q)/\mathrm{SO}_{2m}^\pm(\mathbb{F}_q) = \mathbb{Z}/2$, we may no longer split off $\mathrm{O}_{2m}^\pm(\mathbb{F}_q) = \mathrm{SO}_{2m}^\pm(\mathbb{F}_q) \rtimes \mathbb{Z}/2$ as a product. Still, we may consider for every choice of Lusztig classification $\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)$ -data, the induction of its corresponding representation to $\mathrm{O}_{2m}^\pm(\mathbb{F}_q)$

$$(7) \quad \mathrm{Ind}_{\mathrm{O}_{2m}^\pm(\mathbb{F}_q)}^{\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)}(r^{\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)}[(s), u]).$$

We observe several effects now. First, if two semisimple elements $s, s' \in \mathrm{SO}_{2m}^\pm(\mathbb{F}_q)$ are conjugate in $\mathrm{O}_{2m}^\pm(\mathbb{F}_q)$, then

$$\mathrm{Ind}_{\mathrm{O}_{2m}^\pm(\mathbb{F}_q)}^{\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)}(r^{\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)}[(s), u]) \cong \mathrm{Ind}_{\mathrm{O}_{2m}^\pm(\mathbb{F}_q)}^{\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)}(r^{\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)}[(s'), u])$$

(for any possible choice of u). Otherwise, the two inductions share no common irreducible summands. Second, the representation (7) remains irreducible exactly when s has no 1 or -1 eigenvalues, or both factors of u corresponding to the ± 1 eigenvalues is degenerate. If s has either 1 or -1 as an eigenvalue and the corresponding factor of u is not degenerate, the induction (7) splits into two equi-dimensional irreducible $\mathrm{O}_{2m}^\pm(\mathbb{F}_q)$ -representation. If s has both 1 and -1 as eigenvalues and neither of

the corresponding factor of u is degenerate, the induction (7) splits into four equidimensional pieces. Let us write

$$a(s) = \begin{cases} 1 & \text{if } 1 \text{ is an eigenvalue of } s \\ 0 & \text{else} \end{cases}$$

$$b(s) = \begin{cases} 1 & \text{if } -1 \text{ is an eigenvalue of } s \\ 0 & \text{else} \end{cases}$$

Of course, every irreducible representation is a summand of an induction of the form (7). Taking these effects into account, we may classify the irreducible representations of $O_{2m}^{\pm}(\mathbb{F}_q)$ according to the data of

- (1) An $O_{2m}^{\pm}(\mathbb{F}_q)$ -conjugacy class (s) of a semisimple element satisfying $\det(s) = 1$ (so that $s \in \mathrm{SO}_{2m}^{\pm}(\mathbb{F}_q)$).
- (2) An irreducible unipotent representation u of the centralizer $Z_{\mathrm{SO}_{2m}^{\pm}(\mathbb{F}_q)}(s)$. For the factors corresponding to ± 1 eigenvalues, the two sub-symbols of a degenerate symbol give the same induced $O_{2m}^{\pm}(\mathbb{F}_q)$ -representation, so we only allow one of them $(\lambda_1 < \dots < \lambda_a)_{\lambda_1 < \dots < \lambda_a}^+$ in order to not over-count (see Definition 2 below).
- (3) A choice of an element σ in $\{\pm 1\}^{a(s)+b(s)}$ if $a(s) + b(s) > 0$. If $a(s) > 0$, denote the corresponding coordinate of σ by σ_1 . If $b(s) > 0$, denote the corresponding coordinate of σ by σ_{-1} . If both are positive, we list σ_1 first, writing $\sigma = (\sigma_1, \sigma_{-1})$.

We denote the irreducible $O_{2m}^{\pm}(\mathbb{F}_q)$ -representation corresponding to such a choice of data by

$$r^{O_{2m}^{\pm}(\mathbb{F}_q)}[(s), u]^{\sigma}.$$

For convenience, given an irreducible representation ρ of $O_{2m}^{\pm}(\mathbb{F}_q)$, we write

$$a(\rho) := a(s), \quad b(\rho) := b(s),$$

where s is the choice of semisimple data corresponding to ρ . (In this case, this can be directly observed. We also refer to [3] for a discussion of the general story of parametrizing irreducible characters for a non-connected finite reductive group.)

In each of the above cases, we may additionally use Lusztig's theory of symbols to describe the unipotent data u .

Definition 2. Fix a rank $r \in \mathbb{N}_0$ and consider the equivalence class of a pair of increasing sequences of non-negative integers

$$\begin{pmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{pmatrix}$$

such that $(\lambda_1, \mu_1) \neq (0, 0)$, under switching the two rows.

- (1) This is an irreducible symbol of rank r and type B or C if $a - b$ is odd and $\sum_{i=1}^a \lambda_i + \sum_{j=1}^b \mu_j = r + (a + b - 1)^2/4$.
- (2) This is a symbol of rank r and type D (resp. 2D) if $a - b$ is $0 \pmod{4}$ (resp. $2 \pmod{4}$) and $\sum_{i=1}^a \lambda_i + \sum_{j=1}^b \mu_j = r + (a + b)(a + b - 2)/4$. It is called degenerate if $a = b$ and $\lambda_i = \mu_i$ for every $1 \leq i \leq a$, in which case we consider two irreducible degenerate sub-symbols we denote by $(\lambda_1 < \dots < \lambda_a)^\pm$. Otherwise, the symbol is, itself, called irreducible.

As the terminology suggests, the irreducible symbols of rank r and type B (resp. C) classify the irreducible unipotent representations of $\mathrm{SO}_{2r+1}(\mathbb{F}_q)$ (resp. $\mathrm{Sp}_{2r}(\mathbb{F}_q)$). The irreducible symbols of rank r and type D (resp. 2D) classify the irreducible unipotent representations of $\mathrm{SO}_{2r}^+(\mathbb{F}_q)$ (resp. $\mathrm{SO}_{2r}^-(\mathbb{F}_q)$).

Definition 3. Suppose $G = \mathrm{Sp}_{2N}(\mathbb{F}_q)$, $O_{2m+1}(\mathbb{F}_q)$, or $O_{2m}^\pm(\mathbb{F}_q)$. Consider a subgroup $H \subseteq G^*$ isomorphic to the centralizer of a semisimple element and an irreducible unipotent representation u of (the dual of) H . We then define the set

$$\widehat{G}[H, u]$$

to be the subset of \widehat{G} consisting of irreducible representations corresponding to Lusztig classification data $[(s), u]$ for any (s) with $Z_{G^*}(s) \cong H$ (and any central or extension data).

We note in particular, that if $\rho, \rho' \in \widehat{G}[H, u]$, we must have $\dim(\rho) = \dim(\rho')$.

2.2. The eta correspondence and dimension. Given this set-up, we now begin work to simplify the denominator of each term of (5) and (6) by estimating the dimension of any irreducible representation's image under the eta correspondence. This can be done using the explicit description of the stable eta correspondence found in [10] and the Lusztig's dimension formula for symbols (see for example [15], Appendix).

Lemma 4. Fix a choice of natural numbers $n \leq N$ and a sign α , and consider the corresponding eta correspondence $\eta_{n,\alpha}^{2N}$.

- (1) Say $n = 2m + 1$ is odd. For any irreducible representation ρ of $O_{2m+1}(\mathbb{F}_q)$, the dimension of ρ 's image under the eta correspondence can be expressed as a product of the dimension of ρ

and a polynomial in q with top term $q^{(2m+1)(N-m)}/2$

$$(8) \quad \dim(\eta_{2m+1,\alpha}^{2N}(\rho)) = \dim(\rho) \cdot \left(\frac{q^{(2m+1)(N-m)}}{2} + LOE_q \right).$$

Like $\dim(\rho)$, the dimension $\dim(\eta_{2m+1,\alpha}^{2N}(\rho))$ only depends on the choice of H, u such that $\rho \in \widehat{O_{2m+1}(\mathbb{F}_q)}[H, u]$.

- (2) Say $n = 2m$ is even. For any irreducible representation ρ of $O_{2m}^\alpha(\mathbb{F}_q)$, the dimension of ρ 's image under the eta correspondence can be expressed as a product of the dimension of ρ and a polynomial in q with top term $q^{2m(N-m)+m}/2^{\alpha(\rho)+\beta(\rho)+1}$

$$(9) \quad \dim(\eta_{2m,\alpha}^{2N}(\rho)) = \dim(\rho) \cdot \left(\frac{q^{2m(N-m)+m}}{2} + LOE_q \right).$$

Like $\dim(\rho)$, the dimension $\dim(\eta_{2m,\alpha}^{2N}(\rho))$ only depends on the choice of H, u such that $\rho \in \widehat{O_{2m+1}(\mathbb{F}_q)}[H, u]$.

Given this lemma, we can see that, for an odd orthogonal group $O_{2m+1}(\mathbb{F}_q)$, the character ratio sum (5) can be estimated as

$$(10) \quad \frac{2}{q^{(2m+1)(N-m)}} \sum_{\rho \in \widehat{O_{2m+1}(\mathbb{F}_q)}} \frac{\chi_{\eta_{2m+1,\alpha}^{2N}(\rho)}(g)}{\dim(\rho)},$$

with error terms of the form

$$(11) \quad K_{H,u} \sum_{\pi \in \widehat{O_{2m+1}(\mathbb{F}_q)}[H,u]} \frac{\chi_{\eta_{2m+1,\alpha}^{2N}(\rho)}(g)}{\dim(\rho)}$$

for coefficients $K_{H,u}$ of q -degree strictly less than $-(2m+1)(N-m)$ chosen for each subgroup $H \subseteq \mathrm{Sp}_{2m}(\mathbb{F}_q) = (\mathrm{SO}_{2m+1}(\mathbb{F}_q))^*$ (up to isomorphism) constructible as the centralizer of a semisimple element, and each possible unipotent irreducible H -representation u . We note that the number of possible choices of such H and u does not depend on q .

Similarly, for an even orthogonal group $O_{2m}^\pm(\mathbb{F}_q)$, the character ratio sum (6) can be estimated as

$$(12) \quad \frac{2}{q^{(2m+1)(N-m)}} \left(\sum_{\rho \in \widehat{O_{2m}^+(\mathbb{F}_q)}} \frac{\chi_{\eta_{2m,+}^{2N}(\rho)}(g)}{\dim(\rho)} + \sum_{\rho \in \widehat{O_{2m}^-(\mathbb{F}_q)}} \frac{\chi_{\eta_{2m,-}^{2N}(\rho)}(g)}{\dim(\rho)} \right).$$

with an error consisting of terms

$$(13) \quad K_{H,u}^\pm \sum_{\pi \in \widehat{O_{2m}^\pm(\mathbb{F}_q)}[H,u]} \frac{\chi_{\eta_{2m,\alpha}^{2N}(\rho)}(g)}{\dim(\rho)}$$

for coefficients $K_{H,u}^\pm$ of q -degree strictly less than $-(2m(N-m)+m)$ chosen for each subgroup $H \subseteq \mathrm{SO}_{2m+1}^\pm(\mathbb{F}_q)$ (up to isomorphism) constructible as the centralizer of a semisimple element, and each possible unipotent irreducible H -representation u . Note again that the number of these choices of H and u does not depend on q .

Proof of Lemma 4. We begin with the case of an orthogonal group of odd dimension $2m+1$, and an eta correspondence

$$\eta_{2m+1,\alpha}^{2N} : \widehat{\mathrm{O}_{2m+1}(\mathbb{F}_q)} \leftrightarrow \widehat{\mathrm{Sp}_{2N}(\mathbb{F}_q)}$$

(since N , m , and α are constant for the remainder of this case, we omit the superscript and subscript from now on and just write η). Fix an irreducible representation $\rho = r^{\mathrm{O}_{2m+1}(\mathbb{F}_q)}[(s), u]^{\pm 1}$. Say -1 is an eigenvalue of s of multiplicity 2ℓ , so that its centralizer is of the form

$$Z_{\mathrm{Sp}_{2m}(\mathbb{F}_q)}(s) = \mathrm{Sp}_{2\ell}(\mathbb{F}_q) \times H$$

for a subgroup $H \subseteq \mathrm{Sp}_{2(m-\ell)}$ (consisting of a product of a single symplectic group and factors of A - and 2A -type. Write the representation u as the tensor product of an $\mathrm{Sp}_{2\ell}(\mathbb{F}_q)$ -symbol and an irreducible unipotent H -representation u_H

$$(14) \quad u = \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b} \otimes u_H.$$

Let us write (x_{2n}^\pm) for the conjugacy classes of semisimple elements of $\mathrm{SO}_{2n+1}(\mathbb{F}_q)$ with -1 as an eigenvalue of multiplicity $2n$ and 1 as an eigenvalue of multiplicity 1 such that $Z_{\mathrm{SO}_{2n+1}(\mathbb{F}_q)}(x_{2n}^\pm) = \mathrm{SO}_{2n}^\pm(\mathbb{F}_q)$. According to the extension data of ρ , we consider the new semisimple data $\eta(s) = s \oplus x_{2(N-m)}^\pm \in \mathrm{SO}_{2N+1}(\mathbb{F}_q)$, so that

$$Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(\eta(s)) = \mathrm{SO}_{2(N-m+\ell)}^\pm(\mathbb{F}_q) \times H.$$

Switching the rows of the symbol factor of (14) so that $a-b = \pm 1 \pmod{4}$ and writing $N'_\rho := N - m + \frac{a+b-1}{2}$, we put

$$\eta(u) := \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b < N'_\rho} \otimes u_H,$$

producing an irreducible unipotent representation of $Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(\eta(s))$. Noting that, by our assumption $N \geq 2m$, $\eta(s)$ always has -1 as an eigenvalue, the eta correspondence is constructed by putting

$$\eta(\rho) = r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(\eta(s)), \eta(u), \pm 1],$$

where the central sign data is chosen according to the quadratic character of the eigenvalues of s and the fixed discriminant α of the symmetric bilinear form defining $\mathrm{O}_{2m+1}(\mathbb{F}_q)$.

Computing dimension, we find

$$\begin{aligned} \dim(\eta(\rho)) &= \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{2|\mathrm{SO}_{2(N-m+\ell)}^{\pm}(\mathbb{F}_q) \times H|_{q'}} \cdot \dim\left(\begin{smallmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b < N'_\rho \end{smallmatrix}\right) \cdot \dim(u_H) \\ \dim(\rho) &= \frac{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}}{|\mathrm{Sp}_{2\ell}(\mathbb{F}_q) \times H|_{q'}} \cdot \dim\left(\begin{smallmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{smallmatrix}\right) \cdot \dim(u_H) \end{aligned}$$

Applying Lusztig's formula for the dimension of a symbol, we find that $\dim(\eta(\rho))/\dim(\rho)$ is expressible as

$$(15) \quad \begin{aligned} & \frac{1}{2} \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|_{q'}} \frac{\prod_{i=1}^a (q^{N'_\rho} + q^{\lambda_i}) \prod_{j=1}^b (q^{N'_\rho} - q^{\mu_j})}{\prod_{i=1}^{N'_\rho} (q^{2i} - 1) q^{(a+b-1)^2/4}} = \\ & \frac{1}{2} \frac{\prod_{i=N'_\rho+1}^N (q^{2i} - 1) \prod_{i=1}^a (q^{N'_\rho} + q^{\lambda_i}) \prod_{j=1}^b (q^{N'_\rho} - q^{\mu_j})}{\prod_{i=1}^m (q^{2i} - 1) q^{(a+b-1)^2/4}}. \end{aligned}$$

Applying the formulae for group orders of symplectic and special orthogonal groups, we find that this is a polynomial in q of degree

$$(N - N'_\rho)(N + N'_\rho + 1) + (a + b)N'_\rho - m(m + 1) - \frac{(a + b - 1)^2}{4},$$

which simplifies to $(2m + 1)(N - m)$ by replacing $N'_\rho = N - m + \frac{a+b-1}{2}$. The coefficient of the top $q^{(2m+1)(N-m)}$ is the $1/2$ in (15), proving the claim (8). We also notice that by the construction of $\eta(\rho)$, for any $\rho, \rho' \in \widehat{\mathrm{O}}_{2m+1}(\mathbb{F}_q)[H, u]$, we have $\dim(\eta(\rho)) = \dim(\eta(\rho'))$. We have therefore completed the proof of part (1) of the lemma.

