ON THE LOCAL COHOMOLOGY OF L-SHAPED INTEGRAL FI-MODULES

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Abstract. The local cohomology groups of FI-modules were computed in characteristic 0 by Sam and Snowden, but remain unknown integrally. In this paper we compute them for the Spechtral FI-modules corresponding to the L-shaped Young diagrams (i.e. the partitions $(2, 1, \dots, 1)$). We also find and discuss some interesting features of this computation, including the presence of torsion.

1. Introduction

In representation stability, representations of the category of finite sets and injections, called *FI-modules*, were introduced by Church, Ellenberg, and Farb in [1]. Their applications of this concept include results about the cohomology of configuration spaces on manifolds, diagonal coinvariant algebras, moduli spaces of n-pointed curves, rank varieties of square matrices, and more.

Now for an FI-module M, its *torsion submodule* TM is defined to consist of those elements of M which are sent to 0 by inclusions into large enough finite sets. An important topic in representation stability is the calculation of *local cohomology*, i.e., the derived functors of the FI-torsion of a given FI-module. Rationally, this was solved by Sam and Snowden [9, 10]. Modulo torsion, rationally, the simple objects of the category of FI-modules were called *Spechtral modules* by A. Snowden. Using Schur–Weyl duality, Sam and Snowden [9, 10] reduced the problem of calculating rational local cohomology of Spechtral modules to questions of cohomology of GL-equivariant-quasicoherent sheaves on $\mathbb{P}^\infty$, which can be resolved using the Borel-Weil-Bott Theorem.

Integrally, however, very little is known. The main result of this paper is

Theorem 1. *Let* $\lambda_k = (2, \underbrace{1, \dots, 1}_{k-2})$. *The local cohomology of the integral Spechtral module* M_{λ_k} *is given as follows:*

(1) *If $k \geq 2$ is odd,*

$$(R^iT(M_{\lambda_k}))_n = \begin{cases} \mathbb{Z}^- & \text{if } n = k, \text{ and } i = 2 \\ (\mathbb{Z}/k)^- & \text{if } n = k+1, \text{ and } i = 2 \\ \mathbb{Z} & \text{if } n = 1, \text{ and } i = k \\ 0 & \text{else.} \end{cases}$$

(2) *If $k \geq 2$ is even,*

$$(R^iT(M_{\lambda_k}))_n = \begin{cases} \mathbb{Z}^- & \text{if } n = k, \text{ and } i = 2 \\ (\mathbb{Z}/\frac{k}{2})^- & \text{if } n = k+1, \text{ and } i = 2 \\ \mathbb{Z}/2 & \text{if } n \leq k-1, \text{ and } i = 3 \\ \mathbb{Z} & \text{if } n = 1, \text{ and } i = k \\ 0 & \text{else.} \end{cases}$$

Here the superscript $^-$ denotes the sign representation. The FI-module map from degree n to degree $n+1$ is surjective in the same cohomological degree.

Thus, torsion does occur in FI-local cohomology of Spechtral FI-modules. One could ask how Theorem 1 relates to the local cohomology of integral versions of the objects to which the FI-modules we consider correspond to under Schur–Weyl duality rationally. This very question, however, is ambiguous, precisely due to the fact that Schur–Weyl duality fails integrally. We make some basic comments on this question on the Appendix, focusing mainly on the case of the Spechtral module $M_{(2)}$.

The proof of Theorem 1 starts with the well-known fact that the representation ring of a symmetric group Σ_n is generated by exterior powers of the irreducible representation of rank $(n-1)$ (see, e.g., [5]). Therefore, it makes sense to look for a resolution of Spechtral modules by tensor products of modules of the form $M_{(1^r)}$. For the module M_{λ_k}, it turns out that two tensor factors suffice, one of which satisfies $r = 1$. One can then take advantage of the explicit calculations of Remmel [8]. Based on these calculations, one writes down a rational resolution of M_{λ_k} which has an integral analogue, giving a resolution of an FI-module which I denote by M'_{λ_k}. The proof of Theorem 1 has two major steps: First, we use the resolution we constructed to calculate the local cohomology of M'_{λ_k}, then we prove that $M'_{\lambda_k} \cong M_{\lambda_k}$.

In principle, the same method should apply to the case of Spechtal modules $M_{(p,1^r)}$. (The case of $p = 1$, in fact, is easy and is treated in the Example in Section 2 and in Proposition 6 below.) For $p > 2$, one should be able to construct a resolution based on a tensor product of the form $M_{(1^r)} \otimes M_{(1^s)}$. One would then use the formulas of

Remmel [7] calculating the tensor product of two hook-shaped Specht modules. One sees, however, that the formulas in [7] are considerably more complicated. The complexity increases progressively for larger Young diagrams. In the general case, calculating tensor products of Specht modules is, in fact, an NP-hard problem [3], so other methods may be necessary.

The present paper is organized as follows: We introduce the basic concepts in Section 2. This includes the construction of integral Spechtral modules, the tensor product, and the calculation of the local cohomology of tensor products of Spechtral modules of the form $M_{(1^k)}$, which basically mimics the rational case. In Section 3, we introduce a certain resolution of an integral FI-module M'_{λ_k}, and compute its local cohomology. In Section 4, we prove that $M_{\lambda_k} \cong M'_{\lambda_k}$. In Section 5 (the Appendix), we briefly discuss integral versions of the Schur–Weyl dual of $M_{(2)}$ and their local cohomology.

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2. Preliminaries

2.1. **The construction of the integral Spechtral modules.** First, we shall recall a definition of the integral Specht module of a Young diagram and then use it to construct the integral Spechtral module of a given Young diagram. This will also allow us to establish notation that will be used throughout this paper.

To begin with, denote the symmetric group on n elements by Σ_n. Fix a Young diagram Y. Our notation for Young diagrams will be to write $Y = (\ell_1, \ldots, \ell_k)$ where k is the number of rows of Y, and for each $i = 1, \ldots k$, ℓ_i is the length of the ith row of Y We will use the convention that $i \leq j$ implies $\ell_i \geq \ell_j$.

Suppose Y has n total boxes (meaning $\sum_{i=1}^{k} \ell_i = n$). Of course, Σ_n acts on the boxes of Y by permutation. Denote by Σ_Y^r and Σ_Y^c the subgroups of Σ_n which preserve the rows and columns (respectively) of Y when acting.

To define the integral Specht module of Y, we shall first define a homomorphism of Σ_n-modules

$$\mathbb{Z}(\Sigma_n/\Sigma_Y^r)^* \otimes \mathbb{Z}(\Sigma_n/\Sigma_Y^c) \to \mathbb{Z}^-, \tag{1}$$

where M^* denote the dual of a representation M, i.e.,

$$M^* = Hom_{\mathbb{Z}}(M, \mathbb{Z}),$$

and $\mathbb{Z}^-$ denotes the sign representation.

Note first that both representations $\mathbb{Z}(\Sigma_n/\Sigma_Y^r)$ and $\mathbb{Z}(\Sigma_n/\Sigma_Y^c)$ are self-dual since they are permutation representations. (If a group G acts on a set S, then as G-representations, there is an isomorphism between $\mathbb{Z}(S)$ and its dual by dualizing with respect to the basis S.) In addition, the tensor product

$$\mathbb{Z}(\Sigma_n/\Sigma_Y^r) \otimes \mathbb{Z}(\Sigma_n/\Sigma_Y^c) = \mathbb{Z}(\Sigma_n/\Sigma_Y^r \times \Sigma_n/\Sigma_Y^c)$$

has a well-defined direct summand representation corresponding to the unique free orbit of $\Sigma_n/\Sigma_Y^r \times \Sigma_n/\Sigma_Y^c$. Define (1) as the projection to this direct summand composed with an onto homomorphism to $\mathbb{Z}^-$, which is unique up to sign. By using tensor-hom adjunction (and strong dualizability over $\mathbb{Z}$), we have

$$Hom(\mathbb{Z}(\Sigma_n/\Sigma_Y^r)^* \otimes \mathbb{Z}(\Sigma_n/\Sigma_Y^c), \mathbb{Z}^-) \cong$$
$$Hom(\mathbb{Z}(\Sigma_n/\Sigma_Y^c) \otimes \mathbb{Z}^-, \mathbb{Z}(\Sigma_n/\Sigma_Y^r)).$$

Thus, (1) gives another unique (up to sign) homomorphism of Σ_n-modules

$$\varphi_Y : \mathbb{Z}(\Sigma_n/\Sigma_Y^c) \otimes \mathbb{Z}^- \to \mathbb{Z}(\Sigma_n/\Sigma_Y^r). \tag{2}$$

Then one can define the integral Specht module, which we will denote by S_Y, to be the image $Im(\varphi_Y)$.

