QUANTUM DELANNOY CATEGORIES

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Abstract. The main subject of this paper is the construction of quantum or "$q > 1$" counterparts of the Delannoy category constructed by Harman, Snowden, and Snyder [8]. We investigate the remarkable properties of our new categories. As an application, we find a semisimple pre-Tannakian category of growth $e^{e^{c \cdot n^2}}$, which is the highest known.

Contents

1. Introduction

The main result of this paper is the construction and investigation of certain surprising new categories. A new direction of category theory has recently been developing, which can be viewed as an extension of representation theory. It studies, for the most part, symmetric tensor (abelian) categories which are linear over some field k, whose Hom-sets are finitely generated vector spaces, and which are rigid in the sense that all objects have strong duals [6]. Categories with these properties are often referred to as *pre-Tannakian categories*, in reference to the paper by P. Deligne and J. S. Milne [4], which used the concept of *neutral Tannakian categories*, meaning pre-Tannakian categories with a fiber functor, to construct candidates for categories of motives.

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Pre-Tannakian categories turned out to be of independent interest. Specifically *semisimple* pre-Tannakian categories are mathematical entities which are both difficult to construct and have a rich internal geometry. P. Deligne called them "isolated diamonds." New examples of semisimple pre-Tannakian categories are the main subject of the present paper.

In particular, we construct "$q > 1$" counterparts of the Delannoy category of N. Harman, A. Snowden, and N. Snyder [8]. We call them *quantum Delannoy categories*, extending the terminology of [7]. The quantum Delannoy categories have many striking properties. In particular, they imply the existence of a semisimple pre-Tannakian category with an object of growth at least $e^{e^{c\cdot n^2}}$, which is currently the highest known.

To state our results more precisely and place them in context, we introduce some terminology. A natural generalization of a pre-Tannakian category is an additive category linear over a commutative ring with associative, commutative, unital (ACU) tensor product, finitely generated Hom-modules, and strong duality. We call them *quasi-pre-Tannakian (QPT) categories*. In a QPT category, one can define the *dimension* of an object X as the trace of Id_X, and the *growth* of X as the sequence

$$rank(End(X^{\otimes n})).$$

P. Deligne [2] proved that a semisimple pre-Tannakian category $\mathscr{C}$ has a fiber functor (i.e. a faithful tensor functor) into the category $sVect$ of super vector spaces if and only if every object of $\mathscr{C}$ has at most exponential growth.

QPT categories are much easier to construct than pre-Tannakian ones. For example, the author in [12] constructed a QPT category of arbitrarily high growth. However, when a QPT category is semisimple, it is automatically abelian, and therefore, pre-Tannakian. P. Deligne and J. S. Milne [4] (Subsections 1.26, 1.27) constructed a QPT category $\underline{Rep}(GL_t)$ in characteristic 0 which is free on an object in dimension t. After extending scalars to $\mathbb{C}$, it is semisimple when t is not a non-negative integer. It can be considered as an "interpolation" $\underline{Rep}(GL_t)$ of algebraic representation categories of general linear groups.

The next important advancement in the study of semisimple pre-Tannakian categories was the paper of P. Deligne [3], which constructed, for k a field of characteristic 0, categories $\underline{Rep}(S_t)$ for $t \in k$, "interpolating" the categories of finite dimensional k-representations of the symmetric groups S_n. The categories $\underline{Rep}(S_t)$ thus constructed are semisimple for t not a non-negative integer.

Knop [10, 11], analogously to [3], defined categories $\underline{Rep}(GL_t(\mathbb{F}_q))$ for a finite field $\mathbb{F}_q$ with $q = p^m$ for a prime p, "interpolating" the categories of finite-dimensional k-representations of $GL_n(\mathbb{F}_q)$ for k a field of characteristic 0. Again, these categories are semisimple for generic values of t. In some sense, the category $\underline{Rep}(GL_t(\mathbb{F}_q))$ can be considered a "$q > 1$" analogue of the category $\underline{Rep}(S_t)$. However, while the categories $\underline{Rep}(S_t)$ and $\underline{Rep}(GL_t)$ have objects of growth at most $e^{c\cdot n\cdot ln(n)}$, the category $\underline{Rep}(GL_t(\mathbb{F}_q))$ has objects of growth at least $e^{c\cdot n^2}$ for $c > 0$, which was the highest growth known at the time.