Now consider the case of an orthogonal group $\mathrm{O}_{2m}^\alpha(\mathbb{F}_q)$ of even dimension $2m$ and the eta correspondence

$$\eta_{2m,\alpha}^{2N} : \widehat{\mathrm{O}}_{2m}^\alpha(\mathbb{F}_q) \hookrightarrow \widehat{\mathrm{Sp}}_{2N}(\mathbb{F}_q)$$

(again, we omit the superscripts and subscripts for the remainder of the proof, since they are now fixed). Consider an irreducible representation $\rho = r^{\mathrm{O}_{2m}^\alpha(\mathbb{F}_q)}[(s), u]^\sigma$. Say 1 is an eigenvalue of s of multiplicity $2p$ so that its centralizer is of the form

$$Z_{\mathrm{SO}_{2m}^\alpha(\mathbb{F}_q)}(s) = \mathrm{SO}_{2p}^\pm(\mathbb{F}_q) \times H.$$

We consider new semisimple data $\eta(s) = s \oplus I \in \mathrm{SO}_{2N+1}(\mathbb{F}_q)$, so that

$$Z_{\mathrm{SO}_{2N+1}(\mathbb{F}_q)}(\eta(s)) = \mathrm{SO}_{2(N-m+p)+1}(\mathbb{F}_q) \times H.$$

Suppose $a(\rho) = 1$, so that we may write

$$u = \left(\begin{smallmatrix} \lambda_1 < \dots < \lambda_a \\ \mu_1 < \dots < \mu_b \end{smallmatrix} \right) \otimes u_H$$

for a non-degenerate $\mathrm{SO}_{2p}^{\pm}(\mathbb{F}_q)$ -symbol $\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ and a unipotent H -representation u_H . Writing $N - m + \frac{a+b}{2}$, we may consider this group's unipotent representations

$$\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b < N'_\rho} \otimes u_H \text{ and } \binom{\lambda_1 < \dots < \lambda_a < N'_\rho}{\mu_1 < \dots < \mu_b} \otimes u_H.$$

We choose $\eta(u)$ to be one of them based on the choice of sign data σ_1 . Then $\eta(\rho)$ can be constructed by putting

$$(16) \quad \eta(\rho) = r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(\eta(s)), \eta(u)] \text{ or } r^{\mathrm{Sp}_{2N}(\mathbb{F}_q)}[(\eta(s)), \eta(u), \pm 1],$$

with central sign data needed precisely if $b(\rho) = 1$, in which case we choose the sign ± 1 according to σ_{-1} . For convenience, let us suppose $\eta(\rho)$ is $\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b < N'_\rho} \otimes u_H$ (the case of $\binom{\lambda_1 < \dots < \lambda_a < N'_\rho}{\mu_1 < \dots < \mu_b} \otimes u_H$ can be followed through similarly). Computing dimensions, since we assumed already $a(\rho) = 1$, we then find

$$\begin{aligned} \dim(\eta(\rho)) &= \frac{1}{2^{b(\rho)}} \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{SO}_{2(N-m+p)+1}(\mathbb{F}_q) \times H|_{q'}} \cdot \dim\left(\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b < N'_\rho}\right) \cdot \dim(u_H) \\ \dim(\rho) &= \frac{2}{2^{1+b(\rho)}} \frac{|\mathrm{SO}_{2m}^{\alpha}(\mathbb{F}_q)|_{q'}}{|\mathrm{SO}_{2p}^{\pm}(\mathbb{F}_q) \times H|_{q'}} \cdot \dim\left(\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}\right) \cdot \dim(u_H) \end{aligned}$$

Therefore, applying Lusztig's dimension formula for non-degenerate symbols, we find that the fraction $\dim(\eta(\rho))/\dim(\rho)$ is expressible as

$$(17) \quad \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{2|\mathrm{SO}_{2m}^{\alpha}(\mathbb{F}_q)|_{q'}} \frac{\prod_{i=1}^a (q^{N'_\rho} + q^{\lambda_i}) \prod_{j=1}^b (q^{N'_\rho} - q^{\mu_j})}{\prod_{i=1}^{N'_\rho} (q^{2i} - 1) q^{(a+b)(a+b-2)/4}} = \frac{\prod_{i=N'_\rho+1}^N (q^{2i} - 1) \prod_{i=1}^a (q^{N'_\rho} + q^{\lambda_i}) \prod_{j=1}^b (q^{N'_\rho} - q^{\mu_j})}{2(q^m - \alpha) \prod_{i=1}^{m-1} (q^{2i} - 1) q^{(a+b)(a+b-2)/4}}.$$

This is a polynomial in q of degree

$$(N - N'_\rho)(N + N'_\rho + 1) + (a + b)N'_\rho - m^2 - \frac{(a + b)(a + b - 2)}{4},$$

which can be reduced to $2m(N - M) + m$ by replacing $N'_\rho = N - m + (a + b)/2$. The claimed half coefficient of the top term is the $1/2$ appearing in (17).

Now we consider the case of $a(\rho) = 0$. In this case, we may write

$$u = \binom{\lambda_1 < \dots < \lambda_a}{\lambda_1 < \dots < \lambda_a}^+ \otimes u_H$$

for a degenerate symbol $\binom{\lambda_1 < \dots < \lambda_a}{\lambda_1 < \dots < \lambda_a}^+$ of $\mathrm{SO}_{2p}^+(\mathbb{F}_q)$ (in which we count the case where $p = 0$, considering $\binom{\emptyset}{\emptyset}^+$ as the trivial representation). We put

$$\eta(u) = \binom{\lambda_1 < \dots < \lambda_a < N'_\rho}{\lambda_1 < \dots < \lambda_a}$$

and construct $\eta(\rho)$ as (16), again choosing the central sign according to σ_{-1} when it is needed (which happens precisely when we have σ_{-1} in ρ 's extension sign data). Computing dimensions, since we assumed $a(\rho) = 0$, we then find

$$\begin{aligned} \dim(\eta(\rho)) &= \frac{1}{2^{b(\rho)}} \frac{|\mathrm{Sp}_{2N}(\mathbb{F}_q)|_{q'}}{|\mathrm{SO}_{2(N-m+p)+1}(\mathbb{F}_q) \times H|_{q'}} \cdot \dim\left(\begin{smallmatrix} \lambda_1 < \dots < \lambda_a < N'_\rho \\ \lambda_1 < \dots < \lambda_a \end{smallmatrix}\right) \cdot \dim(u_H) \\ \dim(\rho) &= \frac{2}{2^{b(\rho)}} \frac{|\mathrm{SO}_{2m}^{\pm}(\mathbb{F}_q)|_{q'}}{|\mathrm{SO}_{2p}^{\pm}(\mathbb{F}_q) \times H|_{q'}} \cdot \dim\left(\begin{smallmatrix} \lambda_1 < \dots < \lambda_a \\ \lambda_1 < \dots < \lambda_a \end{smallmatrix}\right)^+ \cdot \dim(u_H) \end{aligned}$$

Applying Lusztig's dimension formula for symbols to reduce $\dim(\eta(\rho))$ and the halved dimension formula for degenerate symbols to reduce $\dim(\rho)$, we then find that $\dim(\eta(\rho))/\dim(\rho)$ is, again, expressible as (17), which as we argued above, gives the claim. We also notice that by the construction of $\eta(\rho)$, for any $\rho, \rho' \in \widehat{\mathrm{O}}_{2m}^{\alpha}(\mathbb{F}_q)[H, u]$, we have $\dim(\eta(\rho)) = \dim(\eta(\rho'))$. We have therefore completed the proof of part (2) of the lemma. \square

3. REASSEMBLING THE CHARACTERS OF THE OSCILLATOR REPRESENTATIONS

Our approach to bounding the sums (10), (11) and (12), (13) will be to ‘‘reassemble’’ the characters of the eta correspondence back into characters of oscillator representation.

3.1. Reassembling the top part of the oscillator representation.

For any orthogonal space and form (W, B) , within the restriction of an oscillator representation, we may consider the ‘‘top part’’

$$\omega^{\mathrm{top}}[V \otimes (W, B)] = \bigoplus_{\rho \in \widehat{\mathrm{O}}(W, B)} \eta_{W, B}^V(\rho) \otimes \rho$$

forming a representation of $\mathrm{Sp}(V) \times \mathrm{O}(W, B)$. In terms of characters, for every $g \in \mathrm{Sp}(V)$, $h \in \mathrm{O}(W, B)$, we have

$$(18) \quad \chi_{\omega^{\mathrm{top}}[V \otimes (W, B)]}(g \otimes h) = \sum_{\rho \in \widehat{\mathrm{O}}(W, B)} \chi_{\eta_{W, B}^V(\rho)}(g) \cdot \chi_{\rho}(h)$$

We can see, then, that it is at least possible to reexpress the terms (10), (11) and (12), (13) in terms of the characters of the corresponding top part of an oscillator representation by elementary character theory:

First we note that, for each conjugacy class $(h) \in \mathrm{O}_{2m+1}(\mathbb{F}_q)$, we may pick a corresponding constant $c_{(h)} \in \mathbb{C}$ such that the following system

of equation holds: For every irreducible representation ρ of $O_{2m+1}(\mathbb{F}_q)$, we have

$$(19) \quad \frac{1}{\dim(\rho)} = \sum_{(h) \in O_{2m+1}(\mathbb{F}_q)} c_{(h)} \chi_\rho(h).$$

Given these coefficients $c_{(h)}$, substituting out each coefficient $1/\dim(\rho)$ and then applying (18), we may re-express (10) as

$$(20) \quad \frac{2}{q^{(2m+1)(N-m)}} \sum_{(h) \in O_{2m+1}(\mathbb{F}_q)} c_{(h)} \cdot \chi_{\omega^{\text{top}}[V \otimes (\mathbb{F}_q^{2m+1}, \alpha)]}(g \otimes h).$$

Each error term (11) can be similarly reduced by picking coefficients $a_{(h)}^{H,u}$ such that

$$(21) \quad \begin{aligned} \frac{1}{\dim(\rho)} &= \sum_{(h) \in O_{2m+1}(\mathbb{F}_q)} a_{(h)}^{H,u} \chi_\rho(h) \text{ for } \rho \in \widehat{O_{2m+1}(\mathbb{F}_q)}[H, u] \\ 0 &= \sum_{(h) \in O_{2m+1}(\mathbb{F}_q)} a_{(h)}^{H,u} \chi_\rho(h) \text{ else.} \end{aligned}$$

Again, by substituting out each coefficient $1/\dim(\rho)$ and using (18), we may then process an error term (11) as

$$(22) \quad K_{H,u} \sum_{(h) \in O_{2m+1}(\mathbb{F}_q)} a_{(h)}^{H,u} \cdot \chi_{\omega^{\text{top}}[V \otimes (\mathbb{F}_q^{2m+1}, \alpha)]}(g \otimes h).$$

We also proceed similarly in the case of even orthogonal groups. For $(h) \in O_{2m}^\pm(\mathbb{F}_q)$ we may pick coefficients $b_{(h)}^\pm \in \mathbb{C}$ such that for every irreducible representation ρ of $O_{2m}^\pm(\mathbb{F}_q)$, we have

$$(23) \quad \frac{1}{\dim(\rho)} = \sum_{(h) \in O_{2m}^\pm(\mathbb{F}_q)} b_{(h)}^\pm \chi_\rho(h).$$

By substituting for $1/\dim(\rho)$ and applying (18), we may then process the sum (12) as

$$(24) \quad \frac{2}{q^{2m(N-m)+m}} \sum_{(h) \in O_{2m}^\pm(\mathbb{F}_q)} b_{(h)}^\pm \cdot \chi_{\omega^{\text{top}}[V \otimes (\mathbb{F}_q^{2m}, \pm)]}(g \otimes h).$$

The error terms (13) will be treated by picking coefficients $a_{(h)}^{\pm, (H, u)}$ such that

$$(25) \quad \begin{aligned} \frac{1}{\dim(\rho)} &= \sum_{(h) \in \widehat{O_{2m}^{\pm}(\mathbb{F}_q)}} a_{(h)}^{\pm, (H, u)} \chi_{\rho}(h) \text{ for } \rho \in \widehat{O_{2m}^{\pm}(\mathbb{F}_q)}[H, u] \\ 0 &= \sum_{(h) \in \widehat{O_{2m}^{\pm}(\mathbb{F}_q)}} a_{(h)}^{\pm, (H, u)} \chi_{\rho}(h) \text{ else} \end{aligned}$$

for every choice of subgroup (up to isomorphism) $H \subseteq \mathrm{SO}_{2m}^{\pm}(\mathbb{F}_q)$ obtainable as a centralizer of a semisimple element and every irreducible unipotent representation u of H . By substituting out the $1/\dim(\rho)$ coefficients and applying (18), we can then process each error term (13) as

$$(26) \quad K_{H, u}^{\pm} \sum_{(h) \in \widehat{O_{2m}^{\pm}(\mathbb{F}_q)}} a_{(h)}^{\pm, (H, u)} \cdot \chi_{\omega^{\mathrm{top}}[V \otimes (\mathbb{F}_q^{2m, \pm})]}(g \otimes h).$$

In summary, for an odd orthogonal group $O_{2m+1}(\mathbb{F}_q)$, we have reduced the character ratio sum (5) to the main term (20), summed with error terms of the form (22). For the even orthogonal groups $O_{2m}^{\pm}(\mathbb{F}_q)$, we have reduced the character ratio sum (6) to the sum of main terms (24) corresponding to each orthogonal group $O_{2m}^{\pm}(\mathbb{F}_q)$, summed with error terms of the form (26). In each case, we have re-expressed the desired character ratio sum in terms of certain coefficients $c_{(h)}$, $a_{(h)}^{H, u}$ or $b_{(h)}^{\pm}$, $a_{(h)}^{\pm, (H, u)}$ and characters of the top part of the relevant oscillator representation.

3.2. The coefficients from character theory. The purpose of this subsection is to write the explicit formulae for the coefficients $c_{(h)}$, $a_{(h)}^{H, u}$ or $b_{(h)}^{\pm}$, $a_{(h)}^{\pm, (H, u)}$ described in the above subsection, and produce a bound for each coefficient's q -degree for any choice of (h) .

By the orthogonality of characters for $O_{2m+1}(\mathbb{F}_q)$, to satisfy the equations (19) and (21), we put

$$(27) \quad \begin{aligned} c_{(h)} &= \frac{|(h)|}{|O_{2m+1}(\mathbb{F}_q)|} \sum_{\rho \in \widehat{O_{2m+1}(\mathbb{F}_q)}} \frac{\overline{\chi_{\rho}(h)}}{\dim(\rho)}, \\ a_{(h)}^{H, u} &= \frac{|(h)|}{|O_{2m+1}(\mathbb{F}_q)|} \sum_{\rho \in \widehat{O_{2m+1}(\mathbb{F}_q)}[H, u]} \frac{\overline{\chi_{\rho}(h)}}{\dim(\rho)} \end{aligned}$$

where $|(h)|$ denotes the order of each conjugacy class (h) . Splitting off the orthogonal group's center $O_{2m+1}(\mathbb{F}_q) = \mathrm{SO}_{2m+1}(\mathbb{F}_q) \times \mathbb{Z}/2$, we find

that (27) can be simplified to

$$(28) \quad \begin{aligned} c_{(h)} &= \frac{|(h)|}{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|} \sum_{\rho \in \widehat{\mathrm{SO}_{2m+1}(\mathbb{F}_q)}} \frac{\overline{\chi_\rho(h)}}{\dim(\rho)} \text{ if } \det(h) = 1 \\ a_{(h)}^{H,u} &= \frac{|(h)|}{|\mathrm{SO}_{2m+1}(\mathbb{F}_q)|} \sum_{\rho \in \widehat{\mathrm{SO}_{2m+1}(\mathbb{F}_q)[H,u]}} \frac{\overline{\chi_\rho(h)}}{\dim(\rho)} \text{ if } \det(h) = 1 \\ c_{(h)} &= a_{(h)}^{H,u} = 0 \text{ if } \det(h) = -1. \end{aligned}$$

for every choice of (H, u) .

Similarly, in the even case, to satisfy the equations (23) and (25) as

$$(29) \quad \begin{aligned} b_{(h)}^\pm &= \frac{|(h)|}{|\mathrm{O}_{2m}^\pm(\mathbb{F}_q)|} \sum_{\rho \in \widehat{\mathrm{O}_{2m}^\pm(\mathbb{F}_q)}} \frac{\overline{\chi_\rho(h)}}{\dim(\rho)}, \\ a_{(h)}^{\pm,(H,u)} &= \frac{|(h)|}{|\mathrm{O}_{2m}^\pm(\mathbb{F}_q)|} \sum_{\rho \in \widehat{\mathrm{O}_{2m}^\pm(\mathbb{F}_q)[H,u]}} \frac{\overline{\chi_\rho(h)}}{\dim(\rho)}. \end{aligned}$$

Though we can no longer reduce to the identity component $\mathrm{SO}_{2m}^\pm(\mathbb{F}_q)$ as easily as for the odd case (28), we will still find that the coefficients $b_{(h)}^\pm$ vanish for h with determinant -1 . Intuitively, this can be seen since the action of the $\mathbb{Z}/2$ corresponding to ± 1 -determinant in $\mathrm{O}_{2m}^\pm(\mathbb{F}_q)$ in an irreducible representation ρ does not affect the value of $1/\dim(\rho)$, but we prove this more formally below.