Now fix a Young diagram $Z = (\ell_1, \ell_2, \ldots, \ell_k)$. Let

$$Z' = (\ell_1 + 1, \ell_2, \ldots, \ell_k).$$

Denote the k-th iteration of this by

$$Z^{(i)} = (\ell_1 + i, \ell_2, \ldots, \ell_k).$$

Definition 2. Let FI be the category of finite sets and injective maps. An FI*-module* is a covariant functor from FI to abelian groups.

Then to construct an FI-module, it suffices to give maps between the Specht modules

$$S_Z \to S_{Z'}$$

which are Σ_n-equivariant and such that in the composition of such maps

$$S_Z \to S_{Z'} \to S_{Z''} \to \cdots \to S_{Z^{(i)}}, \tag{3}$$

the image is fixed under the Σ_i-action on the last i squares in the first row of $Z^{(i)}$.

Thus, consider the group homomorphism

$$\rho : \Sigma_n \to \Sigma_{n+1}$$

given by sending a permutation σ of $1, \dots, n$ (corresponding to a permutation of the boxes of Z) to a permutation of $1, \dots, n+1$ by swapping $1, \dots, n$ according to σ and using the identity on $n+1$ (which corresponds to using the permutation of the boxes of Y except for the $(\ell_1 + 1)$-th one in the first column, on which one uses the identity). A permutation on the boxes of Z which preserves its rows (respectively, columns) will preserve the rows (respectively, columns) of Z' after applying ρ.

Thus, ρ induces maps

$$\Sigma_n/\Sigma_Z^c \to \Sigma_{n+1}/\Sigma_{Z'}^c$$

$$\Sigma_n/\Sigma_Z^r \to \Sigma_{n+1}/\Sigma_{Z'}^r.$$

Hence, after taking free abelian groups, we get a diagram

$$\begin{tikzcd}
\mathbb{Z}(\Sigma_n/\Sigma_Z^c) \otimes \mathbb{Z}^- \arrow[r, "\varphi_Z"] \arrow[d] & \mathbb{Z}(\Sigma_n/\Sigma_Z^r) \arrow[d] \\
\mathbb{Z}(\Sigma_{n+1}/\Sigma_{Z'}^c) \otimes \mathbb{Z}^- \arrow[r, "\varphi_{Z'}"'] & \mathbb{Z}(\Sigma_{n+1}/\Sigma_{Z'}^r).
\end{tikzcd}$$

This diagram commutes because, in Z', preserving columns means preserving the new box, and therefore, in the target of the bottom horizontal map, we are summing over the same terms as in the target of the top horizontal map.

This induces a Σ_n-equivariant map on

$$S_Z = Im(\varphi_Z) \to Im(\varphi_{Z'}) = S_{Z'}. \tag{4}$$

Definition 3. The *integral Spechtral module* M_λ where $\lambda = (\ell_2, \dots, \ell_k)$ is the FI-module

$$S_{(\ell_2,\ell_2,\dots,\ell_k)} \to S_{(\ell_2+1,\ell_2,\dots,\ell_k)} \to S_{(\ell_2+2,\ell_2,\dots,\ell_k)} \to \dots$$

given by the maps (4).

Example: $\lambda = (1^k)$. Then

$$(M_\lambda)_n = S_{(n-k,1^k)}.$$

The image of the generators under (2) are alternating sums of generators corresponding to ordered choices of k elements in a given $k+1$ element subset of $\{1,\dots,n\}$. Consider the $\mathbb{Z}[\Sigma_n]$-module

$$V_n = \mathbb{Z}(\Sigma_n/\Sigma_{n-1}).$$

Let $\widetilde{V}_n = Ker(\mathbb{Z}(\Sigma_n/\Sigma_{n-1}) \to \mathbb{Z})$ be the kernel of the augmentation map. Then we have an embedding $\Lambda^k(V_n) \subseteq (V_n)^{\otimes k}$ given by taking alternating sums, which induces an embedding

$$\Lambda^k(\widetilde{V}_n) \subseteq (\widetilde{V}_n)^{\otimes k}. \tag{5}$$

Therefore, the image of (2) is canonically indentified with the image of (5). Hence,

$$(M_\lambda)_n \cong \Lambda^k(\widetilde{V}_n).$$

In fact, the right hand side $\Lambda^k(\widetilde{V}_n)$ is directly seen to form an FI-module by the emebeddings $\{1,\dots,n\} \subseteq \{1,\dots,n+1\}$. We will also sometimes denote this Spechtral module by $\Lambda^k(\widetilde{V})$.

Note that we can similarly define embeddings

$$\Lambda^k(V_n) \hookrightarrow \Lambda^k(V_{n+1}),$$

thus giving another FI-module, which we will denote by $\Lambda^k(V)$.

2.2. **Finitely generated FI-modules and the tensor product.** A *finitely generated FI-module* is defined to be one that is expressable as a quotient of a finite sum of principal projectives $P(n)$, the m-th term of which (for a fixed n) is

$$P(n)_m = \mathbb{Z}Mor_{FI}([n],[m])$$

where $[n] = \{1,\dots,n\}$. A Spechtral module is, by definition, finitely generated. The *Noetherian property* says that the submodule of a finitely generated module is finitely generated [11].

We define the *tensor product* $M \otimes N$ of FI-modules M, N by letting, at FI-degree n,

$$(M \otimes N)_n = M_n \otimes N_n$$

with FI acting diagonally. (Then also define a tensor product of morphisms of FI-modules at each FI-degree separately in the obvious way).

Lemma 4. *If two FI-modules M, N are finitely generated, then so is their tensor product $M \otimes N$.*

Proof. See [1], Proposition 2.3.6. ∎

2.3. **Local cohomology. First examples.** Define for an FI-module M, its *torsion submodule* TM by

$$TM_n = \{x \in M_n \mid \exists i : \{1, \dots, n\} \to \{1, \dots m\},\ i_*(x) = 0\}. \tag{6}$$

We also sometimes speak of *FI-torsion* to distinguish it from $\mathbb{Z}$-torsion (i.e., torsion in the category of abelian groups).

Definition 5. The *i-th local cohomology* of an FI-module M is defined as the i-th right derived functor R^iTM.

A torsion FI-module M has a resolution by torsion injectives of the form

$$Q(n)_m = Map(Mor_{FI}([m], [n]), J)$$

(for divisible abelian groups J, where Map denotes the abelian group of maps from a set into an abelian group), which are also FI-torsion, and thus

$$R^0TM = M$$
$$R^iTM = 0 \text{ for } i > 0.$$

Now let

$$\Lambda(\widetilde{V}) = \bigoplus_k \Lambda^k(\widetilde{V}), \tag{7}$$

and correspondingly,

$$\Lambda(V) = \bigoplus_k \Lambda^k(V). \tag{8}$$

In degree n, these FI-modules are equal to the exterior algebras $\Lambda(\widetilde{V}_n)$, $\Lambda(V_n)$, respectively. Let, also, $\mathbb{Z}_{\{n\}}$ denote the torsion FI-module that is $\mathbb{Z}$ in FI-degree n and is 0 at all other degrees.

Note that $\Lambda(\widetilde{V})$, $\Lambda(V)$ are not finitely generated. In addition, one has

$$\Lambda(\widetilde{V})_0 = 0, \tag{9}$$

and also $\Lambda(\widetilde{V})_1 = \mathbb{Z}$, $\Lambda(V)_0 = \mathbb{Z}$. (For (9), recall our earlier definition of $\Lambda^k(\widetilde{V})$ as a Spechtral module.)