A major development in the subject was the work by N. Harman and A. Snowden [7], who generalized the interpolation method to a method of constructing locally finite additive k-linear categories with an ACU tensor product and strong duality using oligomorphic groups (see [1]) with "measures." The above examples can be obtained from the oligomorphic groups S_∞, $GL_\infty(\mathbb{F}_q)$. However, a number of other examples exist, including the Delannoy category $\mathscr{D}$ (N. Harman, A. Snowden, N. Snyder, [8]), which comes from a suitable measure on the oligomorphic group $Aut(\mathbb{R}, <)$ of order-preserving bijections of $\mathbb{R}$. This category is semisimple (and hence, semisimple pre-Tannakian) over a field k of any characteristic and the growth of its objects is bounded above and below by the growth of objects of $\underline{Rep}(S_t)$. Another interesting class of examples constructed using oligomorphic groups with measures was recently described in [13].

A. Snowden [14] showed that for a semisimple pre-Tannakian category $\mathcal{C}$ over a finite field k with an object of growth at least $f(n)$, applying a non-zero idempotent of $\mathbb{C}[k, \cdot]$ to the free $\mathbb{C}$-linear category $\mathbb{C}[\mathcal{C}]$ gives a semisimple pre-Tannakian category with objects of growth at least $e^{f(n)}$. Using the Delannoy category, he obtained objects of growth at least $e^{e^{c\cdot n\cdot ln(n)}}$.

The main result of this paper is a construction of "$q > 1$" or *quantum* counterparts of the Delannoy categroy This category combines the ideas of $\underline{Rep}(GL_t(\mathbb{F}_q))$ and $\mathscr{D}$. I first discovered this construction using the formalism of *T-algebras* which I used in [12]. It is a general method for constructing QPT categories by studying the universal algebra structure on $Hom(X^{\otimes m}, X^{\otimes n})$ with varying m, n for some generating object X. The data considered are (tensor) product, partial traces, and units in a particular sense. A T-algebra structure determines a QPT structure completely (assuming finitely generated Hom-modules).

When one studies the T-algebra of $\underline{Rep}(S_t)$, one finds that, denoting by X the "basic object" interpolating the permutation representations of the symmetric group S_n on the set $\{1, \dots, n\}$, the Hom-modules

$Hom(X^{\otimes m}, X^{\otimes n})$ are the free modules on equivalence relations on

$$\{1, \dots, m\} \amalg \{1, \dots, n\}. \tag{1.1}$$

One has $dim(X) := tr(Id_X) = t$. For the Delannoy category, the description of $Hom(X^{\otimes m}, X^{\otimes n})$ is, in our formalism, again given by equivalence relations on (1.1), except that the equivalence classes are (totally) ordered. It turns out that this forces $t = -1$. It is worth noting that, as pointed out it [8], the Delannoy category cannot be obtained by interpolating categories of representations of finite groups.

For $\underline{Rep}(GL_t(\mathbb{F}_q))$, $Hom(X^{\otimes m}, X^{\otimes n})$ can be described as the free module on quotient homomorphisms

$$\mathbb{F}_q^m \oplus \mathbb{F}_q^n \to V. \tag{1.2}$$

The basic object has dimension $dim(X) = q^t$.

I defined the (Borel) quantum Delannoy category by adding the data of a choice of a maximal flag on V to (1.2). As it turns out, this, again, forces $t = -1$. The resulting category $\mathscr{D}_{q,k}$ is semisimple for a target field k of characteristic not dividing $q(q-1)$.

This has an application to growth. Using the method of [14] on the Borel quantum Delannoy category $\mathscr{B}_{q,k}$, we obtain the following

Theorem 1.1. *There exists a semisimple pre-Tannakian category $\mathcal{E}$ over $\mathbb{C}$ with an object X such that $dim(End(X^{\otimes n}))$ has objects of growth at least $e^{e^{c \cdot n^2}}$ for some $c \in \mathbb{R}_{>0}$.*

These are the fastest growing examples of semisimple pre-Tannakian categories currently known.

It turns out that the category $\mathscr{B}_{q,k}$ can also be constructed by the method of [7], using the oligomorphic group which is the semidirect product of $Aut(\mathbb{R}, <)$ with an infinite Borel group.