Lemma 5. *Consider an orthogonal space W of dimension n .*

- (1) *Say $n = 2m + 1$ is odd. Then, for $(h) \in \mathrm{SO}(W)$ with Jordan decomposition $h_s \cdot h_u$, the coefficients $c_{(h)}$ and $a_{(h)}^{H,u}$ satisfy*

$$(30) \quad \begin{aligned} \deg_q(a_{(h)}^{H,u} |Z_{\mathrm{SO}(W)}(h)|), \deg_q(c_{(h)} |Z_{\mathrm{SO}(W)}(h)|) &\leq \\ m - \lceil \frac{\mathrm{mult}_{-1}(h_s)}{4} \rceil & \end{aligned}$$

for any subgroup H obtained as the centralizer of a semisimple element in $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ and any irreducible unipotent representation u of H , where $\mathrm{mult}_{-1}(h_s)$ denotes the multiplicity of -1 as an eigenvalue of h_s .

- (2) *Say $n = 2m$ is even and $\mathrm{O}(W) = \mathrm{O}_{2m}^\pm(\mathbb{F}_q)$. For $(h) \in \mathrm{O}_{2m}^\pm(\mathbb{F}_q)$ with $\det(h) = -1$, we have*

$$(31) \quad b_{(h)}^\pm = a_{(h)}^{\pm,(H,u)} = 0.$$

For (h) with $\det(h) = 1$, writing its Jordan decomposition as $h_s \cdot h_u$, the coefficients $b_{(h)}^\pm$ and $a_{(h)}^{\pm, (H, u)}$ satisfy

$$(32) \quad \deg_q(a_{(h)}^{\pm, (H, u)} |Z_{SO(W)}(h)|), \deg_q(b_{(h)}^\pm |Z_{SO_{2m}^\pm(\mathbb{F}_q)}(h)|) \leq m - \lceil \frac{\text{mult}_{-1}(h_s)}{4} \rceil$$

for any subgroup H obtained as the centralizer of a semisimple element in $SO_{2m}^\pm(\mathbb{F}_q)$ and any irreducible unipotent representation u of H , where $\text{mult}_{-1}(h_s)$ denotes the multiplicity of -1 as an eigenvalue of h_s .

Remark: We note that this statement is somewhat over-kill for our purposes: In most cases of (h) , it will be enough to use the bound m on q -degree instead of $m - \lceil \text{mult}_{-1}(h_s)/4 \rceil$, and otherwise (in the “ -1 eigenvalue cases” of Sections 5 and 6), we will still only use the bound $m - \lceil \text{mult}_{-1}(h)/4 \rceil$ with $\text{mult}_{-1}(h)$ denoting geometric multiplicity of -1 as an eigenvalue of h . instead of $m - \lceil \text{mult}_{-1}(h_s)/4 \rceil$.

Proof. We begin with the case where $\dim(W) = 2m + 1$ is odd and we can write $O(W) = O_{2m+1}(\mathbb{F}_q)$. Fix a conjugacy class $(h) \in SO(W)$ and say its Jordan decomposition is $h = h_s \cdot h_u$ for commuting semisimple and unipotent elements $h_s, h_u \in SO(W)$. First, we recall that the number of group elements in (h) can be expressed as the quotient of the order of the group by the order of h 's centralizer $|SO(W)|/|Z_{SO(W)}(h)|$. Therefore,

$$(33) \quad c_{(h)} \cdot |Z_{SO(W)}(h)| = \sum_{\rho \in \widehat{SO(W)}} \frac{\overline{\chi_\rho(h)}}{\dim(\rho)}.$$

Consider a subgroup $H \subseteq \text{Sp}_{2m}(\mathbb{F}_q) = (SO(W))^*$ (up to isomorphism) obtainable as the centralizer of a semisimple element and an irreducible unipotent representation u of H . Since the number of choices of H, u does not depend on q , it suffices to show for each individual choice of this data, the q -degree of

$$(34) \quad \sum_{\rho \in \widehat{SO(W)}[H, u]} \frac{\overline{\chi_\rho(h)}}{\dim(\rho)}$$

is less than or equal to $m - \lceil \text{mult}_{-1}(h_s)/4 \rceil$. We note that (34) is also precisely $a_{(h)}^{H, u} \cdot |Z_{SO(W)}(h)|$, so this would also prove the claim (30) for $a_{(h)}^{H, u}$.

Fix a choice of H and u . We may write H as a product

$$(35) \quad H = \mathrm{Sp}_{2p}(\mathbb{F}_q) \times \mathrm{Sp}_{2p'}(\mathbb{F}_q) \times \prod_{i=1}^k \mathrm{U}_{n_i}^{\pm}(\mathbb{F}_{q^{r_i}})$$

for $p + \ell + \sum_{i=1}^k n_i r_i = m$ (using the notation that $\mathrm{U}^+ = \mathrm{GL}$). To bound the q -degree of (34), we use the following notation. Let us write \mathcal{S}_H for the set of conjugacy classes $(s) \in \mathrm{Sp}_{2m}(\mathbb{F}_q)$ whose centralizer is isomorphic to H . Given (35), each such conjugacy class (s) is conjugate in $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ to the sum of the identity matrix I_{2p} , the negative identity matrix $-I_{2p'}$, and the sum of n_i copies of a matrix of the form

$$\left(\begin{array}{ccc|ccc} \lambda_i & & & & & \\ & \ddots & & & & \\ & & \lambda_i^{q^{r_i-1}} & & & \\ \hline & & & \lambda_i^{-1} & & \\ & & & & \ddots & \\ & & & & & \lambda_i^{-q^{r_i-1}} \end{array} \right)$$

for some λ_i chosen in $\mu_{q^{r_i \mp 1}}$ (with sign chosen opposite to the \pm superscript of $\mathrm{U}_{n_i}^{\pm}$) not equal to ± 1 and not in any $\mu_{q^r \mp 1}$ for $r < r_i$, for each $i = 1, \dots, k$. We note that replacing λ_i by λ_i^q gives the same conjugacy class (s) . Now we consider for each subset $R \subseteq \{1, \dots, k\}$, a corresponding subset of \mathcal{S}_H defined by

$$\mathcal{S}_H^R = \{(s) \in \mathcal{S}_H \mid \{i \mid \sum_{j=0}^{r_i-1} \lambda_i^{q^j} \neq \pm \alpha_i[\pm]\} = R\},$$

where we put α_i to be a multiplicative generator of $\mu_{q^{\mp 1}}$ with sign chosen opposite to the superscript of the $\mathrm{U}_{n_i}^{\pm}$ factor in H and define $\alpha_i[+] = \alpha_i$ and

$$\alpha_i[-] = \begin{cases} -\alpha_i & \text{if } -1 \in \mu_{q^{\mp 1}}^2 \\ -\alpha_i^{-1} & \text{if } -1 \notin \mu_{q^{\mp 1}}^2. \end{cases}$$

These \mathcal{S}_H^R form a disjoint partition of \mathcal{S}_H .

Lemma 6. *Consider $R \subseteq \{1, \dots, k\}$ with $|R| > m - \lceil \mathrm{mult}_{-1}(h_s)/4 \rceil$. Then*

$$(36) \quad \sum_{(s) \in \mathcal{S}_H^R} \frac{\overline{\chi_{r[(s), u]}(h)}}{\dim(r[(s), u])} = 0$$

where $r[(s), u]$ denotes the irreducible $\mathrm{SO}(W)$ -representaiton corresponding to (s) and our fixed u .

Given this lemma, we may re-express the sum (34) as

$$\sum_{\rho \in \widehat{\mathrm{SO}(W)}[H, u]} \frac{\overline{\chi_\rho(h)}}{\dim(\rho)} = \sum_{|R| \leq m - \lceil \frac{\mathrm{mult}_{-1}(h_s)}{4} \rceil} \sum_{(s) \in \mathcal{S}_H^R} \frac{\overline{\chi_{r[(s), u]}(h)}}{\dim(r[(s), u])}.$$

This has q -degree less than or equal to $m - \lceil \mathrm{mult}_{-1}(h_s)/4 \rceil$ since each summand $\chi_{r[(s), u]}(h)/\dim(r[(s), u])$ has absolute value ≤ 1 , the number of choices of R does not depend on q , and the cardinality of each \mathcal{S}_H^R has q -degree equal to $|R| \leq m - \lceil \mathrm{mult}_{-1}(h_s)/4 \rceil$. Hence, to conclude (30), it suffices to prove Lemma 6.

Proof of Lemma 6. We now must recall more about the classification of irreducible representations of a group such as $\mathrm{SO}(W)$.

In general, for a semisimple conjugacy class $(s) \in G^*$, write $\mathcal{E}(G, (s))$ for the (s) -Lusztig series of irreducible G -representations, consisting of those with characters χ such that

$$\langle \chi, R_G^T(\theta) \rangle \neq 0,$$

where T and θ are a torus and a character on T corresponding to (s) and $R_G^T(\theta)$ denotes the Deligne-Lusztig induction of θ . (We note that the correspondence between tori and their characters in G and semisimple elements and tori in G^* assumes a fixed identification of $\mu_{q \mp 1}$ with its Pontrjagin dual. We write χ_λ for the character corresponding to $\lambda \in \mu_{q \mp 1}$.) In particular, $\mathcal{E}(G, 1)$ is the set of irreducible unipotent representations of G . The classification of irreducible representations follows according to a system of bijections

$$\mathcal{E}(G, (s)) \rightarrow \mathcal{E}(Z_G(s), 1).$$

We will denote these bijections by π_s^G for consistency with the notation of [1]. Denote their inverses by σ_s^G . In the above notation, the irreducible representation $r[(s), u]$ is, by definition, $\sigma_s^{\mathrm{SO}(W)}(u)$. Using the fact that

$$\dim(\sigma_s^{\mathrm{SO}(W)}(u)) = \frac{|\mathrm{SO}(W)|_{q'}}{|H|_{q'}} \dim(u),$$

the denominators of each term of the left hand side of (36) are all equal. Therefore, also by taking complex conjugate, it suffices to show

$$(37) \quad \sum_{(s) \in \mathcal{S}_H^R} \chi_{\sigma_s^{\mathrm{SO}(W)}(u)}(h) = 0.$$

Now we will use the properties of $\pi_s^{\mathrm{SO}(W)}$ (translated to work with $\sigma_s^{\mathrm{SO}(W)}$) described in [1], Theorem 7.1 to calculate with the character

values of $\sigma_s^{\text{SO}(W)}(u)$. First, consider the subgroup

$$L = \text{SO}_{2(p+p')+1}(\mathbb{F}_q) \times \prod_{i=1}^k \text{U}_{n_i}^\pm(\mathbb{F}_{q^{r_i}})$$

of $\text{SO}_{2m+1}(\mathbb{F}_q)$ containing H . Its dual is isomorphic to the fixed points of a standard Levi subgroup of $\text{Sp}_{2m}(\overline{\mathbb{F}}_q)$ under an appropriate twisted Frobenius. We may then write

$$\sigma_s^{\text{SO}(W)}(u) = R_{\text{SO}(W)}^L(\sigma_s^L(u))$$

(see (iv) of Theorem 7.1 of [1]) for every $(s) \in \mathcal{S}_H^R$ where $R_{\text{SO}(W)}^L$ denotes the Deligne-Lusztig induction from L to $\text{SO}(W)$. Now we may apply the character formula for Deligne-Lusztig induction (see, for example, Proposition 12.2 of [2]) to re-express $\chi_{\sigma_s^{\text{SO}(W)}(u)}(h)$ as

$$\frac{1}{|L| \cdot |Z_{\text{SO}(W)}(h_s)|} \sum_{\substack{x \in \text{SO}(W) \\ x^{-1}h_sx \in L}} |Z_{xLx^{-1}}(h_s)| \cdot \sum_{\substack{v \in Z_{xLx^{-1}}(h_s) \\ \text{unipotent}}} Q(h_u, v^{-1}) \cdot \chi_{\sigma_s^L(u)}(x^{-1}h_svx)$$

where Q denotes a certain Green function $Q_{Z_{xLx^{-1}}(h_s)}^{Z_{\text{SO}(W)}(h_s)}$ (again recall that we write $h = h_s \cdot h_u$ for the Jordan decomposition of $h \in \text{SO}(W)$). We can see that the only factor of this expression that depends on the choice of $(s) \in \mathcal{S}_H^R$ is the final factor $\chi_{\sigma_s^L(u)}(x^{-1}h_svx)$. Therefore, the sum over $(s) \in \mathcal{S}_H^R$ can be passed through all other factors; to prove (37) it hence suffices to show for every $x \in \text{SO}(W)$ such that $x^{-1}h_sx \in L$ and for every $v \in Z_{xLx^{-1}}(h_s)$ unipotent, the sum of characters of σ_s^L cancels

$$(38) \quad \sum_{(s) \in \mathcal{S}_H^R} \chi_{\sigma_s^L(u)}(x^{-1}h_svx) = 0.$$

To study the left hand side of (38), let us factor u as a representation of H into a tensor product

$$u = u_{\text{Sp}_{2p} \times \text{Sp}_{2p'}} \otimes \bigotimes_{i=1}^k u_{\text{U}_{n_i}^\pm}$$

where $u_{\text{Sp}_{2p} \times \text{Sp}_{2p'}}$ is an irreducible unipotent representation of $\text{Sp}_{2p}(\mathbb{F}_q) \times \text{Sp}_{2p'}(\mathbb{F}_q)$ and each $u_{\text{U}_{n_i}^\pm}$ is an irreducible unipotent representation of $\text{U}_{n_i}^\pm(\mathbb{F}_{q^{r_i}})$ (recalling (35)). Then $\sigma_s^L(u)$ may always be factored as the tensor product of

$$(39) \quad \sigma_{I_{2p} \oplus -I_{2p}}^{\text{SO}_{2(p+p')+1}(\mathbb{F}_q)}(u_{\text{Sp}_{2p} \times \text{Sp}_{2p'}})$$

as a representation of the $\mathrm{SO}_{2(p+p')+1}(\mathbb{F}_q)$ factor of L , with the tensor product

$$u_{\mathrm{U}_{n_i}^\pm} \otimes \chi_{\mathrm{Tr}(\lambda_i)}(\det)$$

considered as a representation of $\mathrm{U}_{n_i}^\pm(\mathbb{F}_{q^{r_i}})$ over $i = 1, \dots, k$. (This follows by applying (vii) and (iii) of Theorem 7.1 of [1], noting also that each $\mathrm{U}_{n_i}^\pm(\mathbb{F}_{q^{r_i}})$ is self-dual.) Therefore, each term of the left hand side of (38) can be expressed as the product of the character value of (39) at $(x^{-1}h_s vx)_{\mathrm{SO}_{2(p+p')+1}}$ multiplied by

$$\prod_{i=1}^k \chi_{u_{\mathrm{U}_{n_i}^\pm}}((x^{-1}h_s vx)_{\mathrm{U}_{n_i}^\pm}) \cdot \prod_{i=1}^k \chi_{\mathrm{Tr}(\lambda_i)}(\det((x^{-1}h_s vx)_{\mathrm{U}_{n_i}^\pm}))$$

writing $(x^{-1}h_s vx)_K$ for the factor of $x^{-1}h_s vx$ corresponding to each factor K of L and recalling the notation of λ_i for certain eigenvalues determining (s) . Again, the only factor that depends on the choice of $(s) \in \mathcal{S}_H^R$ is the final $\chi_{\lambda_i}(\det)$ factors. Hence, it finally suffices to show the vanishing of

$$(40) \quad \sum_{(s) \in \mathcal{S}_H^R} \prod_{i=1}^k \chi_{\mathrm{Tr}(\lambda_i)}(\det((x^{-1}h_s vx)_{\mathrm{U}_{n_i}^\pm})) = \sum_{(s) \in \mathcal{S}_H^R} \prod_{i=1}^k \chi_{\mathrm{Tr}(\lambda_i)}(\det((x^{-1}h_s x)_{\mathrm{U}_{n_i}^\pm})).$$

We claim that there exists an $j \in R$ such that every eigenvalue of $(x^{-1}h_s x)_{\mathrm{U}_{n_j}^\pm}$ is -1 and $n_j r_j$ is odd. Given this claim, we immediately see the vanishing of (40), since for each $(s) \in \mathcal{S}_H^R$ where λ_j is a square, we may consider $(s') \in \mathcal{S}_H^R$ obtained by replacing the eigenvalues λ_j by $\alpha_j \lambda_j$ (matching up every element of \mathcal{S}_H^R) and the sum of the terms of (40) corresponding to (s) and (s') is a multiple of

$$\chi_{\mathrm{Tr}(\lambda_j)}(\det(-I_{n_j})) + \chi_{\mathrm{Tr}(\alpha_j \lambda_j)}(\det(-I_{n_j})),$$

which is 0 since we assume $n_j r_j$ is odd (and therefore both n_j and r_j are odd).