Proposition 6. *The i-th local cohomology of $\Lambda(\widetilde{V})$ is $\mathbb{Z}$ given by*

$$R^iT\Lambda(\widetilde{V}) = \mathbb{Z}_{\{0\}} \text{ for all } i \geq 1. \tag{10}$$

More specifically,

$$R^iT\Lambda^k(\widetilde{V}) = \begin{cases} \mathbb{Z}_{\{0\}} & \text{for } i = k+1 \\ 0 & \text{else} \end{cases} \tag{11}$$

Proof. Let us denote by $v_1, \dots, v_n$ the elements of Σ_n/Σ_{n-1}, which generate V_n. (Note that then $\widetilde{V}_n$ is generated by $v_i - v_j$ for all $i \neq j$.) In addition, note that for $k \geq 1$, $i_1, \dots, i_k \in \{1, \dots, n\}$ distinct,

$$v_{i_1} \wedge \cdots \wedge v_{i_k} = v_{i_1} \wedge (v_{i_2} - v_{i_1}) \wedge \cdots \wedge (v_{i_k} - v_{i_1}).$$

We will define maps for $k \geq 1$

$$\theta_k : \Lambda^k(V) \to \Lambda^{k-1}(\widetilde{V})$$

by sending $v_{i_1} \wedge \cdots \wedge v_{i_k} = v_{i_1} \wedge (v_{i_2} - v_{i_1}) \wedge \cdots \wedge (v_{i_k} - v_{i_1})$ to

$$(v_{i_2} - v_{i_1}) \wedge \cdots \wedge (v_{i_k} - v_{i_1}).$$

To see that this map is well-defined, we need to show that our definition does not depend on which of the elements v_{i_j} we put first. To this end, consider an alternate expression

$$v_{i_1} \wedge \cdots \wedge v_{i_k} = (-1)^{j-1} v_{i_j} \wedge (v_{i_1} - v_{i_j}) \wedge \dots (\widehat{v_{i_j} - v_{i_j}}) \cdots \wedge (v_{i_k} - v_{i_j})$$

for some $j \in \{1, \dots, k\}$. Our definition of θ_k sends this element to

$$(-1)^{j-1}(v_{i_1} - v_{i_j}) \wedge \dots (\widehat{v_{i_j} - v_{i_j}}) \cdots \wedge (v_{i_k} - v_{i_j}).$$

To show that this is consistent, we need to prove that for every $j = 1, \dots, k$,

$$\begin{aligned} &(v_{i_2} - v_{i_1}) \wedge \cdots \wedge (v_{i_k} - v_{i_1}) = \\ &= (-1)^{j-1}(v_{i_1} - v_{i_j}) \wedge \dots (\widehat{v_{i_j} - v_{i_j}}) \cdots \wedge (v_{i_k} - v_{i_j}), \end{aligned}$$

which follows from the formula

$$(v_{i_0} - v_{i_1}) \wedge \cdots \wedge (v_{i_0} - v_{i_m}) = \sum_{j=0}^{m} (-1)^{m+j} v_{i_0} \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_{i_m}. \tag{12}$$

Also, let

$$\theta_0 : \Lambda^0(V) = \mathbb{Z} \to \mathbb{Z}_{\{0\}}$$

be identity at FI-degree 0 and 0 elsewhere.

Taking the direct sum of all θ_k for $k = 0, 1, \dots, k$ gives an onto map of FI-modules

$$\theta : \Lambda(V) \to \mathbb{Z}_{\{0\}} \oplus \Lambda(\widetilde{V}).$$

One checks by direct computation that

$$\Lambda(\widetilde{V}) \subseteq Ker(\theta) \tag{13}$$

Additionally, both source and target have the same rank in each FI-degree. Thus, the cokernel of (13) is $\mathbb{Z}$-torsion, and hence must be 0, since $\Lambda(V)/\Lambda(\widetilde{V})$ has no $\mathbb{Z}$-torsion.

Therefore, we have a short exact sequence of FI-modules

$$0 \longrightarrow \Lambda(\widetilde{V}) \xrightarrow{\subseteq} \Lambda(V) \xrightarrow{\theta} \mathbb{Z}_{\{0\}} \oplus \Lambda(\widetilde{V}) \longrightarrow 0. \tag{14}$$

Putting copies of (14) together, we get a resolution of $\Lambda(\widetilde{V})$ of the form

$$\Lambda(V) \to \mathbb{Z}_{\{0\}} \oplus \Lambda(V) \to \mathbb{Z}_{\{0\}} \oplus \Lambda(V) \to \dots. \tag{15}$$

One has

$$R^iT(\Lambda(V)) = 0 \text{ for all } i \geq 0$$

because these FI-modules are semi-induced (for a thorough discussion, see [2, 4, 6]). Hence the i-th local cohomology of $\Lambda(\widetilde{V})$ is $\mathbb{Z}_{\{0\}}$ for every $i \geq 1$ (and 0 for $i = 0$). Decomposing using (7), we get the i-th local cohomology

$$R^{k+1}T\Lambda^k(\widetilde{V}) = \mathbb{Z}_{\{0\}}$$

$$R^iT\Lambda^k(\widetilde{V}) = 0 \text{ for } i \neq k+1.$$

□

Proposition 7. *For any choice of $k_1, \dots, k_\ell \geq 0$, the i-th local cohomology of $\Lambda^{k_1}(\widetilde{V}) \otimes \dots \otimes \Lambda^{k_\ell}(\widetilde{V})$ is given by*

$$R^iT(\Lambda^{k_1}(\widetilde{V}) \otimes \dots \otimes \Lambda^{k_\ell}(\widetilde{V})) = \begin{cases} \mathbb{Z}_{\{0\}} & \text{for } i = k_1 + \dots + k_\ell + 1 \\ 0 & \text{else.} \end{cases}$$

Proof. First, recalling (15), the local cohomology of each individual $\Lambda^{k_j}(\widetilde{V})$ can be obtained from its resolution of the form

$$\Lambda^{k_j}(V) \to \Lambda^{k_j-1}(V) \to \Lambda^{k_j-2}(V) \to \dots \to \Lambda^0(V) \to \mathbb{Z}_{\{0\}}. \tag{16}$$

To calculate the local cohomology of $\Lambda^{k_1}(\widetilde{V}) \otimes \dots \otimes \Lambda^{k_\ell}(\widetilde{V})$, we will take the tensor product $\mathcal{C}$ of the resolutions (16) for all $j = 1, \dots, \ell$.

First note that

$$\Lambda^k(V) \otimes \mathbb{Z}_{\{0\}} = \begin{cases} \mathbb{Z}_{\{0\}} & \text{for } k = 0 \\ 0 & \text{else,} \end{cases}$$

$$\mathbb{Z}_{\{0\}} \otimes \mathbb{Z}_{\{0\}} = \mathbb{Z}_{\{0\}}.$$

Additionally, the last map of (16) induces an isomorphism after tensoring with $\mathbb{Z}_{\{0\}}$.

Consider the chain complex

$$\mathcal{I}^\ell = \bigotimes_\ell (Id : \mathbb{Z} \to \mathbb{Z})$$

in homological degrees $0, \dots, \ell$.

The resolution $\mathcal{C}$ of $\Lambda^{k_1}(\widetilde{V}) \otimes \dots \otimes \Lambda^{k_\ell}(\widetilde{V})$ has a last $\mathbb{Z}_{\{0\}}$ in cohomological degree $k_1 + \dots + k_\ell + \ell$, and thus, its torsion part is the homology of

$$\mathcal{I}^\ell_{\leq \ell-1}[-k_1 - k_2 - \dots - k_\ell - \ell] \tag{17}$$

(where in the formula (17), $[-k_1 - k_2 - \dots - k_\ell - \ell]$ denotes a shift in homological degree, and we consider the cohomological degree to be the opposite of the homological degree).

Now to calculate the homology of (17), first note that we have an injective chain map

$$0 \to \mathcal{I}^\ell_{\leq \ell-1} \to \mathcal{I}^\ell,$$

whose cokernel is $\mathbb{Z}[\ell]$ (again, shifting in homological degree), thus giving the short exact sequence of chain complexes

$$0 \to \mathcal{I}^\ell_{\leq \ell-1} \to \mathcal{I}^\ell \to \mathbb{Z}[\ell] \to 0. \tag{18}$$

Taking homology of (18) gives a long exact sequence of the form

$$\dots \to H_r(\mathcal{I}^\ell_{\leq \ell-1}) \to H_r(\mathcal{I}^\ell) \to H_r(\mathbb{Z}[\ell]) \to H_{r-1}(\mathcal{I}^\ell_{\leq \ell-1}) \to \dots .$$

Now we have for all $r \in \mathbb{Z}$, $H_r(\mathcal{I}^\ell) = 0$, and also

$$H_r(\mathbb{Z}[\ell]) = \begin{cases} \mathbb{Z} & \text{for } r = \ell \\ 0 & \text{else.} \end{cases}$$

Hence,

$$H_r(\mathcal{I}^\ell_{\leq \ell-1}) = \begin{cases} \mathbb{Z} & \text{for } r = \ell - 1 \\ 0 & \text{else.} \end{cases}$$