Following a suggestion of P. Deligne [5], we also investigate a variant by using (in a suitably finitary version), the unipotent subgroup instead. The resulting category $\mathscr{U}_{q,k}$ is semisimple for any field k of characteristic not equal to p and can, in some sense, be considered an even more "pure" definition of a quantum Delannoy category. We describe it again using both the T-algebra and oligomorphic group mechanisms. The construction of $\mathscr{U}_{q,k}$ also has a variant where the "basic object" is the projective space $\mathbb{P}(\mathbb{F}_q[\mathbb{R}])$.

We study both categories $\mathscr{B}_{q,k}$ and $\mathscr{U}_{q,k}$ in some detail. In particular, we give the decomposition of the "basic object" into simple summands. In fact, over an algebraically closed field of characteristic not p, one can classify *all the simple objects of* $\mathscr{U}_{q,k}$ (referring to the characters of the p-Sylow subgroup of $GL_n(\mathbb{F}_q)$).

There is a natural tensor functor

$$\mathscr{B}_{q,k} \to \mathscr{U}_{q,k}.$$

Perhaps surprisingly, there also turns out to be a tensor functor

$$\mathscr{D} \to \mathscr{B}_{q,k}$$

(where $\mathscr{D}$ is the Delannoy category), due to a certain striking universal property of the Delannoy category $\mathscr{D}$, which we describe.

The present paper is organized as follows: Section 2 starts by describing the T-algebra method. We show how the T-algebra mechanism applies to the categories $\underline{Rep}(S_t)$, $\mathscr{D}$, and $\underline{Rep}(GL_t(\mathbb{F}_q))$.

In Subsection 2.7, we then construct the Borel quantum Delannoy category $\mathscr{B}_{q,k}$ using T-algebras. In Subsection 2.8, we describe the unipotent analogue for the construction, which gives $\mathscr{U}_{q,k}$.

Section 3 gives a construction of the Borel and unipotent quantum Delannoy categories $\mathscr{B}_{q,k}$ and $\mathscr{U}_{q,k}$ using the framework of oligomorphic groups. We review and verify the technical requirements to apply the results of A. Snowden and N. Harman to get a proof of the semisimplicity of $\mathscr{B}_{q,k}$ and $\mathscr{U}_{q,k}$, and give descriptions of orbits of the corresponding oligomorphic groups using matrices.

An alternative proof of semisimplicity, using the methods of [3] (also giving a calculation of the dimensions of the "top algebra" in the sense of [3]), is given in the Appendix.

Section 4 discusses structural properties of the categories, including the simple decompositions of each category's basic objects, a classification of the simple objects of $\mathscr{U}_{q,k}$, and a universality property for the Delannoy category.

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2. T-Algebra Constructions and the Quantum Delannoy Categories

In this section, we discuss a general mechanism for describing a QPT category generated by a single object X using the modules of homomorphisms between its tensor powers. These modules form a universal algebra which we call a *T-algebra*. The initial T-algebra corresponds to the category $\underline{Rep}(GL_t)$ constructed in [4], [3], Chapter 10. We also treat several other examples using the formalism of T-algebras, including $\underline{Rep}(S_t)$ [3], $\underline{Rep}(GL_t(\mathbb{F}_q))$ [10, 11], and the Delannoy category [8]. For general background on tensor categories, we refer the reader to [6].

In Sections 2.7 and 2.8, we construct the two main examples of the present pape using the T-algebra formalism. The *quantum (or $q > 1$) Delannoy categories* are defined by combining the ideas of $\mathscr{D}$ (Subsection 2.4) and $\underline{Rep}(GL_t(\mathbb{F}_q))$ (Subsection 2.6).

The first version, the *Borel quantum Delannoy category*, denoted by $\mathscr{B}_{q,k}$, is defined in Section 2.7. It is defined for a prime power q over a field k with

$$char(k) \nmid q(q-1) \tag{2.1}$$

is defined by specifying a maximal flag on the target of (2.22).

The second version, the *unipotent quantume Delannoy cateogry*, denoted by $\mathscr{U}_{q,k}$, is defined in Subsection 2.8. It is defined for a prime power $q = p^m$ over a field k of characteristic not p, by specifying a basis of the target of (2.22) modulo the action of the unipotent subgroup of the Borel subgroup of $GL(V)$ (where V denotes the target of (2.22)) of linear transformations preserving the flag. For the remainder of this paper, $q = p^m$ will denote the power of a prime p.