The claim follows from the following elementary combinatorial fact: If $|R| > m' - \lceil t/2 \rceil$, then for any partition $\coprod_{i \in R} P_i = \{1, \dots, m'\}$ there exists a choice of $j \in R$ such that $P_j \subseteq \{1, \dots, t\}$ and $|P_j|$ is odd. This is applied in our current situation by putting $m' = m - (p + p')$ (representing the coordinates of eigenvalues of $x^{-1}g_s x$ not used to form $(x^{-1}g_s x)_{\mathrm{Sp}_{2p} \times \mathrm{Sp}_{2p'}}$), $t = \mathrm{mult}_{-1}(h_s)/2$ (the halving being due to how A- and 2 A-type groups are embedded in $\mathrm{Sp}_{2m}(\mathbb{F}_q)$), and P_i to be the set

of coordinates of eigenvalues of $x^{-1}g_sx$ which will be used to form the eigenvalues of $(x^{-1}g_sx)_{\mathbb{U}_{n_i}^\pm}$, so that $|P_i| = n_i \cdot r_i$. \square

We have now concluded the argument in the case of odd-dimensional W .

Now let us consider the case where $\dim(W) = 2m$ is even, and we can write $O(W) = O_{2m}^\pm(\mathbb{F}_q)$. As in the odd case, we again have

$$\frac{|(h)|}{|O_{2m}^\pm(\mathbb{F}_q)|} = \frac{1}{|Z_{O(W)}(h)|},$$

and thus,

$$(41) \quad b_{(h)}^\pm \cdot |Z_{O(W)}(h)| = \sum_{\rho \in \widehat{O(W)}} \frac{\overline{\chi_\rho(h)}}{\dim(\rho)}.$$

Let us first consider $(h) \in O_{2m}^\pm(\mathbb{F}_q)$ with $\det(h) = -1$. In this case, it suffices to prove that (41) vanishes. Consider semisimple data (s) , consisting of a conjugacy class in $O_{2m}^\pm(\mathbb{F}_q)$ such that $\det(s) = 1$, and unipotent data u , consisting of an irreducible unipotent representation of (the dual of) the centralizer of s . First note that if s has either 1 or -1 as an eigenvalue,

$$\chi_{r[(s),u]^1}(h) + \chi_{r[(s),u]^{-1}}(h) = 0,$$

and if s has both 1 and -1 as eigenvalues, then

$$\chi_{r[(s),u]^{1,1}}(h) + \chi_{r[(s),u]^{-1,1}}(h) + \chi_{r[(s),u]^{1,-1}}(h) + \chi_{r[(s),u]^{-1,-1}}(h) = 0.$$

Each $\rho = r[(s),u]^{\pm 1}$ or $r[(s),u]^{\pm 1, \pm 1}$ appears in (41) with coefficient $1/\dim(\rho)$, which does not depend on the choice of extension sign in ρ 's Lusztig classification data. Therefore, all terms of (41) contributed by irreducible representations $\rho \in \widehat{O_{2m}^\pm(\mathbb{F}_q)}$ corresponding to semisimple data with 1 or -1 eigenvalues can be immediately seen to cancel. The remaining representations ρ are expressible as the inductions of irreducible representations $\bar{\rho}$ of $SO_{2m}^\pm(\mathbb{F}_q)$. Therefore, for such a $\rho = \text{Ind}_{O(W)}^{\text{SO}(W)}(\bar{\rho})$, its character value at h can be calculated as

$$\chi_{\text{Ind}_{O(W)}^{\text{SO}(W)}(\bar{\rho})}(h) = \frac{1}{|\text{SO}(W)|} \sum_{\substack{x \in O(W) \\ xhx^{-1} \in \text{SO}(W)}} \chi_{\bar{\rho}}(xhx^{-1}) = 0,$$

since for any $x \in O(W)$, the determinants satisfy $\det(xhx^{-1}) = \det(h) = -1$, so we will never have $xhx^{-1} \in \text{SO}(W)$.

The proof of (32) for conjugacy classes $(h) \in \mathrm{O}_{2m}^\pm(\mathbb{F}_q)$ proceeds similarly as the proof of (30), noting that the explicit values of the eigenvalues distinct from ± 1 for the semisimple data of an irreducible representation again do not affect the value of $\dim(\rho)$ for irreducible representations ρ of $\mathrm{O}_{2m}^\pm(\mathbb{F}_q)$. \square

We note also that by (28) and (31), we may restrict the sums found in the previous subsection to conjugacy classes of h with determinant 1. In the odd case, the top term (10) is then

$$\frac{2}{q^{(2m+1)(N-m)}} \sum_{(h) \in \mathrm{SO}_{2m+1}(\mathbb{F}_q)} c(h) \cdot \chi_{\omega^{\mathrm{top}}[V \otimes (\mathbb{F}_q^{2m+1}, \alpha)]}(g \otimes h)$$

and the error terms (11) are

$$K_{H,u} \sum_{(h) \in \mathrm{SO}_{2m+1}(\mathbb{F}_q)} a_{(h)}^{H,u} \cdot \chi_{\omega^{\mathrm{top}}[V \otimes (\mathbb{F}_q^{2m+1}, \alpha)]}(g \otimes h).$$

In the even case, the top terms (12) are then

$$\frac{2}{q^{2m(N-m)+m}} \sum_{(h) \in \mathrm{O}_{2m}^\pm(\mathbb{F}_q), \det(h)=1} b_{(h)}^\pm \cdot \chi_{\omega^{\mathrm{top}}[V \otimes (\mathbb{F}_q^{2m}, \pm)]}(g \otimes h)$$

and the error terms (13) are

$$K_{H,u}^\pm \sum_{(h) \in \mathrm{O}_{2m}^\pm(\mathbb{F}_q), \det(h)=1} a_{(h)}^{\pm, (H,u)} \cdot \chi_{\omega^{\mathrm{top}}[V \otimes (\mathbb{F}_q^{2m}, \pm)]}(g \otimes h).$$

3.3. Reducing to a sum in terms of true oscillator representations. Instead of the top parts of the oscillator representation, our goal is to express the character ratio sum only in terms of the characters of full oscillator representations.

For any orthogonal space and form (W, B) such that $\mathrm{O}(W, B)$ has rank m , we recall (see [9]) that the full decomposition of the restriction of an oscillator representation $\omega[V \otimes (W, B)]$ to a product $\mathrm{Sp}(V) \times \mathrm{O}(W, B) \subseteq \mathrm{Sp}(V \otimes W)$ decomposes as

$$\bigoplus_{k=0}^m \bigoplus_{\rho \in \widehat{\mathrm{O}(W[-k], B[-k])}} \eta_{W[-k], B[-k]}^V(\rho) \otimes \mathrm{Ind}_{\mathrm{O}(W, B)}^{P_k}(\rho^-),$$

where, and for each choice of k , P_k denotes the standard maximal parabolic with Levi subgroup $\mathrm{O}(W[-k], B[-k]) \times \mathrm{GL}_k(\mathbb{F}_q)$, and ρ^- denotes the representation obtained from $\rho \otimes \epsilon(\det)$ on the Levi factor of P_k and

taking the unipotent radical act trivially. (In the case of $O(W, B) = O_{2m}^-(\mathbb{F}_q)$, we omit the $k = m$ term, since there $O(W[-m], B[-m])$ does not exist).

Therefore, one can express each top part $\omega^{\text{top}}[V \otimes (W, B)]$ as the alternating sum

$$\bigoplus_{k=0}^m \bigoplus_{k_1+\dots+k_j=k} (-1)^j \text{Ind}_{\text{Sp}(V) \times O(W, B)}^{\text{Sp}(V) \times P_{k_1, \dots, k_j}} (\omega[V \otimes (W[-k], B[-k])]^-),$$

(computed in virtual representations), where P_{k_1, \dots, k_j} denotes the standard maximal parabolic subgroup of $O(W, B)$ with Levi subgroup

$$(42) \quad L_{k_1, \dots, k_j} := O(W[-k], B[-k]) \times \text{GL}_{k_1}(\mathbb{F}_q) \times \dots \times \text{GL}_{k_j}(\mathbb{F}_q),$$

and each $\omega[V \otimes (W[-k], B[-k])]^-$ is considered as the representation

$$\omega[V \otimes (W[-k], B[-k])] \otimes \epsilon(\det) \otimes \dots \otimes \epsilon(\det)$$

of $\text{Sp}(V)$ times this factor, inflated by taking the unipotent radical of P_{k_1, \dots, k_j} to act trivially. For an element $z \in P_{k_1, \dots, k_j}$, let us write $(z)^{L_{k_1, \dots, k_j}}$ for the factor of z contained in the Levi group L_{k_1, \dots, k_j} . Further, we write $(z)^O$ for the factor of $(z)^{L_{k_1, \dots, k_j}}$ in the orthogonal group factor $O(W[-k], B[-k])$ of (42), and write $(z)^{\text{GL}_{k_i}(\mathbb{F}_q)}$ for the factor of $(z)^{L_{k_1, \dots, k_j}}$ corresponding to the $\text{GL}_{k_i}(\mathbb{F}_q)$ factor of (42).

We may therefore express the character value of the top part $\omega^{\text{top}}[V \otimes (W, B)]$ at $g \otimes h$ for $g \in \text{Sp}(V)$, $h \in O(W, B)$ as the sum

$$(43) \quad \sum_{0 \leq k = k_1 + \dots + k_j \leq m} (-1)^j \chi_{\text{Ind}_{\text{Sp}(V) \times O(W)}^{\text{Sp}(V) \times P_{k_1, \dots, k_j}} (\omega[V \otimes (W[-k], B[-k])]^-)} (g \otimes h).$$

To reduce this further, we recall the basic elementary fact that for a subgroup $H \subseteq G$, for an H -representation ρ , the character value of ρ 's induction at an element $g \in G$ can be expressed as

$$(44) \quad \chi_{\text{Ind}_G^H(\rho)}(g) = \frac{1}{|H|} \sum_{\substack{x \in G, \\ xgx^{-1} \in H}} \chi_\rho(xgx^{-1}).$$

We apply this to expand the factors

$$(45) \quad \chi_{\text{Ind}_{\text{Sp}(V) \times O(W)}^{\text{Sp}(V) \times P_{k_1, \dots, k_j}} (\omega[V \otimes (W[-k], B[-k])]^-)} (g \otimes h)$$

appearing in (43) as

$$(46) \quad \frac{1}{|P_{k_1, \dots, k_j}|} \sum_{\substack{x \in O(W), \\ xhx^{-1} \in P_{k_1, \dots, k_j}}} \chi_{\omega[V \otimes (W[-k], B[-k])]^-} (g \otimes (xhx^{-1})^{L_{k_1, \dots, k_j}}).$$

For a fixed choice of $h \in \mathrm{SO}(W)$, we may simplify this further. First note that for each element $h' \in P_{k_1, \dots, k_j}$ such that there exists an element $x_0 \in \mathrm{O}(W)$ such that $h' = x_0 h x_0^{-1}$, the set of elements $x \in \mathrm{O}(W)$ which conjugate h and give h' is in bijection with the centralizer of h

$$\begin{aligned} \{x \in \mathrm{O}(W) \mid x h x^{-1} = h'\} &\cong Z_{\mathrm{O}(W)}(h) \\ x &\mapsto x_0^{-1} x. \end{aligned}$$

Additionally, for a given $h' \in P_{k_1, \dots, k_j}$, the number of elements innerly conjugate (i.e. by another element in P_{k_1, \dots, k_j}) to h' is the quotient of the parabolic's order by the order of h' 's centralizer in the parabolic

$$\frac{|P_{k_1, \dots, k_j}|}{|Z_{P_{k_1, \dots, k_j}}(h')|}.$$

Therefore, the sum (46) can be re-written as

$$(47) \quad \sum_{\substack{(h') \in P_{k_1, \dots, k_j}, \\ \text{conj to } h}} \frac{|Z_{\mathrm{O}(W)}(h)|}{|Z_{P_{k_1, \dots, k_j}}(h')|} \chi_{\omega[V \otimes (W[-k], B[-k])]}(g \otimes (h')^{L_{k_1, \dots, k_j}}).$$

Combining this with the expressions found in Subsection 3.2, we therefore find that the character ratio sums (5) can be estimated as the top term consisting of the coefficient $2/q^{(2m+1)(N-m)}$ multiplied by the sum over $(h) \in \mathrm{SO}_{2m+1}(\mathbb{F}_q)$ and $0 \leq k = k_1 + \dots + k_j \leq m$ of $(-1)^j c_{(h)}$ times (47) for $(W, B) = (\mathbb{F}_q^{2m+1}, \alpha)$. The error terms consist of the coefficients $K_{H,u}$ multiplied by the sum over $(h) \in \mathrm{SO}_{2m+1}(\mathbb{F}_q)$ and $0 \leq k = k_1 + \dots + k_j \leq m$ of $(-1)^j a_{(h)}^{H,u}$ times (47) (still for $(W, B) = (\mathbb{F}_q^{2m+1}, \alpha)$).

Similarly, in the even case, the character ratio sum (6) can be estimated as the top terms consisting of the coefficient $2/q^{2m(N-m)+m}$ multiplied by the sum over $(h) \in \mathrm{O}_{2m}^{\pm}(\mathbb{F}_q)$ with $\det(h) = 1$ and $0 \leq k = k_1 + \dots + k_j \leq m$ of $(-1)^j b_{(h)}^{\pm}$ times (47) for $(W, B) = (\mathbb{F}_q^{2m}, \pm)$. The error terms consist of the coefficients $K_{H,u}^{\pm}$ multiplied by the sum over $(h) \in \mathrm{O}_{2m}^{\pm}(\mathbb{F}_q)$ with $\det(h) = 1$ and $0 \leq k = k_1 + \dots + k_j \leq m$ of $(-1)^j a_{(h)}^{\pm, (H,u)}$ times (47) for $(W, B) = (\mathbb{F}_q^{2m}, \pm)$.

4. THE CHARACTERS OF THE OSCILLATOR REPRESENTATION

The purpose of this section is to bound the character of an oscillator representation at a symplectic group element according to the geometric multiplicity of 1 as an eigenvalue of it.

Proposition 7. *Fix a symplectic space \mathbf{V} . For an element $g \in Sp(\mathbf{V})$, let us write $\text{mult}_1(g)$ for the geometric multiplicity of 1 as an eigenvalue of g . Then*

$$(48) \quad \deg_q(|\chi_{\omega[\mathbf{V}]}(g)|) \leq \frac{\text{mult}_1(g)}{2}.$$

Given this proposition, we use the relationship of Kronecker product and Jordan normal form to conclude the following

Corollary 8. *Consider a symplectic space V of dimension $2N$ and an orthogonal space W . Fix an element $h \in O(W, B)$. Let us write $\text{mult}_\lambda(h)$ for the geometric multiplicity of each eigenvalue λ of h (including eigenvalues in a field extension). Write $\mathcal{J}_{1 \times 1}^{(\pm 1)}$ for the number of 1×1 Jordan blocks with eigenvalue ± 1 appearing in h 's Jordan normal form (we put $\mathcal{J}_{1 \times 1}^{(1)} = 0$ if no such Jordan blocks are present).*

(1) *For any $g \neq I \in Sp(V)$, we find that*

$$(49) \quad \begin{aligned} & \deg_q(|\chi_{\omega[V \otimes (W, B)]}(g \otimes h)|) \leq \\ & \frac{1}{2} \cdot (\max\{2N\text{mult}_1(h) - \mathcal{J}_{1 \times 1}^{(1)}, 2N\text{mult}_\lambda(h), \\ & (2N - 1)\text{mult}_1(h) + \text{mult}_\lambda(h), \text{ for } \lambda \neq 1\}) \end{aligned}$$

(2) *For any $g \neq \pm I \in Sp(V)$, we find that*

$$(50) \quad \begin{aligned} & \deg_q(|\chi_{\omega[V \otimes (W, B)]}(g \otimes h)|) \leq \\ & \frac{1}{2} \cdot (\max\{2N\text{mult}_1(h) - \mathcal{J}_{1 \times 1}^{(1)}, (2N - 1)\text{mult}_1(h) + \text{mult}_\lambda(h), \\ & 2N\text{mult}_{-1}(h) - \mathcal{J}_{1 \times 1}^{(-1)}, (2N - 1)\text{mult}_{-1}(h) + \text{mult}_\lambda(h), \\ & 2N\text{mult}_\lambda(h), \text{ for } \lambda \neq 1\}) \end{aligned}$$

This corollary is the final ingredient we will need to prove Theorem 1, since it can be used to bound the factors of the oscillator representation characters appearing in (47). The remaining coefficient of these terms This will be done in Section 5 below.

In the present section, we recall a model of the oscillator representation and prove Proposition 7 in Subsection 4.1 and conclude Corollary 8 using some elementary facts about matrices in Subsection 4.2.