Thus, after converting to cohomological grading by reversing the signs of all degrees,

$$T(\Lambda^{k_1}(\widetilde{V}) \otimes \dots \otimes \Lambda^{k_\ell}(\widetilde{V}))$$

has cohomology $\mathbb{Z}_{\{0\}}$ exactly in degree $k_1 + \dots + k_\ell + \ell - (\ell - 1) = k_1 + \dots + k_\ell + 1$ and 0 else, i.e.,

$$R^i T(\Lambda^{k_1}(\widetilde{V}) \otimes \dots \otimes \Lambda^{k_\ell}(\widetilde{V})) = \begin{cases} \mathbb{Z}_{\{0\}} & \text{for } i = k_1 + \dots + k_\ell + 1 \\ 0 & \text{else.} \end{cases}$$

$\square$

Lemma 8. *The map* $\wedge : \Lambda^{k-1}(\widetilde{V}) \otimes \widetilde{V} \to \Lambda^k(\widetilde{V})$ *induces an isomorphism in local cohomology*

$$R^iT(\Lambda^{k-1}(\widetilde{V}) \otimes \widetilde{V}) \cong R^iT(\Lambda^k(\widetilde{V}).$$

Proof. We will prove this by proving the existence of a chain map from a resolution of $\Lambda^{k-1}(\widetilde{V}) \otimes \widetilde{V}$ to a resolution of $\Lambda^k(\widetilde{V})$ which induces a quasiisomorphism after applying T.

Now as in (16), there is a resolution $\mathcal{B}$ of $\Lambda^{k-1}(\widetilde{V})$ of the form

$$\Lambda^{k-1}(V) \to \Lambda^{k-2}(V) \to \cdots \to \Lambda^0(V) \to \mathbb{Z}_{\{0\}}. \tag{19}$$

Applying this to $\Lambda^1(\widetilde{V}) = \widetilde{V}$, we get that $\widetilde{V}$ has a resolution of the form

$$V \to \mathbb{Z} \to \mathbb{Z}_{\{0\}}.$$

As in the proof of Proposition 7, we can obtain a resolution of

$$\Lambda^{k-1}(\widetilde{V}) \otimes \widetilde{V}$$

by tensoring these two resolutions as chain complexes. Denote this resolution by $\mathcal{C}$:

$$\begin{tikzcd}
\Lambda^{k-1}(V) \otimes \mathbb{Z}_{\{0\}} \arrow[r] & \dots \arrow[r] & \Lambda^0(V) \otimes \mathbb{Z}_{\{0\}} \arrow[r] & \mathbb{Z}_{\{0\}} \\
\Lambda^{k-1}(V) \arrow[r] \arrow[u] & \dots \arrow[r] & \Lambda^0(V) \arrow[r] \arrow[u] & \mathbb{Z}_{\{0\}} \arrow[u] \\
\Lambda^{k-1}(V) \otimes V \arrow[r] \arrow[u] & \dots \arrow[r] & \Lambda^0(V) \otimes V \arrow[r] \arrow[u] & \mathbb{Z}_{\{0\}} \otimes V \arrow[u]
\end{tikzcd} \tag{20}$$

Note that $\Lambda^1(V) \otimes \mathbb{Z}_{\{0\}} = \mathbb{Z}_{\{0\}} \otimes V = 0$. There is a morphism from $\mathcal{C}$ to

$$\Lambda^k(V) \to \Lambda^{k-1}(V) \to \cdots \to \Lambda^0(V) \to \mathbb{Z}_{\{0\}}$$

where we send the top row of (20) to 0, and the bottom two rows are sent by the map

$$(-1)^{k-1} Id \oplus \wedge : \Lambda^i(V) \oplus \Lambda^{i-1}(V) \otimes V \to \Lambda^i(V),$$

$$(-1)^{k-1} : \mathbb{Z}_{\{0\}} \to \mathbb{Z}_{\{0\}}.$$

Clearly, this induces an isomorphism in cohomology after applying T. $\square$

3. The local cohomology of M'_{λ_k}

The main goal of this paper is to compute the local cohomology of the Spechtral module M_{λ_k}. In this section, we will define another FI-module M'_{λ_k} and compute its local cohomology, and in the next section, we will prove that

$$M_{\lambda_k} \cong M'_{\lambda_k}.$$

3.1. **Certain chain complexes of FI-modules.** Let $\mathcal{D}$ denote the chain complex

$$\Lambda^k(V) \otimes V \to \Lambda^{k-1}(V) \otimes V \to \cdots \to \Lambda^0(V) \otimes V$$

where the differential is given by

$$d(v_{i_1} \wedge \cdots \wedge v_{i_j} \otimes v_\ell) = \\ = \begin{cases} (-1)^{j-s-1} v_{i_1} \wedge \dots \widehat{v_{i_s}} \cdots \wedge v_{i_j} \otimes v_\ell & \text{if } i_s = \ell \text{ for some } s \\ 0 & \text{else.} \end{cases}$$

The FI-degree n summand of this chain complex can be alternately described as

$${}_n\mathcal{D} = \bigoplus_{i=1}^n \Lambda[v_1, \dots, \widehat{v_i}, \dots, v_n] \otimes (\mathbb{Z}\{v_i \otimes v_i\} \xrightarrow{\cong} \mathbb{Z}\{v_i\}),$$

where the isomorphism $\mathbb{Z}\{v_i \otimes v_i\} \to \mathbb{Z}\{v_i\}$ is given by sending $v_i \otimes v_i$ to v_i, and the chain complex is indexed homologically, putting the term $\Lambda^k(V) \otimes V$ in homological degree k. Thus, for every FI-degree n, $H_*({}_n\mathcal{D}) = 0$.

At every FI-degree n and homological degree k, the differential of ${}_n\mathcal{D}$ sends

(21)
$$(v_{i_2} - v_{i_1}) \wedge \cdots \wedge (v_{i_k} - v_{i_1}) \otimes v_{i_1} \mapsto -(v_{i_2} - v_{i_k}) \wedge \cdots \wedge (v_{i_{k-1}} - v_{i_k}) \otimes v_{i_1}.$$

Hence, by always restricting differentials to $\Lambda^k(\widetilde{V}) \otimes V$, there is a subcomplex $\widetilde{\mathcal{D}}$ of the form

$$\Lambda^k(\widetilde{V}) \otimes V \xrightarrow{d_k^{\widetilde{\mathcal{D}}}} \Lambda^{k-1}(\widetilde{V}) \otimes V \xrightarrow{d_{k-1}^{\widetilde{\mathcal{D}}}} \dots \xrightarrow{d_1^{\widetilde{\mathcal{D}}}} \Lambda^0(\widetilde{V}) \otimes V.$$

Now in the case of $n > 1$, for every $m \in \mathbb{Z}$,

$$H_m({}_n\widetilde{\mathcal{D}}) = 0, \tag{22}$$

since, recalling formula (12), $\Lambda^k(\widetilde{V}) \otimes V$ is generated by elements in the source of (21) and the target of (21) (where in the target, k is replaced by $k+1$).

If $n = 1$, then the chain complex ${}_1\widetilde{\mathcal{D}}$ is of the form

$$0 \to 0 \to \cdots \to 0 \to \mathbb{Z}$$

(where the $\mathbb{Z}$ is in homological degree 0). Hence, in this case,

$$H_m({}_1\widetilde{\mathcal{D}}) = \begin{cases} \mathbb{Z} & \text{for } m = 0 \\ 0 & \text{else.} \end{cases} \tag{23}$$

3.2. **The definition of the FI-module M'_{λ_k}.** Let K be the FI-module defined as the kernel

$$0 \to K \to \Lambda^{k-1}(\widetilde{V}) \otimes \widetilde{V} \to \Lambda^k(\widetilde{V}) \tag{24}$$

of the canonical map $\Lambda^{k-1}(\widetilde{V}) \otimes \widetilde{V} \xrightarrow{\wedge} \Lambda^k(\widetilde{V})$. Consider the map

$$\delta : K \to \Lambda^{k-2}(\widetilde{V}) \otimes V$$

that is the restriction of the differential of $\widetilde{\mathcal{D}}$. We put

$$M'_{\lambda_k} = Ker(\delta).$$

Thus, we have a candidate for a resolution of M'_{λ_k} of the form

$$K \xrightarrow{d_0^{\mathcal{E}}} \Lambda^{k-2}(\widetilde{V}) \otimes V \xrightarrow{d_1^{\mathcal{E}}} \Lambda^{k-3}(\widetilde{V}) \otimes V \xrightarrow{d_2^{\mathcal{E}}} \dots \xrightarrow{d_{k-2}^{\mathcal{E}}} \Lambda^0(\widetilde{V}) \otimes V$$

where $d_0^{\mathcal{E}} = \delta$. Denote it by $\mathcal{E}$. Let us use chain cohomological grading for $\mathcal{E}$, i.e., $\mathcal{E}^0 = K$, $\mathcal{E}^1 = \Lambda^{k-2}(\widetilde{V}) \otimes V$, etc., and

$$d_m^{\mathcal{E}} : \mathcal{E}^m \to \mathcal{E}^{m+1}.$$

We will see that $\mathcal{E}$ is "close enough" to a resolution to calculate the local cohomology of M'_{λ_k}. To do this, we begin by calculating the chain cohomology of $\mathcal{E}$. By definition, $H^0(\mathcal{E}) = M'_{\lambda_k}$.