2.1. Definition of a T-algebra.

A k-linear abelian category $\mathscr{C}$ for a field k with an ACU tensor product and strong duality (generated by an object X) is, in fact, determined by the structure of the morphism k-vector spaces

$$\mathscr{C}_{S,T} := Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T}) \tag{2.2}$$

for all finite sets S, T, which can be axiomatized by a type of a universal algebra structure which we call a *T-algebra*, which we define as follows:

Definition 2.2. *Fix a field k. A* T-algebra *consists of the data of*

(1) *A system of k-vector spaces $\mathcal{T}_{S,T}$ for all finite sets S and T, functorial with respect to bijections in the S and T coordinates,*

(2) *for all subsets $S' \subseteq S$, $T' \subseteq T$ and any choice of a bijection*

$$\phi : S' \xrightarrow{\cong} T'$$

a specified $S_{S\smallsetminus S'} \times S_{T\smallsetminus T'}$-equivariant trace map

$$\tau_\phi : \mathcal{T}_{S,T} \to \mathcal{T}_{S\smallsetminus S', T\smallsetminus T'}$$

(where S_U denotes the symmetric group on a set U),

(3) *for finite sets S_1, S_2, T_1, T_2, product maps*

$$\pi : \mathcal{T}_{S_1,T_1} \otimes \mathcal{T}_{S_2,T_2} \to \mathcal{T}_{S_1 \amalg S_2, T_1 \amalg T_2}, \tag{2.3}$$

(4) *an element $1 \in \mathcal{T}_{\emptyset,\emptyset}$ and an element $\iota \in \mathcal{T}_{\{1\},\{1\}}$.*

We require that this data satisfy the following axioms, which encode the functoriality of traces with respect to bijections in the remaining coordinates, ACU properties of the tensor product maps, compatibility of tensor product with traces, and "compositon unitality" for ι:

(1) *For finite sets S, T with subsets $S', S'' \subseteq S$, $T', T'' \subseteq T$ such that $S' \cap S'' = T' \cap T'' = \emptyset$, for choices of bijections*

$$\phi : S' \to T'$$
$$\phi' : S'' \to T'',$$

we have

$$\tau_{\phi \amalg \phi'} = \tau_\phi \circ \tau_{\phi'} = \tau_{\phi'} \circ \tau_\phi$$

as maps

$$\mathcal{T}_{S,T} \to \mathcal{T}_{S\smallsetminus(S' \amalg S''), T\smallsetminus(T' \amalg T'')}.$$

(2) *The product maps π satisfy the clear commutativity and associativity axioms, and are unital with respect to 1, which means, for finite sets S, T,*

$$\begin{array}{ccc} \mathcal{T}_{S,T} & \xrightarrow{Id_{\mathcal{T}_{S,T}} \otimes 1} & \mathcal{T}_{S,T} \otimes \mathcal{T}_{\emptyset,\emptyset} \\ & {\scriptstyle Id_{\mathcal{T}_{S,T}}} \searrow & \downarrow {\scriptstyle \pi} \\ & & \mathcal{T}_{S,T} \end{array}$$

(3) *For finite sets S_1, S_2, T_1, and T_2, and subsets $S' \subseteq S_1 \amalg S_2$, $T' \subseteq T_1 \amalg T_2$, for a choices of bijection*

$$\phi : S' \to T'$$

such that $\phi(S' \cap S_i) = T' \cap T_i$, we have

$$\tau_\phi \circ \pi = \pi \circ (\tau_{\phi|_{S_1}} \otimes \tau_{\phi|_{S_2}})$$

(both mapping

$$\mathcal{T}_{S_1,T_1} \otimes \mathcal{T}_{S_2,T_2} \to \mathcal{T}_{(S_1 \amalg S_2) \smallsetminus S', (T_1 \amalg T_2) \smallsetminus T'}).$$

(4) *For any $x \in \mathcal{T}_{\{1\},\{1\}}$, "composing" with ι gives x, meaning that if we take the product of ι and x, consider it as an element of $\mathcal{T}_{\{1,2\},\{1,2\}}$, and take the partial trace $\tau_{\{1\} \to \{2\}}$ with respect to the bijection sending 1 to 2, we recover x (and similarly if we take the product of x and ι).*

It is clear that for a QPT category $\mathscr{C}$, the Hom-spaces $\mathscr{C}_{S,T}$, together with partial traces and (tensor) products on morphisms, form a T-algebra.