4.1. The dualized Schrödinger model and the generators of the symplectic group. We begin by recalling the structure of an oscillator representation over a finite field. We shall use a “dualized” version of the usual Schrödinger model. Fix a $2N$ -dimensional symplectic space

\mathbf{V} over \mathbb{F}_q . Let us assume that the symplectic form is in standard form

$$\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right)$$

and let us write Λ for the Lagrangian subspace spanned by the first \mathbf{N} standard basis vectors. Then $\mathrm{Sp}(\mathbf{V}) = \mathrm{Sp}_{2\mathbf{N}}(\mathbb{F}_q)$ has a standard set of generators

$$(51) \quad \left(\begin{array}{c|c} I & A \\ \hline 0 & I \end{array} \right), \left(\begin{array}{c|c} (B^T)^{-1} & 0 \\ \hline 0 & B \end{array} \right), \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right)$$

for symmetric $\mathbf{N} \times \mathbf{N}$ -matrices A and invertible $\mathbf{N} \times \mathbf{N}$ -matrices B . Then an oscillator representation $\omega_\psi[\mathbf{V}]$ associated to a non-trivial additive character $\psi : \mathbb{F}_q \rightarrow \mathbb{C}$ can be described as the vector space $\mathbb{C}\Lambda$ with the action given by

$$(52) \quad \begin{aligned} \omega_\psi \left(\begin{array}{c|c} I & A \\ \hline 0 & I \end{array} \right) (v) &= \psi\left(\frac{v^T A v}{2}\right) \cdot (v) \\ \omega_\psi \left(\begin{array}{c|c} (B^T)^{-1} & 0 \\ \hline 0 & B \end{array} \right) (v) &= \epsilon(\det(B)) \cdot (Bv) \\ \omega_\psi \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right) (v) &= \frac{1}{\sum_{u \in \Lambda} \psi(u^T u)} \sum_{w \in \Lambda} \psi(w^T u) \cdot (w) \end{aligned}$$

for every basis element (v) corresponding to a choice of $v \in \Lambda$.

We also note now that every matrix in $\mathrm{Sp}(\mathbf{V})$ is in fact conjugate to a product of generators (51) of the form

$$(53) \quad \begin{aligned} &\left(\begin{array}{c|c} (B^T)^{-1} & 0 \\ \hline 0 & B \end{array} \right) \left(\begin{array}{c|c} I & A \\ \hline 0 & I \end{array} \right) \text{ or} \\ &\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right) \left(\begin{array}{c|c} (B^T)^{-1} & 0 \\ \hline 0 & B \end{array} \right) \left(\begin{array}{c|c} I & A \\ \hline 0 & I \end{array} \right) \end{aligned}$$

for some symmetric A and invertible B .

Proof of Proposition 7. Since both the character value of the oscillator representation and the geometric multiplicity of 1 as an eigenvalue of a certain symplectic group element are invariant under conjugation, it suffices to prove (48) for a representative of every conjugacy class in $\mathrm{Sp}(\mathbf{V})$, i.e. for the products (53).

We begin by proving the claim for

$$(54) \quad g = \left(\begin{array}{c|c} (B^T)^{-1} & 0 \\ \hline 0 & B \end{array} \right) \left(\begin{array}{c|c} I & A \\ \hline 0 & I \end{array} \right) = \left(\begin{array}{c|c} (B^T)^{-1} & (B^T)^{-1}A \\ \hline 0 & B \end{array} \right).$$

According to (52), for $v \in \Lambda$, this element acts on the oscillator representation as

$$\omega_\psi\left(\left(\frac{(B^T)^{-1} \mid 0}{0 \mid B}\right)\left(\frac{I \mid A}{0 \mid I}\right)\right)(v) = \epsilon(\det(B)) \cdot \psi\left(\frac{v^T A v}{2}\right) \cdot (Bv).$$

Only the basis elements (v) satisfying $Bv = v$ will contribute to the trace of this action. These are precisely the vectors in the 1-eigenspace of B , which we shall denote by $E_1(B)$. Therefore, we may express the character value of the oscillator representation at this group element as

$$\chi_{\omega_\psi[\mathbf{V}]}\left(\left(\frac{(B^T)^{-1} \mid 0}{0 \mid B}\right)\left(\frac{I \mid A}{0 \mid I}\right)\right) = \sum_{v \in E_1(B)} \psi\left(\frac{v^T A v}{2}\right).$$

Considering this as a Gaussian sum on $E_1(B)$, the q -degree of its absolute value is

$$(55) \quad \frac{\dim(E_1(B)) + \dim(\text{Ker}(A|_{E_1(B)}))}{2}.$$

The dimension of the eigenspace $E_1(B)$ is equal to the dimension of the eigenspace $E_1((B^T)^{-1})$ of 1 as an eigenvalue of $(B^T)^{-1}$, and we may rewrite $\text{Ker}(A|_{E_1(B)})$ as $\text{Ker}(A) \cap E_1(B)$. In particular, we find that the numerator of (55) is less than or equal to the dimension of the 1-eigenspace of (54), which consists of vectors $(v^T \mid w^T)^T \in \mathbf{V}$ satisfying

$$\left(\frac{(B^T)^{-1}(v + Aw)}{Bw}\right) = \left(\frac{(B^T)^{-1} \mid (B^T)^{-1}A}{0 \mid B}\right) \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix},$$

which in particular contains the subspace of $v \in E_1((B^T)^{-1})$ and $w \in \text{Ker}(A) \cap E_1(B)$. Hence, (55) is less than or equal to $\text{mult}_1(g)/2$, giving the claim.

We proceed similarly in the other case of (53). Consider

$$(56) \quad g = \left(\frac{0 \mid I}{-I \mid 0}\right) \left(\frac{(B^T)^{-1} \mid 0}{0 \mid B}\right) \left(\frac{I \mid A}{0 \mid I}\right) = \left(\frac{0 \mid B}{-(B^T)^{-1} \mid -(B^T)^{-1}A}\right).$$

According to (52), the action of this element on the oscillator representation is

$$\begin{aligned} \omega_\psi\left(\left(\begin{array}{c|c} 0 & I \\ -I & 0 \end{array}\right) \left(\begin{array}{c|c} (B^T)^{-1} & 0 \\ 0 & B \end{array}\right) \left(\begin{array}{c|c} I & A \\ 0 & I \end{array}\right)\right)(v) = \\ \frac{\epsilon(\det(B))}{\sum_{u \in \Lambda} \psi(u^T u)} \sum_{w \in \Lambda} \psi(w^T Bv) \psi\left(\frac{v^T A w}{2}\right) \cdot (w). \end{aligned}$$

The terms for $v = w \in \Lambda$ are those which contribute to the trace of this action. Therefore, we find

$$\begin{aligned} \chi_{\omega_\psi[\mathbf{V}]}\left(\left(\begin{array}{c|c} 0 & I \\ -I & 0 \end{array}\right) \left(\begin{array}{c|c} (B^T)^{-1} & 0 \\ 0 & B \end{array}\right) \left(\begin{array}{c|c} I & A \\ 0 & I \end{array}\right)\right) = \\ \frac{\epsilon(\det(B)) \sum_{v \in \Lambda} \psi(v^T (B + \frac{A}{2})v)}{\sum_{u \in \Lambda} \psi(u^T u)} = \frac{\epsilon(\det(B)) \sum_{v \in \Lambda} \psi(v^T (\frac{B + B^T + A}{2})v)}{\sum_{u \in \Lambda} \psi(u^T u)} \end{aligned}$$

reducing

$$v^T (B + \frac{A}{2})v = v^T (\frac{B + B^T + A}{2})v$$

since $(B - B^T)/2$ is antisymmetric and hence $v^T (\frac{B - B^T}{2})v = 0$. Considering the Gaussian sums, we therefore find that the q -degree of the absolute value of the character value is

$$(57) \quad -\frac{N}{2} + \frac{N + \dim(\text{Ker}(\frac{B + B^T + A}{2}))}{2} = \frac{\dim(\text{Ker}(B + B^T + A))}{2}.$$

On the other hand, the eigenspace for 1 as an eigenvalue of (56) consists of vectors $(v^T \mid w^T)^T \in \mathbf{V}$ satisfying

$$\left(\begin{array}{c|c} Bw & \\ \hline -(B^T)^{-1}(v + Aw) & \end{array}\right) = \left(\begin{array}{c|c} 0 & B \\ \hline -(B^T)^{-1} & (B^T)^{-1}A \end{array}\right) \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix},$$

i.e. satisfying $v = Bw$ and, substituting out v ,

$$-(B^T)^{-1}(Bw + Aw) = w.$$

This is a restatement of the condition that $w \in \text{Ker}(B + B^T + A)$. Therefore $\text{mult}_1(g) = \dim(\text{Ker}(B + B^T + A))$, and hence (57) gives the claimed bound. \square

4.2. Kronecker products and geometric multiplicity. For $\lambda \in \mathbb{F}_q^\times$, consider a $k \times k$ Jordan block $J_k(\lambda)$ with eigenvalue λ and an $\ell \times \ell$ Jordan block $J_\ell(\lambda^{-1})$ with eigenvalue λ^{-1} . We recall the elementary fact that the geometric multiplicity of 1 as an eigenvalue of the Kronecker product of these two blocks is

$$(58) \quad \text{mult}_1(J_k(\lambda) \otimes J_\ell(\lambda^{-1})) = \min(k, \ell).$$

One immediate consequence of this is the following

Lemma 9. *Consider a matrix A with all eigenvalues equal to a fixed $\lambda \in \mathbb{F}_q^\times$.*

(1) *For any $d \times d$ matrix B with all eigenvalues equal to λ^{-1} ,*

$$(59) \quad \text{mult}_1(A \otimes B) \leq d \text{mult}_\lambda(A)$$

(2) *Fix a sign \pm . If $\lambda = \pm 1$, then for any $d \times d$ matrix $B \neq \pm I$ with all eigenvalues equal to ± 1 ,*

$$(60) \quad \text{mult}_1(A \otimes B) \leq d \text{mult}_1(A) - \mathcal{J}_{1 \times 1}^{(\pm 1)},$$

where $\mathcal{J}_{1 \times 1}^{(\pm 1)}$ is the number of 1×1 Jordan blocks with eigenvalue ± 1 appearing in A .

Combining Proposition 7 with this gives Corollary 8:

Proof of Corollary 8. Suppose we are given the notation of the statement. By Proposition 7, to prove part (1), it suffices to prove that for every non-trivial symplectic group element $g \neq I \in \text{Sp}(V)$, the geometric multiplicity of 1 as an eigenvalue of $g \otimes h$ is less than or equal to the maximum

$$(61) \quad \max\{2N \text{mult}_1(h) - \mathcal{J}_{1 \times 1}^{(1)}, 2N \text{mult}_\lambda(h), \\ (2N - 1) \text{mult}_1(h) + \text{mult}_\lambda(h) \text{ for } \lambda \neq 1\}.$$

Similarly, to prove part (2), it suffices to prove that for every element of the symplectic group $g \neq \pm I \in \text{Sp}(V)$, the geometric multiplicity of 1 as an eigenvalue of $g \otimes h$ is less than or equal to the maximum

$$(62) \quad \max\{2N \text{mult}_1(h) - \mathcal{J}_{1 \times 1}^{(1)}, (2N - 1) \text{mult}_1(h) + \text{mult}_\lambda(h), \\ 2N \text{mult}_{-1}(h) - \mathcal{J}_{1 \times 1}^{(-1)}, (2N - 1) \text{mult}_{-1}(h) + \text{mult}_\lambda(h), \\ 2N \text{mult}_\lambda(h), \text{ for } \lambda \neq 1\}.$$

In fact, we shall prove that these bounds hold for every $2N \times 2N$ matrix g not equal to I , resp. $\pm I$, and every $n \times n$ matrix h with all non-zero eigenvalues.

First, we prove part (1) of the corollary by verifying the geometric multiplicity bound (61) for $g \neq I$. By Lemma 9, (1), we see that for all choices of $g \neq I$ with eigenvalues λ of some algebraic multiplicities $0 \leq d_\lambda \leq 2N$ such that $d_1 \leq 2N-1$, the maximal geometric multiplicity of 1 as an eigenvalue of $g \otimes h$ is

$$(63) \quad \sum_{\lambda \in \mathbb{F}_q^\times} d_\lambda \text{mult}_{\lambda^{-1}}(h).$$

In the case where g has 1 as an eigenvalue of algebraic multiplicity $2N$, Lemma 9, (2) gives that the geometric multiplicity of 1 as an eigenvalue of $g \otimes h$ is less than or equal to

$$(64) \quad 2N \text{mult}_1(h) - \mathcal{J}_{1 \times 1}.$$

It suffices to show the maximum of (63) and (64) is equal to (61), for every choice of h .

Consider an eigenvalue λ_0 of h whose geometric multiplicity is greater than or equal to the geometric multiplicity of every other eigenvalue.

Case 1: Say $\lambda_0 \neq 1$. Then (61) is equal to $2N \text{mult}_{\lambda_0}(h)$. On the other hand, for every choice of algebraic multiplicities d_λ with $d_1 \leq 2N-1$, we have

$$(65) \quad \sum_{\lambda \in \mathbb{F}_q} d_{\lambda^{-1}} \text{mult}_\lambda(h) \leq \sum_{\lambda \in \mathbb{F}_q^\times} d_{\lambda^{-1}} \text{mult}_{\lambda_0}(h) = 2N \text{mult}_{\lambda_0}(h),$$

with equality in the case of $d_{\lambda_0^{-1}} = 2N$ and $d_\lambda = 0$ for $\lambda \neq \lambda_0^{-1}$. Since $\text{mult}_1(h) \leq \text{mult}_{\lambda_0}$, we also always have

$$2N \text{mult}_1(h) - \mathcal{J}_{1 \times 1} \leq 2N \text{mult}_{\lambda_0}(h).$$

Hence, the maximum of (63) and (64) in this case is $2N \text{mult}_{\lambda_0}(h)$, matching the maximum (61) by the condition on λ_0 (since $\text{mult}_1(h)$, $\text{mult}_\lambda(h) \leq \text{mult}_{\lambda_0}(h)$ clearly gives that

$$2N \text{mult}_1(h) - \mathcal{J}_{1 \times 1}, (2N-1) \text{mult}_1(h) + \text{mult}_\lambda(h)$$

are less than or equal to $2N \text{mult}_{\lambda_0}(h)$).

Case 2: Say $\lambda_0 = 1$. Let us write μ_0 for an eigenvalue not equal to 1 such that $\text{mult}_\lambda(h) \leq \text{mult}_{\mu_0}(h)$ for every $\lambda \neq 1$. Using a similar argument as (65), we find that choice of natural numbers d_λ with $d_1 \leq 2N-1$ maximizing (63) will be $d_1 = 0$ and $d_{\mu_0} = 2N$ or $d_1 = 2N-1$ and $d_{\mu_0} = 1$, giving

$$(66) \quad 2N \text{mult}_{\mu_0}(h)$$

or

$$(67) \quad (2N-1) \text{mult}_1(h) + \text{mult}_{\mu_0}(h),$$

respectively. Therefore, the maximum of (63) and (64) is precisely equal to

$$\max\{2N\text{mult}_1(h) - \mathcal{J}_{1 \times 1}, 2N\text{mult}_{\mu_0}(h), (2N - 1)\text{mult}_1(h) + \text{mult}_{\mu_0}(h)\},$$

which matches (61) by the assumption on μ_0 .

The proof of part (2) of the corollary and the verification of the geometric multiplicity bound (62) for $g \neq \pm I$ follows exactly similarly. \square

For completeness, we include a proof of Lemma 9 now.

Proof of Lemma 9. We may assume A and B are in Jordan normal form. Let us write $\mathcal{J}_A, \mathcal{J}_B$ for indexing sets for the Jordan blocks appearing in A, B (so that $|\mathcal{J}_A| = \text{mult}_\lambda(A)$ and $|\mathcal{J}_B| = \text{mult}_{\lambda^{-1}}(B)$). Write $\dim(J)$ for the size of a Jordan block J .

Part (1) follows immediately, since, by (58),

$$\begin{aligned} \text{mult}_1(A \otimes B) &= \sum_{\substack{J \in \mathcal{J}_A \\ K \in \mathcal{J}_B}} \min(\dim(J), \dim(K)) \leq \sum_{\substack{J \in \mathcal{J}_A \\ K \in \mathcal{J}_B}} \dim(K) = \\ & \sum_{J \in \mathcal{J}_A} d = d \text{mult}_\lambda(A) \end{aligned}$$

as claimed in (59). (Note that equality is attained, for example, by $B = \lambda^{-1} \cdot I$.)

Now we consider part (2) of the statement. Suppose without loss of generality the sign \pm is $+$. let us further write \mathcal{J}_A^1 and \mathcal{J}_B^1 for the sets of 1×1 Jordan blocks appearing in A and B (so that $n = |\mathcal{J}_A^1|$). Then

$$\text{mult}_1(A \otimes B) = \sum_{\substack{J \in \mathcal{J}_A \\ K \in \mathcal{J}_B}} \min(\dim(J), \dim(K)).$$

Every choice of $K \in \mathcal{J}_B^1$ with any of the $\text{mult}_1(A)$ choices of $J \in \mathcal{J}_A$ contributes a 1 term in the sum, so this can be re-written as

$$(68) \quad \text{mult}_1(A) \cdot |\mathcal{J}_B^1| + \sum_{\substack{J \in \mathcal{J}_A \\ K \in \mathcal{J}_B \setminus \mathcal{J}_B^1}} \min(\dim(J), \dim(K)).$$

Similarly, every choice of $J \in \mathcal{J}_A^1$ with any $K \in \mathcal{J}_B \setminus \mathcal{J}_B^1$ contributes a 1 term, and every remaining term is less than or equal to $\dim(K)$.