3.3. **The chain cohomology of $\mathcal{E}$.**

Theorem 9. *Let $k \geq 2$.*

(1) *For $m > 1$,*

$$H^m({}_n\mathcal{E}) = \begin{cases} 0 & \text{if } n > 1, \text{ and } m \geq 2 \\ 0 & \text{if } n = 1, \text{ and } 2 \leq m < k-1 \\ \mathbb{Z} & \text{if } n = 1, \text{ and } m = k-1. \end{cases}$$

(2) *We have*

$$H^1({}_n\mathcal{E}) = \begin{cases} \mathbb{Z}^- & \text{if } n = k \\ (\mathbb{Z}/k)^- & \text{if } n = k+1 \\ \mathbb{Z}/2 & \text{if } n > k+1 \text{ and } k \text{ is even} \\ \mathbb{Z} & \text{if } n = 1, k = 2 \\ 0 & \text{else} \end{cases}$$

where $\mathbb{Z}^-$, $(\mathbb{Z}/k)^-$ denote the sign representations of Σ_n.

Comment: The penultimate case of (2) is part of the last case of (1).

Proof of the cases $m > 1$ and $m = 1$, $n \leq k$. First, note that, for $m > 1$,

$$H^m(\mathcal{E}) = Ker(d_m^{\mathcal{E}})/Im(d_{m-1}^{\mathcal{E}}) = H_{k-1-m}(\widetilde{\mathcal{D}}),$$

thus proving (1) of the Theorem by (22), (23).

It remains to calculate the first chain chomology $H^1({}_n\mathcal{E})$ for every $n \geq 1$.

First, suppose $n \leq k-1$. In this case, $dim(\widetilde{V}) = n-1 < k-1$, and therefore $\Lambda^{k-1}(\widetilde{V}) \otimes \widetilde{V} = 0$. Hence, $K = 0$. Then the chain complex ${}_n\mathcal{E}$ is of the form

$$0 \to \Lambda^{k-2}(\widetilde{V}) \otimes V \to \Lambda^{k-3}(\widetilde{V}) \otimes V \to \cdots \to \Lambda^0(\widetilde{V}) \otimes V,$$

which is isomorphic to $\widetilde{\mathcal{D}}[1-k]$. Hence, $H^1({}_n\mathcal{E}) = H_{k-2}({}_n\widetilde{\mathcal{D}}) = 0$, except in the case $n = 1$, $k = 2$, which was already discussed.

Now suppose $n = k$. In this case, $dim(\widetilde{V}) = n-1$. Thus, $\Lambda^k(\widetilde{V}) = 0$, and (24) gives

$$K \cong \Lambda^{k-1}(\widetilde{V}) \otimes \widetilde{V}.$$

The chain complex ${}_n\mathcal{E}$ is of the form

$$\Lambda^{k-1}(\widetilde{V}_n) \otimes \widetilde{V}_n \to \Lambda^{k-2}(\widetilde{V}_n) \otimes V_n \to \cdots \to \Lambda^0(\widetilde{V}_n) \otimes V_n.$$

Thus, it forms a short exact sequence

$$0 \to {}_n\mathcal{E} \to {}_n\widetilde{\mathcal{D}}[1-k] \to \mathbb{Z}^- \to 0$$

(where $\mathbb{Z}^-$ represents the cochain complex

$$\mathbb{Z}^- \to 0 \to \cdots \to 0,$$

and, again, $[1-k]$ denotes the shift in homological degree). This gives a long exact sequence

$$H^0(\widetilde{\mathcal{D}}[1-k]) \to H^0(\mathbb{Z}^-) \to H^1(\mathcal{E}) \to H^1(\widetilde{\mathcal{D}}[1-k]) \to \ldots.$$

Hence, since

$$H^0(\widetilde{\mathcal{D}}[1-k]) = H_{k-1}(\widetilde{\mathcal{D}}) = 0,$$
$$H^1(\widetilde{\mathcal{D}}[1-k]) = H_{k-2}(\widetilde{\mathcal{D}}) = 0,$$

we have

$$H^1({}_n\mathcal{E}) = H^0(\mathbb{Z}^-) = \mathbb{Z}^-.$$

□

3.4. **The proof of Theorem 9, part (2) in the case $n \geq k+1$.**

Proof of Theorem 9, (2) for $n = k+1$. Note that we have canonical isomorphisms

$$\Lambda^k(\widetilde{V}_{k+1}) \cong \mathbb{Z}^-,\ \Lambda^{k-1}(\widetilde{V}_{k+1}) \cong \mathbb{Z}^- \otimes \widetilde{V}^*_{k+1}. \tag{25}$$

Thus, we have a short exact sequence of chain complexes

$$0 \to {}_{k+1}\mathcal{E} \to {}_{k+1}\widetilde{\mathcal{D}}[1-k] \to \mathcal{K} \to 0 \tag{26}$$

where $\mathcal{K}$ is of the form

$$\Lambda^k(\widetilde{V}_{k+1}) \otimes V_{k+1} \xrightarrow{\delta} (\Lambda^{k-1}(\widetilde{V}_{k+1}) \otimes V_{k+1})/K \tag{27}$$

in homological degrees $0, -1$. By (25), we have an idenfication

$$\mathcal{K}_0 = \mathbb{Z}^- \otimes V_{k+1}, \tag{28}$$

which has dimension $k+1$.

To understand $\mathcal{K}$, first note that we can identify

$$\begin{gathered} K_{k+1} = \mathbb{Z}^- \otimes \{f \in Hom(\widetilde{V}, \widetilde{V}) | tr(f) = 0\}, \\ \Lambda^{k-1}(\widetilde{V}) \otimes V = \mathbb{Z}^- \otimes Hom(\widetilde{V}, V). \end{gathered} \tag{29}$$

Writing $V/\widetilde{V} = \mathbb{Z}$, we thus get a short exact sequence

$$0 \longrightarrow \mathbb{Z}^- \longrightarrow \mathcal{K}_{-1} \xrightarrow{\pi} \mathbb{Z}^- \otimes \widetilde{V}^* \longrightarrow 0. \tag{30}$$

In particular, $dim(\mathcal{K}_{-1}) = dim(\mathcal{K}_0) = k+1$. By the long exact sequence in cohomology associated with (26), we need to show that the cokernel of δ is isomorphic to $(\mathbb{Z}/k)^-$.

By Noether's isomorphism theorem, we have

$$\begin{gathered} Coker(\delta) = \\ \mathbb{Z}/(tr((Im(d_0^{\widetilde{\mathcal{D}}[1-k]}) \subseteq \mathbb{Z}^- \otimes Hom(\widetilde{V}_{k+1}, V_{k+1}))\cap \\ (\mathbb{Z}^- \otimes Hom(\widetilde{V}_{k+1}, \widetilde{V}_{k+1})))) \end{gathered}$$

Now define $e_i \in \widetilde{V}^*_{k+1}$ by setting, for $j \neq k$,

$$e_i(v_j - v_k) = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } k = j \\ 0 & \text{else.} \end{cases}$$

We have

$$d_0^{\widetilde{\mathcal{D}}[1-k]}(v_i) = e_i \otimes v_i.$$

Thus,

$$Im(d_0^{\widetilde{\mathcal{D}}[1-k]}) \cap (\mathbb{Z}^- \otimes Hom(\widetilde{V}_{k+1}, \widetilde{V}_{k+1}))$$

is generated by

$$d_0^{\widetilde{\mathcal{D}}[1-k]}(v_1 + \cdots + v_{k+1}) = \sum_{i=1}^{k} e_i(v_i - v_{k+1}) \tag{31}$$

(using $v_{k+1} = -v_1 - \cdots - v_k \in \widetilde{V}_{k+1}$).