Conversely, the axioms of a T-algebra $\mathcal{T}$ encode the morphisms of a category $\mathscr{C}_\mathcal{T}$ by taking

$$Hom_{\mathscr{C}_\mathcal{T}}(X^{\otimes m_1} \otimes (X^\vee)^{\otimes n_1}, X^{\otimes m_2} \otimes (X^\vee)^{\otimes n_2}) = \\ = \mathcal{T}_{\{1,\dots m_1+n_2\},\{1,\dots,m_2+n_1\}}$$

The axioms precisely encode the structure of an additive category with ACU tensor product and strong duality in this caegory.

In fact, one sees from axiom (1) that the traces are determined by "elementary traces"

$$\tau_{i,j} : \mathcal{T}_{S,T} \to \mathcal{T}_{S \smallsetminus \{i\}, T \smallsetminus \{j\}} \tag{2.4}$$

for choices of coordinates $i \in S$, $j \in T$.

As in axiom (4), the composition in a category $\mathscr{C}_\mathcal{T}$ for a T-algebra $\mathcal{T}$ is described by taking

$$\mathcal{T}_{S,T} \otimes \mathcal{T}_{T,R} \to \mathcal{T}_{S,R}$$

by sending a choice of $f \in \mathcal{T}_{S,T}$, $g \in \mathcal{T}_{T,R}$ to the element obtained by taking the product $\pi(f,g) \in \mathcal{T}_{S \amalg T, T \amalg R}$ and applying τ_{Id_T}.

Comments: 1. T-algebras being a universal algebra means that they cannot encode local finiteness. However in explicit examples, this property is readily verified.

2. When discussing semisimplicity, one usuall needs to create new objects which are images of idempotents. This step, known as the *pseudo-abelian* or *Karoubian* envelope described in [3], will be involved automatically without explicit mention.

3. Two complementary cases of the T-algebra formalism stand out. One is the case of *graded T-algebras* when $\mathcal{T}_{S,T} = 0$ unless $|S| = |T|$. An example is the *unital T-algebra* constructed in [4], Section 1.26. (For more examples, see also [12].)

The other extreme is when the basic object is self-dual so we have an isomorphism

$$\mathcal{T}_{S,T} \cong \mathcal{T}_{\emptyset, S \amalg T}$$

compatible with products and partial traces. This latter case is in fact the one of interest in all the examples in the present paper, which is why our definitions will only depend on $S \amalg T$.

We will begin by describing some examples of tensor categories which inform our constructions of the Borel and unipotent quantum Delannoy categories, from the point of view of T-algebras.

2.3. **The interpolated category of representations of the symmetric group.** Let us first discuss the example of the category $\underline{Rep}(S_t)$ (see [3]) from the T-algebra point of view.

Let X denote the basic generating object of $\underline{Rep}(S_t)$, which can be thought of as interpolating the action of the symmetric group S_n on the set $[n] = \{1, \dots, n\}$. We may choose bases of the morphism spaces between its tensor powers as partitions of the disjoint union of the indexing sets of the source and target:

(2.5)
$$\begin{gathered}\underline{Rep}(S_t)_{S,T} = Hom_{\underline{Rep}(S_t)}(X^{\otimes S}, X^{\otimes T}) = \\ = k\{\{U_1, \dots, U_\ell\} \mid \ell \leq |S| + |T|,\ \emptyset \neq U_i \subseteq S \amalg T, \\ U_i \cap U_j = \emptyset \text{ for } i \neq j, \text{ and } \textstyle\coprod_{i=1}^{\ell} U_i = S \amalg T\}.\end{gathered}$$

It is important in (2.5) to note that the set $\{U_1, \dots, U_\ell\}$ is unordered, i.e. is equivalent to the data of an equivalence relation on $S \amalg T$.

For $i \in S$, $j \in T$, the partial trace map $\tau_{i,j}$ applied to a partition $\{U_1, \dots, U_\ell\}$ of $S \amalg T$ is defined by taking it to be 0 if i and j are not in the same equivalence class U_s, and, if $i,j \in U_s$, taking the partition

$$\{U_1, \dots, U_s \smallsetminus \{i,j\}, \dots, U_\ell\}$$

when

$$U_s \smallsetminus \{i,j\} \neq \emptyset \tag{2.6}$$

and

$$\{U_1, \dots, U_{s-1}, U_{s+1}, \dots, U_\ell\}$$

when

$$U_s = \{i,j\}, \tag{2.7}$$

multiplied by a certain constant.