Therefore, (68) is less than or equal to

$$(69) \quad \text{mult}_1(A) \cdot |\mathcal{J}_B^1| + n \cdot (\text{mult}_1(B) - |\mathcal{J}_B^1|) + \sum_{\substack{J \in \mathcal{J}_A \setminus \mathcal{J}_A^1 \\ K \in \mathcal{J}_B \setminus \mathcal{J}_B^1}} \dim(K).$$

We note that

$$\sum_{K \in \mathcal{J}_B \setminus \mathcal{J}_B^1} \dim(K) = d - |\mathcal{J}_B^1|,$$

and therefore (69) can be simplified as

$$\begin{aligned} & \text{mult}_1(A) |\mathcal{J}_B^1| + n(\text{mult}_1(B) - |\mathcal{J}_B^1|) + (\text{mult}_1(A) - n)(d - |\mathcal{J}_B^1|) = \\ & d \text{mult}_1(A) - n(d - \text{mult}_1(B)), \end{aligned}$$

which is less than or equal to (60) since $\text{mult}_1(B) \leq d - 1$ by the assumption that $B \neq I$. We note that equality is attained when B consists of a single 2×2 Jordan block and $(d - 2) 1 \times 1$ Jordan blocks. \square

5. THE ODD RANK CASE OF THEOREM 1

Let us write $(W, B) = (\mathbb{F}_q^{2m+1}, \alpha)$. According to the work done in Sections 2 and 3, we must prove the limit of the main term consisting of the coefficient $2/q^{(2m+1)(N-m)}$ multiplied by the sum over all conjugacy classes $(h) \in \text{SO}(W, B)$ and all choices of $0 \leq k = k_1 + \dots + k_j \leq m$ of terms

$$\sum_{\substack{(h') \in P_{k_1, \dots, k_j} \\ \text{conj to } h}} (-1)^j \frac{c(h) \cdot |Z_{O(W)}(h)|}{|Z_{P_{k_1, \dots, k_j}}(h')|} \chi_{\omega[V \otimes (W[-k], B[-k])]}(g \otimes (h')^{L_{k_1, \dots, k_j}})$$

vanishes, along with the limit of the error terms consisting of $K_{H,u}$ multiplied by the sum over all conjugacy classes $(h) \in \text{SO}(W, B)$ and all choices of $0 \leq k = k_1 + \dots + k_j \leq m$ of terms

$$\sum_{\substack{(h') \in P_{k_1, \dots, k_j} \\ \text{conj to } h}} (-1)^j \frac{a_{(h)}^{H,u} \cdot |Z_{O(W)}(h)|}{|Z_{P_{k_1, \dots, k_j}}(h')|} \chi_{\omega[V \otimes (W[-k], B[-k])]}(g \otimes (h')^{L_{k_1, \dots, k_j}}).$$

For $h' \in P_{k_1, \dots, k_j}$, recall that we write $(h')^{\circ}$ for the factor of $(h')^{L_{k_1, \dots, k_j}}$ in the orthogonal group factor $O(W[-k], B[-k])$ of the Levi subgroup L_{k_1, \dots, k_j} . We note that the absolute value of the character factor appearing in the above expressions is equal to

$$|\chi_{\omega[V \otimes (W[-k], B[-k])]}(g \otimes (h')^{\circ})|.$$

Proposition 10. *Assume the above established notation. For every choice of $g \neq \pm I \in O_{2m+1}(\mathbb{F}_q)$, the coefficient $2/q^{(2m+1)(N-m)}$ multiplied by the sum over all conjugacy classes $(h) \in SO(W, B)$ and all choices of $0 \leq k = k_1 + \cdots + k_j \leq m$ of terms*

$$(70) \quad \sum_{\substack{(h') \in P_{k_1, \dots, k_j} \\ \text{conj to } h}} \frac{|c_{(h)}| \cdot |Z_{O(W)}(h)|}{|Z_{P_{k_1, \dots, k_j}}(h')|} |\chi_{\omega[V \otimes (W[-k], B[-k])]}(g \otimes (h')^{\circ})|$$

where, as in (47), the sum runs over distinct P_{k_1, \dots, k_j} -conjugacy classes (h') for h' which are $O_{2m+1}(\mathbb{F}_q)$ -conjugate (h) , has strictly negative q -degree.

By the triangle inequality, this clearly proves that the main term of the character ratio sum (5) vanishes as q goes to infinity. The proof of Proposition 10 can be followed through exactly similarly with $K_{H,u}$ instead of the coefficient $2/q^{(2m+1)(N-m)}$ (recalling $\deg_q(K_{H,u}) < -(2m+1)(N-m)$) and $a_{(h)}^{H,u}$ instead of $c_{(h)}$ (the bound of q -degree we have for $c_{(h)} \cdot |Z_{O(W)}(h)|$ also applies to $a_{(h)}^{H,u} \cdot |Z_{O(W)}(h)|$). Doing this gives that the error term of $K_{H,u}$ times the sum over $(h) \in SO(W, B)$ and $0 \leq k = k_1 + \cdots + k_j \leq m$ of terms

$$\sum_{\substack{(h') \in P_{k_1, \dots, k_j} \\ \text{conj to } h}} \frac{|a_{(h)}^{H,u}| \cdot |Z_{O(W)}(h)|}{|Z_{P_{k_1, \dots, k_j}}(h')|} |\chi_{\omega[V \otimes (W[-k], B[-k])]}(g \otimes (h')^{\circ})|$$

has q -degree strictly negative. Therefore the triangle inequality can again be applied to see that the limit of each error term vanishes as q goes to infinity. Hence, to conclude the odd case of Theorem 1, it suffices to prove Proposition 10.

We proceed by grouping terms according to the q -degree of the factor

$$(71) \quad |\chi_{\omega[V \otimes (W[-k], B[-k])]}(g \otimes (h')^{\circ})|.$$

- Specifically, if the geometric multiplicity of 1 as an eigenvalue of $(h')^{\circ} \in O(W[-k], B[-k])$ is greater than or equal to the geometric multiplicity of every other eigenvalue, then the q -degree of (71) is less than or equal to

$$(72) \quad N \text{mult}_1((h')^{\circ}) - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2} \text{ or} \\ \frac{2N-1}{2} \text{mult}_1((h')^{\circ}) + \text{mult}_{\lambda}((h')^{\circ})$$

for some other eigenvalue $\lambda \neq 1$.

- Similarly, if the geometric multiplicity of -1 as an eigenvalue of $(h')^{\circ}$ is greater than or equal to the geometric multiplicity of every other eigenvalue, then the q -degree of (71) is less than or equal to

$$(73) \quad N \text{mult}_{-1}((h')^{\circ}) - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2} \text{ or} \\ \frac{2N-1}{2} \text{mult}_{-1}((h')^{\circ}) + \frac{\text{mult}_{\lambda}((h')^{\circ})}{2}$$

for some eigenvalue $\lambda \neq -1$.

- Finally, if the geometric multiplicity of some eigenvalue $\lambda \neq \pm 1$ is greater than or equal to the geometric multiplicity of any other eigenvalue (including ± 1), then the q -degree of (71) is less than or equal to

$$(74) \quad N \text{mult}_{\lambda}((h')^{\circ}).$$

We will proceed by separating the terms corresponding to each choice of (h') into the above cases. Within each case (e.g. the case where 1 is the eigenvalue with maximal geometric multiplicity), we shall separate further into cases indexed by what this maximal geometric multiplicity is (e.g. $\text{mult}_1((h')^{\circ}) = 2p$ for a choice of $p \leq m$).

We first obtain a bound on the q -degree of a term of (70) contributed by any (h') in each case. Adding this q -degree bound to the q -degree of the number of (h') falling into each case is always strictly negative. Arguing this is the purpose of Subsections 5.1, 5.2, and 5.3. Since the number of cases does not depend on q , this proves Proposition 10, concluding the proof of the odd case of Theorem 1.

5.1. The case of 1 eigenvalues. Fix a choice of natural number $1 \leq p \leq m$. In this subsection, we consider the product of $2/q^{(2m+1)(N-m)}$ and the sum of all terms of (70) corresponding to $(h') \in P_{k_1, \dots, k_j}$ for some $0 \leq k = k_1 + \dots + k_j \leq m$ such that the orthogonal group part $(h')^{\circ} \in O(W[-k])$ satisfies

$$(75) \quad \text{mult}_{\lambda}((h')^{\circ}) \leq \text{mult}_1((h')^{\circ}) = 2p + 1$$

for every eigenvalue $\lambda \neq 1$.

In this case, we automatically have $O_{2p+1}(\mathbb{F}_q) \subseteq Z_{P_{k_1, \dots, k_j}}(h')$, and therefore

$$\deg_q \left(\frac{1}{|Z_{P_{k_1, \dots, k_j}}(h')|} \right) \leq -(2p^2 + p).$$

Applying Lemma 5, we also automatically have

$$\deg_q(|c_{(h)}| \cdot |Z_{O(W)}(h)|) \leq m$$

for every (h) corresponding to an (h') in this case.

Writing $\mathcal{J}_{1 \times 1}^{(1)}$ for the number of 1×1 Jordan blocks in $(h')^{\circ}$ with eigenvalue 1, (72) can be applied to find that the q -degree of the absolute value of the character value of $\omega[V \otimes (W[-k], B[-k])] at $g \otimes (h')^{\circ}$ is less than or equal to$

$$(76) \quad \max\left((2p+1)N - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2}, (2N-1)\frac{2p+1}{2} + \frac{\text{mult}_{\lambda}((h')^{\circ})}{2}\right)$$

We consider two subcases differentiated by which component of (76) attains the maximum.

Case A: First, consider all (h') satisfying (75) such that the maximum (76) is attained by $(2p+1)N - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2}$. Each term of (70) (not counting the global coefficient of $2/q^{(2m+1)(N-m)}$) corresponding to such an (h') has q -degree less than or equal to

$$(77) \quad m - (2p^2 + p) + (2p+1)N - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2} = m - 2p^2 - p + (2p+1)N - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2}.$$

The number $\mathcal{J}_{1 \times 1}^{(1)}$ of 1×1 Jordan blocks of 1 appearing in h' satisfies

$$2p+1 \leq \mathcal{J}_{1 \times 1}^{(1)} + \frac{2(m-k) + 1 - \mathcal{J}_{1 \times 1}^{(1)}}{2}$$

since the geometric multiplicity of 1 as an eigenvalue of h' would be maximized if the remaining $2(m-k) - \mathcal{J}_{1 \times 1}^{(1)} + 1$ coordinates of h' consisted of 2×2 Jordan blocks. Simplifying and rounding down gives that we must have

$$2p - m + k < \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2},$$

and therefore, (77) is strictly less than

$$2m - 2p^2 - 3p + (2p+1)N - k.$$

Therefore, after multiplying with the coefficient $2/q^{(2m+1)(N-m)}$, we find that the total q -degree of the term contributed of (70) by (h') is less than or equal to

$$3m - 2p^2 - 3p - 2(m-p)N + 2m^2 - k.$$

Since $p \leq m - k$ and the stability condition ensures $N \geq 2m + 1$, this q -degree is less than or equal to the expression obtained by replacing N by $2m + 1$, which can be simplified to

$$(78) \quad -2(m-p)^2 + m - k - p.$$

We have proved so far that the product of $2/q^{(2m+1)(N-m)}$ and the term of (70) contributed by each (h') in the present case has q -degree strictly less than (78).

Now let us consider the number of (h') falling into the present case. This number is a polynomial in q of degree $m-p$. Hence, the product of $2/q^{(2m+1)(N-m)}$ with the sum of all terms of (70) contributed by (h') in the present case has total q -degree strictly less than

$$-2(m-p)^2 + 2(m-p) - k,$$

which is always less than or equal to 0 since we must have $(m-p)^2 \geq m-p$. Therefore, the contribution of this case has total q -degree strictly negative. This verifies this case's contribution for Proposition 10.

Case B: Consider all (h') satisfying (75) such that the maximum (76) is attained by $(2N-1)\frac{2p+1}{2} + \frac{r}{2}$ where $r \geq 1$ is the geometric multiplicity of an eigenvalue λ of $(h')^O$ not equal to 1. Note that we must have $r+2p+1 \leq 2m+1$. Each term of (70) contributed by such (h') (not counting the global $2/q^{(2m+1)(N-m)}$ coefficient) has q -degree less than or equal to

$$(79) \quad \begin{aligned} m - (2p^2 + p) + \frac{2N-1}{2}(2p+1) + \frac{r}{2} = \\ m - 2p^2 - 2p + (2p+1)N - \frac{1}{2} + \frac{r}{2}. \end{aligned}$$

Multiplying by the coefficient $2/q^{(2m+1)(N-m)}$, we find that the total q -degree of the term contributed by (h') is less than or equal to

$$-2(m-p)N + 2m^2 + 2m - 2p^2 - 2p - \frac{1}{2} + \frac{r}{2}.$$

Since $p \leq m-k$ and the stability condition ensures $N \geq 2m+1$, this q -degree is less than or equal to the expression obtained by replacing N by $2m+1$, which simplifies to

$$-2(m-p)^2 + p - \frac{1}{2} + \frac{r}{2}.$$

This is strictly less than

$$(80) \quad -2(m-p)^2 + \frac{r}{2}.$$

We have proved so far that the product of $2/q^{(2m+1)(N-m)}$ and the term of (70) contributed by each (h') in the present case has q -degree strictly less than (80).

Now let us consider the number of (h') falling into the present case. The number of these (h') is a polynomial in q of degree $1+m-p-\frac{r}{2}$. Hence, the product of $2/q^{(2m+1)(N-m)}$ with the sum of all terms of (70)

contributed by (h') in the present case has total q -degree strictly less than

$$-2(m-p)^2 + (m-p) + 1,$$

which is always less than or equal to 0 since we must have $m-p \geq 1$ in this case. Therefore, the contribution of this case has total q -degree strictly negative. This verifies this case's contribution for Proposition 10.

5.2. The case of -1 eigenvalues. Fix a natural number $1 \leq \ell \leq m$. In this subsection, we consider the product of $2/q^{(2m+1)(N-m)}$ and the sum of all terms of (70) corresponding to $(h') \in P_{k_1, \dots, k_j}$ for some $0 \leq k = k_1 + \dots + k_j \leq m$ such that the orthogonal group part $(h')^{\circ} \in \text{O}(W[-k], B[-k])$ satisfies

$$(81) \quad \text{mult}_{\lambda}((h')^{\circ}) \leq \text{mult}_{-1}((h')^{\circ}) = 2\ell$$

for every eigenvalue $\lambda \neq -1$. In this case, we automatically have $\text{O}_{2\ell}^{\pm}(\mathbb{F}_q) \subseteq Z_{P_{k_1, \dots, k_j}}(h')$, and therefore

$$\deg_q\left(\frac{1}{|Z_{P_{k_1, \dots, k_j}}(h')|}\right) \leq -(2\ell^2 - \ell).$$

Applying Lemma 5, we also have

$$\deg_q(|c(h)| \cdot |Z_{\text{O}(W)}(h)|) \leq m - \lceil \ell/2 \rceil$$

for every (h) corresponding to an (h') in the present case (since the geometric multiplicity of -1 as an eigenvalue of h is at least the geometric multiplicity of h').

Writing $\mathcal{J}_{1 \times 1}^{(-1)}$ for the number of 1×1 Jordan blocks in $(h')^{\circ}$ with eigenvalue -1 , (73) can be applied to see that the q -degree of the absolute value of the character of $\omega[V \otimes (W[-k], B[-k])]$ at $g \otimes (h')^{\circ}$ is less than or equal to

$$(82) \quad \max\left(2\ell N - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2}, (2N-1)\ell + \frac{\text{mult}_{\lambda}((h')^{\circ})}{2}\right)$$

We consider two subcases differentiated by which component of (82) attains the maximum.

Case A: First consider all (h') satisfying (81) such that the maximum (82) is attained by $2\ell N - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2}$. In this case, each corresponding term of (70) contributed by such an (h') (not counting the global $2/q^{(2m+1)(N-m)}$) has q -degree less than or equal to

$$(83) \quad m - (2\ell^2 - \ell) - \lceil \frac{\ell}{2} \rceil + 2\ell N - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2} = m - 2\ell^2 + 2\ell N - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2} + \lfloor \frac{\ell}{2} \rfloor.$$

Similarly, as in Case A of Subsection 5.1, we can see that

$$2\ell - m + k \leq \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2}.$$

Therefore, (83) is less than or equal to

$$2m - 2\ell^2 - 2\ell + \lfloor \frac{\ell}{2} \rfloor + 2\ell N - k.$$

Multiplying by the coefficient $2/q^{(2m+1)(N-m)}$, we find that the total q -degree of the term contributed by an (h') in this case is less than or equal to

$$3m - 2\ell^2 - 2\ell + \lfloor \frac{\ell}{2} \rfloor - (2(m - \ell) + 1)N + 2m^2 - k.$$

Since $\ell \leq m - k$ and the stability condition ensures $N \geq 2m + 1$, this expression is less than or equal to the expression obtained by replacing N by $2m + 1$, which simplifies as

$$(84) \quad -2(m - \ell)^2 - k - m - 1 + \lfloor \frac{\ell}{2} \rfloor.$$

We have proved so far that the q -degree of the product of $2/q^{(2m+1)(N-m)}$ and the term of (70) contributed by (h') in the present case has q -degree less than or equal to (84).