Now for $i = 1, \ldots, k$,

$$e_i(v_i - v_{k+1}) = 1,$$

$$e_i(v_j - v_{k+1}) = 0, \text{ for } i \neq j.$$

Thus, the trace of the right hand side of (31) is k, as claimed.

$\square$

Next, we have the following

Lemma 10. *For every $n \geq k+1$, K_n is generated by elements of the form*

$$(v_{i_1} - v_{i_2}) \wedge (v_{i_1} - v_{i_3}) \wedge \cdots \wedge (v_{i_1} - v_{i_{k-1}}) \wedge (v_i - v_j) \otimes (v_i - v_j) \tag{32}$$

where $i \neq j$ and all $i_1, \ldots, i_{k-1}$ are different, and

$$|\{i, j\} \cap \{i_1, \ldots, i_{k-1}\}| = 0 \text{ or } 1. \tag{33}$$

Proof. The statement that the set of the elements (32) for any

$$i, j, i_1, i_2, \ldots, i_{k-1} \in \{1, \ldots, n\}$$

generates K_n follows from the fact that for variables $w_1, \ldots, w_s$,

$$\Lambda_{\mathbb{Z}}[w_1, \ldots, w_s] = T[w_1, \ldots, w_s]/I$$

where $T[w_1, \ldots, w_s]$ denotes the tensor algebra on $w_1, \ldots, w_s$, and I is generated by $w_i \otimes w_i$, $w_i \otimes w_j + w_j \otimes w_i$, and additionally, for any $i, j, i', j' \in \{1, \ldots, n\}$,

$$\begin{gathered}(v_i - v_j) \otimes (v_{i'} - v_{j'}) + (v_{i'} - v_{j'}) \otimes (v_i - v_j) = \\ = -(v_i - v_{i'}) \otimes (v_i - v_{i'}) + (v_i - v_{j'}) \otimes (v_i - v_{j'}) + \\ +(v_j - v_{i'}) \otimes (v_j - v_{i'}) - (v_j - v_{j'}) \otimes (v_j - v_{j'}).\end{gathered}$$

Now in view of formula (12), unless $i_1, \dots, i_{k-1}$ are different and $i \neq j$, and (33) holds, the expression (32) is 0.

$\square$

Now denote by
$$\epsilon : \Lambda^{k-2}(V_n) \otimes V_n \to \mathbb{Z}/2$$
the map which sends
$$v_{i_1} \wedge \dots \wedge v_{i_{k-2}} \otimes v_j \mapsto 1$$
for $i_1, \dots, i_{k-2}$ different. Let $\widetilde{\epsilon}$ denote the composition
$$\Lambda^{k-2}(\widetilde{V}_n) \otimes V_n \hookrightarrow \Lambda^{k-2}(V_n) \otimes V_n \xrightarrow{\epsilon} \mathbb{Z}/2.$$

We have

Lemma 11. *For k even, the compostion*
$$K_n \xrightarrow{d_0^{\mathcal{E}}} \Lambda^{k-2}(\widetilde{V}_n) \otimes V_n \xrightarrow{\widetilde{\epsilon}} \mathbb{Z}/2$$
is 0.

Proof. This follows from the fact that, by formula (12), the $d_0^{\mathcal{E}}$ images of the generators (32) with the condition (33) can be expressed as a sum of evenly many terms of the form
$$v_{i_1} \wedge \dots \wedge v_{i_{k-2}} \otimes v_j \tag{34}$$
for $i_1, \dots, i_{k-2}$ different.

$\square$

Proof of Theorem 9, (2) for $n > k+1$. First note that we have
$$\begin{gathered} d_0^{\mathcal{E}}((v_{i-1} - v_{i_2}) \wedge \dots \wedge (v_{i_1} - v_{i_k}) \otimes (v_{i_1} - v_{i_2})) = \\ = (-1)^k((v_{i_1} - v_{i_3}) \wedge \dots \wedge (v_{i_1} - v_{i_k}) \otimes v_{i_2} + \\ +(v_{i_2} - v_{i_3}) \wedge \dots \wedge (v_{i_2} - v_{i_k}) \otimes v_{i_1}) \end{gathered} \tag{35}$$
for $i_1, \dots i_k \in \{1, \dots, n\}$ different. Also, for $i_1, \dots, i_{k+1} \in \{1, \dots, n\}$ different, we have
$$\begin{gathered} d_0^{\mathcal{E}}((v_{i_1} - v_{i_2}) \wedge (v_{i_3} - v_{i_4}) \wedge \dots \wedge (v_{i_3} - v_{i_{k+1}}) \otimes (v_{i_1} - v_{i_2})) = \\ = (-1)^k((v_{i_3} - v_{i_4}) \wedge \dots \wedge (v_{i_3} - v_{i_{k+1}}) \otimes (v_{i_1} + v_{i_2})). \end{gathered} \tag{36}$$
Since $n \geq k+2$, the set
$$S = \{1, \dots n\} \smallsetminus \{i_3, i_4, \dots, i_{k+1}\}$$
has cardinality ≥ 3, and thus, the elements of the form (36) generate an index 2 subgroup in
$$\langle (v_{i_3} - v_{i_4}) \wedge \dots \wedge (v_{i_3} - v_{i_{k+1}}) \otimes v_j | j \in S \rangle \tag{37}$$

Additionally, elements of the right hand side of (35) identify, in

$$Ker(d_1^{\mathcal{E}})/Im(d_0^{\mathcal{E}}),$$

the generators (37) for different choices of $\{i_3, \dots, i_{k+1}\}$. Since elements (37) generate $Ker(d_1^{\mathcal{E}})$, this represents $H^1({}_n\mathcal{E})$ as a factor of $\mathbb{Z}/2$.

If k is even, then the onto map

$$\mathbb{Z}/2 \to H^1({}_n\mathcal{E})$$

must be an isomorphism (since $H^1({}_n\mathcal{E})$ surjects onto $\mathbb{Z}/2$ by Lemma 11).

If k is odd, we also know from the case $n = k+1$ that the k-multiple of

$$(v_{i_3} - v_{i_4}) \wedge \dots \wedge (v_{i_3} - v_{i_{k+1}}) \otimes v_j \tag{38}$$

for $j \neq i_3, \dots i_{k+1}$ is $0 \in H^1({}_n\mathcal{E})$, and since the 2-multiple of (38) is also 0 in $H^1({}_n\mathcal{E})$, which is generated by elements of this form, we know that

$$H^1({}_n\mathcal{E}) = 0,$$

as claimed.

□

3.5. The local cohomology of M'_{λ_k}.

Theorem 12. (1) *If $k \geq 2$ is odd, we have*

$$R^iT(M'_{\lambda_k}) = \begin{cases} \mathbb{Z}^- & \text{if } n = k, \text{ and } i = 2 \\ (\mathbb{Z}/k)^- & \text{if } n = k+1, \text{ and } i = 2 \\ \mathbb{Z} & \text{if } n = 1, \text{ and } i = k \\ 0 & \text{else.} \end{cases}$$

(2) *If $k \geq 2$ is even, we have*

$$R^iT(M'_{\lambda_k}) = \begin{cases} \mathbb{Z}^- & \text{if } n = k, \text{ and } i = 2 \\ (\mathbb{Z}/\frac{k}{2})^- & \text{if } n = k+1, \text{ and } i = 2 \\ \mathbb{Z}/2 & \text{if } n \leq k-1, \text{ and } i = 3 \\ \mathbb{Z} & \text{if } n = 1, \text{ and } i = k \\ 0 & \text{else.} \end{cases}$$

Proof. We will begin by proving (1). Suppose k is odd. The chain complex of FI-modules

$$(K \to \Lambda^{k-2}(\widetilde{V}) \otimes V \to \Lambda^{k-3}(\widetilde{V}) \otimes V \to \dots \to \Lambda^0(\widetilde{V}) \otimes V)_{n \geq k+2} \tag{39}$$

(where the subscript means restricting to FI-degree $\geq k+2$) gives a resolution of M_{λ_k}, by Theorem 9, since the chain cohomology is only nonzero in FI-degree $\leq k+1$.

Now in general, if $\mathcal{C}$ denotes a resolution of an object M in an abelian category with enough injectives, and T is a left exact functor into another abelian category, we have a spectral sequence

$$E_1^{p,q} = R^q T(\mathcal{C}^p)$$

that converges to $R^{p+q}T(M)$, (meaning that there exists a decreasing filtration $F_p(R^m T(M))$ such that

$$E_\infty^{p,q} = F_p(R^{p+q}T(M))/F_{p+1}(R^{p+q}T(M))).$$

In this case, of course, we take the category of FI-modules, T is the torsion functor (6), $\mathcal{C}$ is (39), and M is M'_{λ_k}.

In addition, for an FI-module M, if

$$\forall q \geq 0 \quad R^q T(M) = 0, \tag{40}$$

then we have

$$R^q T(M_{n\geq s}) = \begin{cases} M/M_{n\leq s-1} & \text{if } q=1 \\ 0 & \text{else} \end{cases} \tag{41}$$

by the short exact sequence of FI-modules

$$0 \to M_{n\geq s} \to M \to M_{n\leq s-1} \to 0$$

(where the subscripts again mean restricting to the indicated FI-degrees).

Note that K satisfies (40) by Proposition 7 (and the map

$$\Lambda^{k-1}(\widetilde{V}) \otimes \widetilde{V} \to \Lambda^k(\widetilde{V})$$

induces an isomorphism in local cohomology, see Lemma 8), and the FI-module $\Lambda^k(\widetilde{V}) \otimes V$ satisfies (40) because $\mathbb{Z}_{\{0\}} \otimes V = 0$.

So, in this case, the E_1-term is nonzero only for $q=1$, (and therefore the spectral sequence collapses to E_2), and is equal to the chain complex at $n \leq k+1$, which is

$$K_{n\leq k+1} \to (\Lambda^{k-2}(\widetilde{V}) \otimes V)_{n\leq k+1} \to \cdots \to (\Lambda^0(\widetilde{V}) \otimes V)_{n\leq k+1}.$$

The cohomology of this chain complex is given by Theorem 9, thus implying our result.

Now we shall prove (2). Suppose k is even. Let $\widehat{\mathcal{E}}$ denote the chain complex

$$K \to \Lambda^{k-2}(\widetilde{V}) \otimes V \to \mathbb{Z}/2_{n\geq 0} \oplus \Lambda^{k-3}(\widetilde{V}) \otimes V \to \cdots \to \Lambda^0(\widetilde{V}) \otimes V$$

(the maps $\Lambda^{k-2}(\widetilde{V_n}) \otimes V_n \to \mathbb{Z}/2$ are given by the number of terms (34) of the source modulo 2). Then $\widehat{\mathcal{E}}_{n\geq k+2}$ gives a resolution of M'_{λ_k}, by Theorem 9.

By the discussion in the previous case, the resulting spectral sequence is only non-trivial at $q = 1$ (hence, it collapses again to E_2). For $q = 1$, we get the chain complex $\widehat{\mathcal{E}}_{n\leq k+1}$. Now we have a short exact sequence

$$0 \to \mathbb{Z}/2_{n\leq k+1}[-2] \to \widehat{\mathcal{E}}_{n\leq k+1} \to \mathcal{E}_{n\leq k+1} \to 0.$$

Thus, we get a long exact sequence in local cohomology which is

$$\begin{tikzcd} 0 \arrow[r] & R^1T\widehat{\mathcal{E}}_{n\leq k+1} \arrow[r] & R^1T\mathcal{E}_{n\leq k+1} \arrow[r, "\delta"] & \mathbb{Z}/2_{n\leq k+1} \arrow[d] \\ & & & R^2T\widehat{\mathcal{E}}_{n\leq k+1} \arrow[d] \\ & & & 0 \end{tikzcd} \tag{42}$$

(the connecting map δ would go to R^2 which is shifted down by 2 on the FI-torsion module $\mathbb{Z}/2_{n\leq k+1}$).

The term $R^1T\mathcal{E}_{n\leq k+1}$ is calculated by Theorem 9. The map δ is onto in FI-degrees k and $k+1$ by the definition of the connecting map. Thus, our result follows.

□

4. Proof that M'_{λ_k} is isomorphic to M_{λ_k}

Theorem 13. *For all $k \geq 2$, there is an isomorphism*

$$\psi : M_{\lambda_k} \xrightarrow{\cong} M'_{\lambda_k}.$$

Proof. We begin by describing explicitly the integral Specht module

$$S_{(n-k,\,2,\,1^{k-2})} = (M_{\lambda_k})_n.$$

The generators are certain sums in $\mathbb{Z}(Q_n^r)$ where Q_n^r is the set of all ordered choices of different elements $i_1, \dots, i_{k-2} \in \{1, \dots, n\}$ and a 2-element set $I \subseteq \{1, \dots, n\} \smallsetminus \{i_1, \dots, i_{k-2}\}$ (which is identified with Σ_n/Σ^r). They are images of elements of $\mathbb{Z}(\Sigma_n/\Sigma^c)$, which can be identified with $\mathbb{Z}(Q_n^c)$, where Q_n^c is the set of different elements $\iota_1, \dots, \iota_{n-k-2} \in \{1, \dots n\}$, and a 2-element set

$$J \subseteq \{1, \dots, n\} \smallsetminus \{\iota_1, \dots, \iota_{n-k-2}\}.$$

The sum ranges over all compatible choices in Q_n^r with a given element of Q_n^c with signs determined by the sign of the overall permutation of $\{1, \dots, n\}$ which the choice determines. Note that up to sign, these generators actually only depend on the sets $J, J' \in \{1, \dots, n\}$, $|J| = 2$, $|J'| = k$, $J \cap J' = \emptyset$, where $J' = \{1, \dots, n\} \smallsetminus (\{\iota_1, \dots, \iota_{n-k-2}\} \cup J)$.

We will denote the corresponding generator by $\overline{x}_{J,J'}$. Putting

$$J = \{j_1 < j_2\},\ J' = \{j'_1 < \dots < j'_k\},$$

we can write

$$(43)\quad \overline{x}_{J,J'} = \sum_{\sigma \in \Sigma_2, \tau \in \Sigma_k} \operatorname{sign}(\sigma) \cdot \operatorname{sign}(\tau) \cdot ((j'_{\tau(3)}, \dots, j'_{\tau(k)}), \{j_{\sigma(1)}, j'_{\tau(2)}\}).$$

Now the element (43) determines (up to sign) an element of

$$Ker(\wedge : \Lambda^{k-1}(V_n) \otimes V_n \to \Lambda^k(V_n))$$

given by

(44)

$$\begin{aligned} x_{J,J'} = & \sum_{p=1,\dots,k,\ q=1,2} (-1)^{p+q}(k-1) v_{j'_1} \wedge \dots \widehat{v_{j'_p}} \dots \wedge v_{j'_k} \otimes v_{j_q} \\ + & \sum_{1 \le p_1 \ne p_2 \le k,\ q=1,2} (-1)^{p_1+q} v_{j'_1} \wedge \dots \widehat{v'_{j_{p_1}}} \dots \widehat{v'_{j_{p_2}}} \wedge v_{j_q} \dots \wedge v_{j'_k} \otimes v_{j'_{p_2}}. \end{aligned}$$

We claim that the elements (43), (44) span isomorphic representations. In fact, the passage from (43) to (44) can be described as follows: (43) can be considered as an element of $V_n^{\otimes k-2} \otimes Sym^2(V_n)$, where the $Sym^2(V_n)$ corresponds to the $\{j_{\sigma(1)}, j'_{\tau(2)}\}$ coordinate of (43). Embedding into $V_n^{\otimes k-2} \otimes V_n^{\otimes 2} = V_n^{\otimes k}$, however, provides an element in

$$\Lambda^{k-1}(V_n) \otimes V_n \hookrightarrow V_n^{\otimes k}$$

given by anti-symmetrization, which is given (up to sign) by (44).

Now by definition,

$$(45)\qquad x_{J,J'} \in Ker(d_{k-1}^{\widetilde{\mathcal{D}}} : \Lambda^{k-1}(V_n) \otimes V_n \to \Lambda^{k-2}(V_n) \otimes V_n)$$

(since no element v_j occurs on both sides of the $\otimes$ in any of the summands). We claim also that

$$(46)\qquad x_{J,J'} \in \Lambda^{k-1}(\widetilde{V}_n) \otimes \widetilde{V}_n.$$

This is obvious for the first summand (44), whose term for a given p is

$$e_{J' \smallsetminus \{j'_p\}} \otimes e_J$$

where $e_{(i_0,\dots,i_m)}$ denotes the element (12) and we write

$$e_{\{i_0<\dots<i_m\}} = e_{(i_0,\dots,i_m)}.$$

The sum of the other summand of (44) is seen to be in $\Lambda^{k-1}(V_n)\otimes\widetilde{V}_n$ by grouping the terms with given $\{p_1,p_2\}$, q, and in $\Lambda^{k-1}(\widetilde{V}_n)\otimes V_n$ by grouping the terms with a given p_2. Thus, (46) holds. By (45), (46), we have produced a map of Σ_n-representations

$$\psi_n : (M_{\lambda_k})_n \to (M'_{\lambda_k})_n. \tag{47}$$

This map is non-zero and hence injective (since the representations $(M_{\lambda_k})_n$ are irreducible after tensoring with $\mathbb{Q}$).