The number of (h') falling into the present case is a polynomial in q of degree $m - \ell$. Hence, the total q -degree of the product of $2/q^{(2m+1)(N-m)}$ and the sum of all terms of (70) contributed by (h') in the present case is less than or equal to

$$-2(m - \ell)^2 - k - \lceil \frac{\ell}{2} \rceil - 1,$$

which we clearly see is always strictly negative since $\ell \leq m$. This verifies this case's contribution for Proposition 10.

Case B: Consider all (h') satisfying (81) such that the maximum (82) is attained by $(2N - 1)\ell + \frac{r}{2}$ where $r \geq 1$ is the geometric multiplicity of an eigenvalue λ of $(h')^{\mathcal{O}}$ not equal to -1 . Note that we must have $r + 2\ell \leq 2m + 1$. Each term of the sum (70) contributed by an (h') in the present case (not counting the global $2/q^{(2m+1)(N-m)}$ coefficient) has q -degree less than or equal to

$$(85) \quad \begin{aligned} m - (2\ell^2 - \ell) - \lceil \frac{\ell}{2} \rceil + (2N - 1)\ell + \frac{r}{2} = \\ m - 2\ell^2 - \lceil \frac{\ell}{2} \rceil + 2N\ell + \frac{r}{2}. \end{aligned}$$

Multiplying by the coefficient $2/q^{(2m+1)(N-m)}$, we find that the total q -degree of the term contributed by (h') is less than or equal to

$$-2((m - \ell) + 1)N + 2m^2 + 2m - 2\ell^2 - \lceil \frac{\ell}{2} \rceil + \frac{r}{2}.$$

Since $\ell \leq m - k$ and the stability condition ensures $N \geq 2m + 1$, this q -degree is less than or equal to the expression obtained by replacing N by $2m + 1$, which simplifies to

$$(86) \quad -2(m - \ell)^2 - 2(m - \ell) - 1 - \lceil \frac{\ell}{2} \rceil + \frac{r}{2}.$$

To summarize, we have proved so far that the product of $2/q^{(2m+1)(N-m)}$ and the term of (70) contributed by each (h') in the present case has q -degree strictly less than (86).

Now let us consider the number of (h') falling into the present case. The number of these (h') is a polynomial in q of degree $1 + m - \ell - \frac{r}{2}$. Hence, the product of $2/q^{(2m+1)(N-m)}$ with the sum of all terms of (70) contributed by (h') in the present case has total q -degree strictly less than

$$-2(m - \ell)^2 - (m - \ell) - \lceil \frac{\ell}{2} \rceil,$$

which we can see is always strictly negative. Therefore, the contribution of this case has total q -degree strictly negative. This verifies this case's contribution for Proposition 10.

5.3. The case of generic eigenvalues. Fix a choice of natural numbers $n, r \geq 1$ such that $nr \leq m - k$. In this subsection, we consider the product of $2/q^{(2m+1)(N-m)}$ and the sum of all terms of (70) corresponding to $(h') \in P_{k_1, \dots, k_j}$ for some $0 \leq k = k_1 + \dots + k_j \leq m$ such that for every eigenvalue μ of the orthogonal group part $(h')^O \in O(W[-k], B[-k])$

$$(87) \quad \text{mult}_\mu((h')^O) \leq \text{mult}_\lambda((h')^O) = n$$

for an eigenvalue $\lambda \neq \pm 1$ contained in a degree r field extension of \mathbb{F}_q (and not in any lower-degree field extension). In this case, we automatically have $U_n^\pm(\mathbb{F}_{q^r}) \subseteq Z_{P_{k_1, \dots, k_j}}(h')$ and therefore

$$\deg_q \left(\frac{1}{|Z_{P_{k_1, \dots, k_j}}(h')|} \right) \leq -n^2 r.$$

Applying Lemma 5, we also have

$$\deg_q(|c(h)| \cdot |Z_{O(W)}(h)|) \leq m$$

for every (h) corresponding to an (h') in the present case. We also apply (74) in this case to see that the q -degree of the absolute value

of the character of $\omega[V \otimes (W[-k], B[-k])]$ at $g \otimes (h')^O$ is less than or equal to nN .

In total, this gives that the term of (70) contributed by (h') (not counting the global $2/q^{(2m+1)(N-m)}$ coefficient has q -degree less than or equal to

$$(88) \quad m - n^2r + nN.$$

Combining with the coefficient $2/q^{(2m+1)(N-m)}$, we find that the total q -degree of the term contributed by (h') is less than or equal to

$$(n - 2m - 1)N + 2m^2 + 2m - n^2r.$$

Since $m_\lambda \leq m$ and the stability condition ensures $N \geq 2m + 1$, this is less than or equal to the expression obtained by replacing N by $2m + 1$, which simplifies as

$$2nm + n - 2m^2 - 2m - 1 - n^2r = -2m(m - n) + n - n^2r - 2m - 1.$$

The number of these (h') falling into this case (i.e. satisfying (87) for some $\lambda \neq \pm 1$) is a polynomial in q of degree

$$r + m - nr.$$

Hence, the total q -degree of the product of $2/q^{(2m+1)(N-m)}$ with the sum of all terms of (70) contributed by (h') satisfying (87) for some $\lambda \neq \pm 1$ for any $\lambda \neq \pm 1$, is less than or equal to

$$-2m(m - n) + (1 - n)r + n(1 - nr) - m - 1.$$

Since $r, nr \geq 1$, this is less than or equal to $-2m(m - n) - m - 1$, which is always strictly negative. This verifies this case's contribution for Proposition 10

Combining the work of this subsection with the work done in Subsections 5.1 and 5.2, we have concluded the proof of Proposition 10, since the number of cases (i.e. the number of choices of ℓ , p , or n and r , and $k = k_1 + \dots + k_j$) does not depend on q . Therefore, the sum of each case's contribution of strictly negative q -degree has total q -degree which is strictly negative, as well. Hence, as describe in the introduction of this section, we may conclude the odd case of Theorem 1.

6. THE EVEN RANK CASE OF THEOREM 1

Let us write $W^\pm = (\mathbb{F}_q^{2m}, \pm)$ and write $W[-k]^\pm = (\mathbb{F}_q^{2(m-k)}, \pm)$ for the bilinear form obtained by removing k 2-dimensional hyperbolics from W^\pm . Again, by applying the work done in Section 2, we have

reduced the main term of the character ratio sum (6) to the coefficient $2/q^{2m(N-m)+m}$ multiplied by the sum over both signs \pm , all conjugacy classes $(h) \in O(W^\pm)$, and all choices of $0 \leq k = k_1 + \cdots + k_j \leq m$ of terms

$$(89) \quad \sum_{\substack{(h') \in P_{k_1, \dots, k_j} \\ \text{conj to } h}} (-1)^j \frac{b_{(h)}^\pm \cdot |Z_{O(W^\pm)}(h)|}{|Z_{P_{k_1, \dots, k_j}}(h')|} \chi_{\omega[V \otimes W[-k]^\pm]}(g \otimes (h')^{L_{k_1, \dots, k_j}})$$

(running over distinct P_{k_1, \dots, k_j} -conjugacy classes (h') for h' which are $O_{2m}^\pm(\mathbb{F}_q)$ -conjugate to h). We claim the following

Proposition 11. *Assume the above established notation. For every $g \neq \pm I \in Sp_{2N}(\mathbb{F}_q)$, the following hold:*

- (1) *For each of the signs \pm , the product of $2/q^{-(2m(N-m)+m)}$ and the sum over $(h) \in O(W^\pm)$ with $\det(h) = 1$, and $0 \leq k = k_1 + \cdots + k_j \leq m$ of terms*

$$(90) \quad \sum_{\substack{(h') \in P_{k_1, \dots, k_j} \\ \text{conj to } h}} \frac{|b_{(h)}^\pm| \cdot |Z_{O(W^\pm)}(h)|}{|Z_{P_{k_1, \dots, k_j}}(h')|} |\chi_{\omega[V \otimes W[-k]^\pm]}(g \otimes (h')^O)|$$

has q -degree less than or equal to 0.

- (2) *The product of $2/q^{-(2m(N-m)+m)}$ and the sum over both signs \pm , conjugacy classes $(h) \in O(W^\pm)$ with $\det(h) = 1$, and $0 \leq k = k_1 + \cdots + k_j \leq m$ of terms (89) has strictly negative q -degree.*

Part (2) of Proposition 11 immediately implies that the limit of the main term of the character ratio sum vanishes as q goes to infinity.

As in the odd case, the proof of part (1) of Proposition 11 can be followed through with $b_{(h)}^\pm$ replaced by each $a_{(h)}^{\pm, (H, u)}$ and $2/q^{-(2m(N-m)+m)}$ replaced by $K_{H, u}^\pm$ to obtain the vanishing of the error terms consisting of the coefficient $K_{H, u}^\pm$ times the sum over all conjugacy classes $(h) \in O(W^\pm)$, and all choices of $0 \leq k = k_1 + \cdots + k_j \leq m$ of terms

$$\sum_{\substack{(h') \in P_{k_1, \dots, k_j} \\ \text{conj to } h}} (-1)^j \frac{a_{(h)}^{\pm, (H, u)} \cdot |Z_{O(W^\pm)}(h)|}{|Z_{P_{k_1, \dots, k_j}}(h')|} \chi_{\omega[V \otimes W[-k]^\pm]}(g \otimes (h')^{L_{k_1, \dots, k_j}})$$

as q goes to infinity, since the degree of $K_{H, u}^\pm$ is *strictly less* than $-(2m(N-m)+m)$ and the bound on q -degree we use for $b_{(h)}^\pm$ applies exactly the same for $a_{(h)}^{\pm, (H, u)}$, meaning that following the proof through for the above sum gives strictly negative q -degree (note that

we may eliminate the sign and work with the sum of the absolute values of the summands by the triangle inequality).

As in the previous section, we will begin by grouping terms according to the q -degree of the factor contributed by the character value of the signed oscillator representation

$$(91) \quad \chi_{\omega[V \otimes W[-k]^\pm]}(g \otimes (h')^{L_{k_1, \dots, k_j}}),$$

which, by definition, is calculated as the character value of the true oscillator representation $\chi_{\omega[V \otimes W[-k]^\pm]}(g \otimes (h')^O)$, multiplied by the sign

$$\epsilon\left(\prod_{i=1}^j \det((h')^{\text{GL}_{k_i}(\mathbb{F}_q)})\right).$$

Of course, this sign does not affect q -degree, so Corollary 8 can still be applied to bound the q -degree

$$(92) \quad \begin{aligned} \deg_q(\chi_{\omega[V \otimes W[-k]^\pm]}(g \otimes (h')^{L_{k_1, \dots, k_j}})) = \\ \deg_q(|\chi_{\omega[V \otimes W[-k]^\pm]}(g \otimes (h')^O)|) \end{aligned}$$

- Specifically, if the geometric multiplicity of 1 as an eigenvalue of $(h')^O \in O(W[-k]^\pm)$ is greater than or equal to the geometric multiplicity of every other eigenvalue, then the q -degree of (92) is less than or equal to

$$(93) \quad \begin{aligned} N \text{mult}_1((h')^O) - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2} \text{ or} \\ \frac{2N-1}{2} \text{mult}_1((h')^O) + \frac{\text{mult}_\lambda((h')^O)}{2} \end{aligned}$$

for some other eigenvalue $\lambda \neq 1$.

- Similarly, if the geometric multiplicity of -1 as an eigenvalue of $(h')^O$ is greater than or equal to the geometric multiplicity of every other eigenvalue, then the q -degree of (92) is less than or equal to

$$(94) \quad \begin{aligned} N \text{mult}_{-1}((h')^O) - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2} \text{ or} \\ \frac{2N-1}{2} \text{mult}_{-1}((h')^O) + \frac{\text{mult}_\lambda((h')^O)}{2} \end{aligned}$$

for some eigenvalue $\lambda \neq -1$.

- Finally, if the geometric multiplicity of some eigenvalue $\lambda \neq \pm 1$ of $(h')^O$ is greater than or equal to the geometric multiplicity of any other eigenvalue (including ± 1), then the q -degree of (92) is less than or equal to

$$(95) \quad N \text{mult}_\lambda((h')^O).$$

We will proceed by separating the terms corresponding to each choice of (h') into the above cases. Within each case (e.g. the case where 1 is the eigenvalue with maximal geometric multiplicity), we shall separate further into cases indexed by what this maximal geometric multiplicity is (e.g. $\text{mult}_1((h')^O) = 2p$ for a choice of $p \leq m$).

We first obtain a bound on the q -degree of a term of (90) contributed by any (h') in each case. Adding this q -degree bound to the q -degree of the number of (h') falling into each case is always less than or equal to 0, and is almost always strictly negative. Arguing this is the purpose of Subsections 6.1, 6.2, and 6.3. Since the number of cases does not depend on q , this implies part (1) of Proposition 11.

To prove part (2) of Proposition 11, first note that the contribution corresponding to this case of (h') in (89) has q -degree less than or equal to the q -degree of the contribution in (90). Thus, it suffices to consider only the terms of (89) contributed by a case of (h') such that the sum of the q -degree bound for (90) and the q -degree of the number of (h') is equal to 0. We keep track of such cases while doing the work to prove part (1) of the proposition in Subsections 6.1, 6.2, and 6.3. In Subsection 6.4, we prove that for those final cases of (h') , the product of $2/q^{-(2m(N-m)+m)}$ with their contribution in the sum of terms (89) vanishes as q goes to infinity, concluding the proof of part (2) of Proposition 11.

6.1. The case of 1 eigenvalues. Fix a choice of natural number $1 \leq p \leq m$. In this subsection, we consider the sum of all terms of (90) corresponding to (h') contained in some parabolic P_{k_1, \dots, k_j} for $0 \leq k = k_1 + \dots + k_j \leq m$ such that for the orthogonal group factor $(h')^O \in O(W[-k]^\pm)$,

$$\text{mult}_\lambda((h')^O) \leq \text{mult}_1((h')^O) = 2p$$

for every eigenvalue $\lambda \neq 1$. We immediately note that then the centralizer of h' satisfies

$$\deg_q\left(\frac{1}{|Z_{P_{k_1, \dots, k_j}}(h')|}\right) \leq -(2p^2 - p)$$

and

$$\deg_q(|b_{(h)}^\pm| \cdot |Z_{O(W^\pm)}(h)|) \leq m,$$

by Lemma 5. Write $\mathcal{J}_{1 \times 1}^{(1)}$ for the number of 1×1 Jordan blocks with eigenvalue 1 in $(h')^O$. According to (93), we then find that the q -degree of the character of $\omega[V \otimes W[-k]^\pm]^-$ at $g \otimes (h')^O$ is less than or equal

to

$$(96) \quad \max\left(2Np - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2}, (2N - 1)p + \frac{\text{mult}_\lambda((h')^{\circ})}{2}\right)$$

Consider subcases based on which component in (96) attains the maximum.

Case A: Consider all (h') where (96) is $2Np - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2}$. Each term of (90) corresponding to such an (h') has q -degree less than or equal to

$$m - (2p^2 - p) + 2Np - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2}.$$

We must have that $\mathcal{J}_{1 \times 1}^{(1)}/2 \geq 2p - m + k$, so this is less than or equal to

$$m - 2p^2 + 2Np - p + m - k.$$

Now, multiplying by the coefficient $q^{-(2m(N-m)+m)}$ gives total q -degree of each term less than or equal to

$$-2(m - p)N + 2m^2 - 2p^2 - p + m - k.$$

We may use the stability condition $N \geq 2m$ to see that this is less than or equal to the expression with N replaced by $2m$, which can be simplified to

$$-2(m - p)^2 + (m - p) - k.$$

Now the number of (h') satisfying the condition of this case is a polynomial in q of degree less than or equal to $m - p$. In total, therefore, we have proved that the sum over all terms of (90) corresponding to (h') in this case is less than or equal to

$$(97) \quad -2(m - p)^2 + 2(m - p) - k,$$

which is less than or equal to 0 for every possible choice of p and k . This verifies this case's contribution for part (1) of Proposition 11.

The degree (97) is strictly negative in every case where $k > 0$ and in every case where $p \leq m - 2$ if $k = 0$. Therefore, only the cases where $p = m$ or $m - 1$ and $k = 0$ must be considered separately to prove part (2) of Proposition 11. We defer this discussion to Subsection 6.4.