We first claim that $\psi_n\otimes\mathbb{Q}$ is an isomorphism. To this end, Remmel [8], Theorem 2.1 proves that rationally,

$$\Lambda^{k-1}(\widetilde{V}_n)\otimes\widetilde{V}_n \cong (M_{\lambda_k})_n\oplus(M_{\lambda_{k-1}})_n\oplus\Lambda^k(\widetilde{V}_n)\oplus\Lambda^{k-1}(\widetilde{V}_n)\oplus\Lambda^{k-2}(\widetilde{V}_n).$$

Inductively, it follows that rationally,

$$\begin{aligned} L_k &:= Ker(d^{\widetilde{\mathcal{D}}}_{k-2} : \Lambda^{k-2}(\widetilde{V}_n)\otimes V_n \to \Lambda^{k-3}(\widetilde{V}_n)\otimes V_n) \\ &\cong (M_{\lambda_{k-1}})_n\oplus\Lambda^{k-1}(\widetilde{V}_n)\oplus\Lambda^{k-2}(\widetilde{V}_n). \end{aligned}$$

On the other hand, rationally,

$$K_n \cong (M_{\lambda_k})_n\oplus(M_{\lambda_{k-1}})_n\oplus\Lambda^{k-1}(\widetilde{V}_n)\oplus\Lambda^{k-2}(\widetilde{V}_n)$$

so we have a short exact sequence

$$0\to(M_{\lambda_k})_n\to K_n\to L_k\to 0,$$

as claimed.

Thus, the cokernel of (47) is torsion, and we need to show that the torsion is 0. We will use induction on n and k. By the induction hypothesis, the elements $x_{J,J'}$ with $n\notin J\cup J'$ generate

$$(M'_{\lambda_k})_{n-1}\subseteq\Lambda^{k-1}(\widetilde{V}_{n-1})\otimes\widetilde{V}_{n-1}.$$

Thus, it suffices to consider the images of the elements

$$x_{J,J'},\ n\in J \tag{48}$$

$$x_{J,J'},\ n\in J' \tag{49}$$

in

$$\Lambda^{k-2}(\widetilde{V}_{n-1})\wedge\{v_n\}\otimes\widetilde{V}_{n-1}\oplus\Lambda^{k-1}(\widetilde{V}_{n-1})\otimes\{v_n\}. \tag{50}$$

However, the condition that these elements be in K_n determines the second summand (50) from the first summand, so we may further replace (50) by

$$\Lambda^{k-2}(\widetilde{V}_{n-1})\wedge\{v_n\}\otimes\widetilde{V}_{n-1}\cong\Lambda^{k-2}(\widetilde{V}_{n-1})\otimes\widetilde{V}_{n-1} \tag{51}$$

Now for $n \in J'$, (with appropriate signs), the image of an element

$$x_{\{j_1,j_2\},J'} \pm x_{\{j_1,n\},J' \smallsetminus \{n\} \cup \{j_2\}} \pm x_{\{j_2,n\},J' \smallsetminus \{n\} \cup \{j_2\}} \tag{52}$$

becomes the generator

$$x_{\{j_1,j_2\},J' \smallsetminus \{n\}} \in (M_{\lambda_{k-1}})_{n-1}. \tag{53}$$

On the other hand, the image of an element $x_{\{i,n\},\{j'_1,\dots,j'_k\}}$ in (51) is, up to sign,

$$\sum_{s=1}^{k} (-1)^s e_{\{j'_1,\dots,\widehat{j'_s},\dots,j'_k\}} \otimes v_{j'_s}. \tag{54}$$

Additionally, in any characteristic $p > 0$, the elements (53) are linearly independent of the elements (54) by the characteristic p Pieri rule. Thus, if a linear combination of such elements is divisible by p, the sums of the (53)-summands and (54)-summands would have to be divisible by p separately. For the (53)-summands, the element is a p-multiple of a linear combination of the elements (52) by the induction hypothesis. The (54)-summands actually do not depend on i, but otherwise, for different choices of $\{j_1, \dots, j_k\}$, are linearly independent in characteristic p. Thus, our result follows.

□

5. Appendix: Comments on the Relationshipe with Schur–Weyl Correspondence

As mentioned in the Introduction, rationally, Schur–Weyl duality gives an equivalence between the category of FI-modules and the category of $GL(V)$-equivariant graded $Sym(V)$-modules deegree-wise polynomial in the Schur functors, where $V = \mathbb{Q}\{x_0, x_1, \dots\}$. (To distinguish the two meanings of x_i, we write the basis elements of V as dx_i.) In fact, there is an equivalence between the category of generic FI-modules (meaning the Serre quotient of FI-modules over torsion FI-modules) and a certain category of "quasi-coherent sheaves" on $\mathbb{P}^\infty$. Further it is easy to show that under this duality, $M_{(\underbrace{1,\dots,1}_{k})}$ corresponds to the coherent sheaf $\underline{\Omega}^k$ of k-forms on $\mathbb{P}^\infty_{\mathbb{Q}}$. In fact, this statement for $M_{(\underbrace{1,\dots,1}_{k})}$ holds on the level of graded modules, if we take the saturated model of $\underline{\Omega}^k$.

Integrally, this correspondence fails. Let us discuss briefly what happens in the above Theorem for $k = 2$. Rationally, on the $Sym(V)$-modules side, $M_{(2)}$ corresponds to the equivariant graded $Sym(V)$-module $\underline{S}_{(2)}$ defined as $Ker(d)$ in the resolution
(55)

$$V^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^\infty}(-2) \xrightarrow{d} \Lambda^2 V \otimes \mathcal{O}_{\mathbb{P}^\infty}(-2) \oplus V \otimes \mathcal{O}_{\mathbb{P}^\infty}(-1) \longrightarrow W$$

where W is torsion equal to V in degree 1 and $\Lambda^2 V$ in degree 2 and 0 elsewhere, and we have

$$x_i : dx_j \mapsto dx_i \wedge dx_j \in W.$$

The first map (55) is a sum of the projection with the $\mathcal{O} = \mathbb{Q}[x_0, x_1, \dots]$-module map given by

$$dx_i \otimes dx_j \mapsto x_i dx_j.$$

Since the first two terms of the resolution have 0 local cohomology, we see that this matches the result of Theorem 1.

Now integrally, if we still denote $W = Coker(d)$ in (55), then W is the same in degrees 1 and 2, but is equal to $\Lambda^n V/2$ in degree ≥ 3. If we denote by Λ the graded $Sym(V)$-module where $\Lambda_n = \Lambda^n(V)/2$ for $n \in \mathbb{N}_0$, considered as a quotient of $Sym(V)$, then W coincides with Λ in degrees ≥ 3. We see that this object is spurious in the sense that it is "Schur–Weyl dual" to the constant $\mathbb{Z}/2$ FI-module, as is $Sym(V)/2$.

Now one can show that Λ has 0 local cohomology in the category of graded $Sym(V)$-modules. (To see this, let, for a finite $I \subseteq \mathbb{N}_0$, let N_I be the $Sym(V)$-module on monomials of the form

$$x_{i_1} \wedge \cdots \wedge x_{i_n} x_{j_1}^{-1} \dots x_{j_m}^{-1} \tag{56}$$

where $i_1, \dots, i_n \notin I$, $j_r \neq i_s$. Multiplication of (56) by a x_k-monomial which results in a monomial not of this form is 0. Then N_I is injective, torsion-free, and Λ has a "cube-like" resolution by $N_I x_{i_1}^{-1} \dots x_{i_N}^{-1}$ where $I = \{i_1 < \cdots < i_N\}$).

Plugging in this computation of local cohomology, we actually get that $\underline{S}_{(2)}$ has the same local cohomology as M_2 integrally. Similarly, for $k > 2$ even, $R^3 T S_{\lambda_k}$ will contain the part of the module Λ in degrees $\leq k - 1$.

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