Case B: Consider all (h') where (96) is equal to $(2N - 1)p + \frac{r}{2}$ where $r \geq 1$ is the geometric multiplicity of an eigenvalue λ of $(h')^{\circ}$ distinct from 1. Each term of (90) corresponding to such an (h') has q -degree less than or equal to

$$m - (2p^2 - p) + (2N - 1)p + \frac{r}{2},$$

which can be simplified as

$$m - 2p^2 + 2Np + \frac{r}{2}.$$

Multiplying by the coefficient $q^{-(2m(N-m)+m)}$ gives total q -degree of each term

$$-2(m-p)N + 2m^2 - 2p^2 + \frac{r}{2},$$

which, by using the stability condition $N \geq 2m$, we can see is less than or equal to

$$-2(m-p)^2 + \frac{r}{2}.$$

The number of (h') satisfying the condition of this case for r and $\lambda = -1$ has q -degree less than or equal to $m - p - \frac{r}{2}$. In total, then, the sum over all terms corresponding to such (h') is less than or equal to

$$-2(m-p)^2 + (m-p),$$

which is always strictly negative, since $m - p \geq r \geq 1$.

Finally, the number of (h') satisfying the condition of this case for a certain r and any $\lambda \neq -1$ is a polynomial in q of degree less than or equal to $m - p - r + 1$ (the extra 1 is contributed by the choice of eigenvalue $\lambda \in \mathbb{F}_q$; choosing λ in a field extension of q will only give a smaller q -degree number of choices of (h')). In total, then, the sum over all terms corresponding to (h') in this case is less than or equal to

$$-2(m-p)^2 + (m-p) - \frac{r}{2} + 1,$$

which is strictly less than $-2(m-p)^2 + (m-p) + 1$. Now, this is always less than or equal to 0 since we must still have $m - p \geq 1$.

In total, we can sum the two terms of strictly negative q -degrees corresponding to $\lambda = -1$ or any $\lambda \neq -1$, to get that the total q -degree of the sum of terms corresponding to (h') satisfying the conditions of this case for r is of strictly negative q -degree. Since the number of choices of r does not depend on q , the sum of terms corresponding to (h') satisfying the conditions of this case for any r is of strictly negative q -degree. This verifies this case's contribution for part (1) of Proposition 11. There are no cases which must be considered separately for part (2) of Proposition 11.

6.2. The case of -1 eigenvalues. Now fix a choice of natural number $1 \leq \ell \leq m$. In this subsection, we shall consider the sum of all terms of (90) corresponding to (h') contained in some parabolic P_{k_1, \dots, k_j} for $0 \leq k = k_1 + \dots + k_j \leq m$ such that

$$\text{mult}_\lambda((h')^0) \leq \text{mult}_{-1}((h')^0) = 2\ell$$

for every eigenvalue $\lambda \neq -1$. Similarly as before, we immediately can conclude that the centralizer of h' satisfies

$$\deg_q\left(\frac{1}{|Z_{P_{k_1, \dots, k_j}}(h')|}\right) \leq -(2\ell^2 - \ell)$$

and

$$\deg_q(b_{(h)}^\pm \cdot |Z_{O(W^\pm)}(h)|) \leq m - \lceil \ell/2 \rceil,$$

by Lemma 5. Again, write $\mathcal{J}_{1 \times 1}^{(-1)}$ for the number of 1×1 Jordan blocks with eigenvalues -1 in $(h')^O$. In this case, (94) can be applied to find that the q -degree of the character of the signed oscillator representation $\omega[V \otimes W[-k]^\pm]^-$ at $g \otimes (h')^O$ is less than or equal to

$$(98) \quad \max\left(2N\ell - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2}, (2N - 1)\ell + \frac{\text{mult}_\lambda((h')^O)}{2}\right)$$

Entirely similarly as in Subsection 6.1, consider subcases based on which component in (98) attains the maximum.

Case A: First, consider all (h') where (98) is $2N\ell - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2}$. Each term of (90) (not counting the global coefficient $2/q^{2m(N-m)+m}$) corresponding to such an (h') has q -degree less than or equal to

$$m - \lceil \frac{\ell}{2} \rceil - (2\ell^2 - \ell) + 2N\ell - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2} = m - 2\ell^2 + 2N\ell - \frac{\mathcal{J}_{1 \times 1}^{(-1)}}{2} + \lfloor \frac{\ell}{2} \rfloor.$$

We must have that $\mathcal{J}_{1 \times 1}^{(-1)}/2 \geq 2\ell - m + k$, so this is less than or equal to

$$m - 2\ell^2 + 2N\ell - 2\ell + m - k + \lfloor \frac{\ell}{2} \rfloor.$$

Now, multiplying by the coefficient $2/q^{2m(N-m)+m}$ gives total q -degree of each term less than or equal to

$$-2(m - \ell)N + 2m^2 - 2\ell^2 - 2\ell + m - k + \lfloor \frac{\ell}{2} \rfloor,$$

which, by the stability condition $N \geq 2m$, is less than or equal to the expression with N replaced by $2m$. This can be simplified to

$$-2(m - \ell)^2 - 2\ell + m - k + \lfloor \frac{\ell}{2} \rfloor = -2(m - \ell)^2 + (m - \ell) - (k + \lceil \frac{\ell}{2} \rceil).$$

Now the number of (h') satisfying the condition of this case is a polynomial in q of degree less than or equal to $m - \ell$. In total, therefore, we have proved that the sum over all terms corresponding to (h') in this case is less than or equal to

$$-2(m - \ell)^2 + 2(m - \ell) - (k + \lceil \frac{\ell}{2} \rceil).$$

Since $\ell \geq 1$, this is strictly less than $-2(m - \ell)^2 + 2(m - \ell)$, which is always less than or equal to 0. Therefore, the contribution of this case has total q -degree strictly negative. This verifies this case's contribution for part (1) of Proposition 11. There are no cases which must be considered separately for part (2) of Proposition 11

Case B: Consider all (h') where (98) is equal to $(2N - 1)\ell + \frac{r}{2}$ where $r \geq 1$ is the geometric multiplicity of an eigenvalue λ of $(h')^O$ not equal to -1 . Each term of (90) (not counting the global coefficient of $2/q^{2m(N-m)+m}$) corresponding to such an (h') has q -degree less than or equal to

$$m - \lceil \frac{\ell}{2} \rceil - (2\ell^2 - \ell) + (2N - 1)\ell + \frac{r}{2} = m - 2\ell^2 + 2N\ell - \lceil \frac{\ell}{2} \rceil + \frac{r}{2}.$$

Taking into account the coefficient $2/q^{2m(N-m)+m}$ gives total q -degree of each term

$$-2(m - \ell)N + 2m^2 - 2\ell^2 - \lceil \frac{\ell}{2} \rceil + \frac{r}{2}.$$

Again, this is less than or equal to the expression with N replaced by $2m$ by the stability condition $N \geq 2m$, which is

$$-2(m - \ell)^2 - \lceil \frac{\ell}{2} \rceil + \frac{r}{2}.$$

The number of (h') satisfying the condition of this case for r and $\lambda = 1$ has q -degree less than or equal to $m - \ell - \frac{r}{2}$, which gives a total contribution of q -degree less than or equal to

$$-2(m - \ell)^2 - \lceil \frac{\ell}{2} \rceil + (m - \ell),$$

which is always strictly negative, since both $m - \ell$ and ℓ are at least 1.

Finally, the number of (h') satisfying the condition of this case for a certain r and any $\lambda \neq 1$ has q -degree less than or equal to $m - \ell - r + 1$ (according to the same reasoning as in the corresponding case of Subsection 6.1). In total, then, the sum over all terms corresponding to (h') in this case is less than or equal to

$$-2(m - \ell)^2 - \lceil \frac{\ell}{2} \rceil + (m - \ell) - \frac{r}{2} + 1,$$

which is strictly less than $-2(m - \ell)^2 + (m - \ell) + 1$, which is always less than or equal to 0 since, again, we have $m - \ell \geq 1$.

Since the number of subcases we have considered (i.e. the choice of r and the dichotomy of if $\lambda = 1$ or not) does not depend on q , we see the total q -degree of the contribution from this case is strictly negative.

Again, this verifies this case's contribution for part (1) of Proposition 11. There are no cases which attain q -degree 0 and which must be considered separately for part (2) of Proposition 11.

6.3. The case of generic eigenvalues. Fix a choice of natural numbers $n, r \geq 1$. In this subsection, we consider the sum of all terms of (90) corresponding to (h') in a parabolic P_{k_1, \dots, k_j} for $0 \leq k = k_1 + \dots + k_j \leq m$ such that for every eigenvalue μ of $(h')^\circ$,

$$\text{mult}_\mu((h')^\circ) \leq \text{mult}_\lambda((h')^\circ) = n$$

for an eigenvalue $\lambda \neq \pm 1$ contained in a degree r field extension of \mathbb{F}_q (and not in any lower-degree field extension). Similarly as in the previous subsections, for each (h') in this case, we have $U_n^\pm(\mathbb{F}_{q^r}) \subseteq Z_{P_{k_1, \dots, k_j}}(h')$, we always have that the q -degree of $1/|Z_{P_{k_1, \dots, k_j}}(h')|$ is less than or equal to $-n^2r$. Lemma 5 can again be applied to see that the q -degree of the numerator $|b_{(h)}^\pm| \cdot |Z_{O(W^\pm)}(h)|$ is less than or equal to m . In total, this gives that the term of (90) contributed by (h') (not counting the global coefficient $2/q^{2m(N-m)+m}$), has q -degree less than or equal to $-n^2r + m + nN$. Combining this with the coefficient $2/q^{2m(N-m)+m}$, we find that the total q -degree of the term contributed by (h') is less than or equal to

$$-(2m - n)N + 2m^2 - n^2r.$$

Using the stability condition $N \geq 2m$ (and the fact that we must have $n \leq 2m$), this is less than or equal to

$$(99) \quad -2m^2 + 2mn - n^2r.$$

Now the number of terms falling into this case, i.e. the number of (h') such that the geometric multiplicity of every eigenvalue of $(h')^\circ$ is less than or equal to n , which is the geometric multiplicity of an eigenvalue not equal to ± 1 contained in a degree r field extension of \mathbb{F}_q (and not in any smaller field extension). The number of such (h') is a polynomial in q of degree

$$r + m - nr$$

(with the r summand contributed by the choice of λ). Therefore, summing with the degree (99) of each term, the total contribution of the terms in this case has q -degree less than or equal to

$$(100) \quad -2m^2 + 2mn - n^2r + r + m - nr.$$

To see why this is negative, first suppose $nr < m$, which we note is forced for $k \geq 1$. Note that we can use $r \geq 1$ to see that this is less

than or equal to

$$\begin{aligned} -2m^2 + 2mnr - n^2r + r + m - nr = \\ (-2m + 1)(m - nr) - r(n^2 - 1). \end{aligned}$$

We can immediately see that this is negative, since because $r(n^2 - 1) \geq 0$, it is less than or equal to

$$(-2m + 1)(m - nr) < 0$$

(since $m - nr > 0$ and $-2m + 1 < 0$).

Now, supposes $nr = m$, which can only occur if $k = 0$. Then we see (100) simplifies as

$$-2r^2n^2 + rn^2 + r = rn^2(-r + 1) + r(-rn^2 + 1).$$

The first summand is less than or equal to 0 since $r, n \geq 1$ and the second summand is strictly negative since $rn^2 = rm \geq m > 1$.

This verifies this case's contribution for part (1) of Proposition 11. There are no cases which attain q -degree 0 and which must be considered separately for part (2) of Proposition 11.

Note that combining the work done here with the work in Subsections 6.1 and 6.2, we have concluded the proof of part (1) of Proposition 11, since the number of cases (i.e. the number of choices of ℓ , p , or n and $k = k_1 + \dots + k_j$) does not depend on q . Therefore, the sum each case's contribution of q -degree less than or equal to 0 has total q -degree less than or equal to 0, as well.

6.4. Concluding part (2) of Proposition 11. All that remains to prove part (2) of Proposition 11 is to consider the contribution of all terms (89) corresponding to the cases where $k = 0$ and $p = m - 1$ or m , falling into Case A of Subsection 6.1.

Case A(I): First let us consider the case of $p = m$, which only occurs for $h' = h = I$. Since, in this case, h has no eigenvalues $\lambda \neq 1$, the maximum (96) is automatically equal to $2Nm - \frac{\mathcal{J}_{1 \times 1}^{(1)}}{2} = 2Nm - m$. Therefore this case, indeed, falls into Case A of Subsection 6.1. Now, the only $g \neq I \in \text{Sp}(V)$ attaining

$$(101) \quad \deg_q(\chi_{\omega[V \otimes W^{\pm}]}(g \otimes I)) = 2Nm - m$$

are those which are conjugate to a *transvection*, meaning

$$\left(\begin{array}{c|c} I & E_a \\ \hline 0 & I \end{array} \right)$$

where E_a denotes the $N \times N$ diagonal matrix with a single entry $a \in \mathbb{F}_q^\times$ in the first row and column, and all other entries equal to 0. This follows since to attain (101), g must have a Jordan normal form with $2(m-1)$ 1×1 Jordan blocks and a single 2×2 Jordan block (all with eigenvalue 1), meaning that it must be conjugate to a transvection (see Corollary 3.6 of [14]).

We may consider the terms of the sums (89) for both the groups $O_{2m}^\pm(\mathbb{F}_q)$ and $h = h' = I$. By the argument completed in Case A of Subsection 6.1 above, the q -degree of these terms is less than or equal to 0. Our strategy will be to prove that the terms of top q -degree contributed by $h = h' = I$ for $O_{2m}^+(\mathbb{F}_q)$ and $O_{2m}^-(\mathbb{F}_q)$ cancel with each other, leaving only a contribution with q -degree strictly less than 0 (which hence will vanish as q goes to infinity). Since the top q -degree terms of the coefficients of the character values $\chi_{\omega[V \otimes (\mathbb{F}_q^{2m, \pm})]}(g \otimes I)$ are equal for the two signs, it suffices to prove

$$(102) \quad \chi_{\omega[V \otimes (\mathbb{F}_q^{2m, +})]}(g \otimes I) + \chi_{\omega[V \otimes (\mathbb{F}_q^{2m, -})]}(g \otimes I) = 0.$$

Consider the symmetric bilinear form defining $(\mathbb{F}_q^{2m, \pm})$ as a sum of $(m-1)$ copies of the hyperbolic $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the two-dimensional form

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha^\pm \end{pmatrix}$$

where we put $\alpha^+ = 1$ (forming another hyperbolic) and $\alpha^- \in \mathbb{F}_q^\times$ such that $-\alpha^- \notin (\mathbb{F}_q^\times)^2$. Therefore, we may re-express the left hand side of (102) as

$$\chi_{\omega_1[V]}(g)^m \cdot \chi_{\omega_{-1}[V]}(g)^{m-1} \cdot (\chi_{\omega_{\alpha^+}[V]}(g) + \chi_{\omega_{\alpha^-}[V]}(g)),$$

and hence, it suffices to prove

$$\chi_{\omega_{\alpha^+}[V]}(g) + \chi_{\omega_{\alpha^-}[V]}(g).$$

This follows since, recalling (52), we compute that for $2N$ -dimensional V and q the ℓ -th power of a prime,

$$\chi_{\omega_{\alpha^\pm}[V]}(g) = (-1)^{N(\ell+1)} \cdot \epsilon(\alpha^\pm) \cdot \epsilon(-1)^{N/2} \cdot q^{N-\frac{1}{2}},$$

which cancel with each other, since $\epsilon(\alpha^-) = -\epsilon(-1) = -\epsilon(\alpha^+)$.

Case A(II): Now let us consider the case of $p = m - 1$. Note that we must have $m \geq 2$ for this case to arise. This means every term corresponding to $(h') = (h)$ in $P_0 = O(W^\pm)$ such that

$$\text{mult}_1(h') = 2(m-1)$$

Now, for each such $x \in \mathrm{SO}_4^+(\mathbb{F}_q)$, since the term of top q -degree of the coefficient of $\chi_{\omega[V \otimes (\mathbb{F}_q^{2m, \pm})]}(g \otimes (x \oplus I))$ in the corresponding term of (89) do not depend on x , it suffices to prove that

$$(104) \quad \chi_{\omega[V \otimes (\mathbb{F}_q^{2m, +})]}(g \otimes (x \oplus I)^+) + \chi_{\omega[V \otimes (\mathbb{F}_q^{2m, -})]}(g \otimes (x \oplus I)^-) = 0.$$

This follows since the left hand side of this equation can be expressed as the product of $\chi_{\omega[V \otimes (\mathbb{F}_q^4, +)]}(g \otimes x)$ with

$$\chi_{\omega[V \otimes (\mathbb{F}_q^{2(m-1), +})]}(g \otimes I) + \chi_{\omega[V \otimes (\mathbb{F}_q^{2(m-1), -})]}(g \otimes I) = 0$$

(applying the argument in Case A(I) to prove (102) with m replaced by $m - 1$).

Thus, we have concluded the proof of Proposition 11, and as described in the introduction to this section, have hence proved the even case of Theorem 1.

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