The constant is determined by the number of choices how the given configuration can arise when $t = N$ is a large integer, where $\{U_1, \dots U_\ell\}$ corresponds to the orbit

$$S_{S\amalg T}/S_{U_1} \times \cdots \times S_{U_\ell}. \tag{2.8}$$

realized as the idempotent in $End(k[1, \dots, N]^{S\amalg T})$ which is identity on the orbit consisting of tuples whose coordinates belonging to the same set U_s are equal and 0 on other orbits. Therefore, the coefficient is 1 in the case of (2.6) and

$$t - \ell + 1 \tag{2.9}$$

in the case of (2.7).

The products of $\{U_1, \dots, U_\ell\}$ and $\{V_1, \dots, V_{\ell'}\}$ are described by "gluings," which are specified by surjections

$$\kappa : \{1, \dots, \ell\} \amalg \{1, \dots, \ell'\} \twoheadrightarrow \{1, \dots, \ell + \ell' - h\} \tag{2.10}$$

which are injective on each of the discrete summands in the source. The gluing is accomplished by forming a new equivalence relation with classes whose jth class is the union of U_s, resp. $V_{s'}$ (whichever apply) where $\kappa(s) = j$, resp. $\kappa(s') = j$. Keep in mind, however, that the sets of equivalence classes are unordered, so gluing data are considered equal when they produce the same sets of equivalence classes. Compatibility of traces with product can be checked directly, but (since they correspond to numerical identities) also follow from considering $t = N$ a large integer. Since one encounters factorials, the field k must be of characteristic 0.

Semisimplicity was proved in [3], Sections 3-5, assuming that

$$t(t-1)\dots(t-n+1) \tag{2.11}$$

are invertible for all $n \in \mathbb{N}$, i.e. that t is not a non-negative integer. The proof proceeds by considering the object U_n given by the idempotent in $End(X^{\otimes n})$ corresponding to the partition

$$\{\{1_1, 1_2\}, \{2_1, 2_2\}, \dots, \{n_1, n_2\}\} \tag{2.12}$$

(where the subscript indicates which disjoint summand of

$$\{1, \dots, n\} \amalg \{1, \dots, n\}$$

we are in; note that projecting to the factor corresponding to this idempotent eliminates morphisms where two or more elements of $[n]_i$ are in the same equivalence class for $i = 1$ or 2). One proves semisimplicity of the additive subcategory generated by U_m, $m \leq n$. Using Proposition 3.8 of [3], one splits off U_n a direct sum of simple objects occurring for $m < n$, and one is left with the group algebra $k[S_n]$, which is semisimple since k has characteristic 0.

2.4. **The Delannoy category.** To describe the Delannoy category $\mathscr{D}$, for finite sets S, T, take the representations

$$\mathscr{D}_{S,T} = Hom_{\mathscr{D}}(X^{\otimes S}, X^{\otimes T})$$

to be the free k-modules on partitions of $S \amalg T$, similarly as in the case of $\underline{Rep}(S_t)$, but with a total ordering of the components of the partition.

As noted in [8], the Delannoy category cannot be considered an interpolation of categories of representations of finite groups. More precisely, for finite sets S, T, $\mathscr{D}_{S,T}$ is

$$\begin{gathered} k\{(U_1, \dots, U_\ell) \mid \ell \leq |S| + |T|,\ \emptyset \neq U_i \subseteq S \amalg T, \\ U_i \cap U_j = \emptyset \text{ for } i \neq j, \text{ and } \textstyle\coprod_{i=1}^{\ell} U_i = S \amalg T\}. \end{gathered} \tag{2.13}$$

Thus, the only difference with (2.5) is that the tuple $(U_1, \dots, U_\ell)$ is ordered.

Again, for $i \in S$, $j \in T$, the partial trace map $\tau_{i,j}$ applied to an ordered partition $(U_1, \dots, U_\ell)$ of $S \amalg T$ is defined by taking it to be 0 if i and j are not elements of the same U_s, and, if $i, j \in U_s$, taking

$$(U_1, \dots, U_s \smallsetminus \{i, j\}, \dots, U_\ell) \tag{2.14}$$

when (2.6) occurs and

$$(U_1, \dots, U_{s-1}, U_{s+1}, \dots, U_\ell) \tag{2.15}$$

when (2.7) occurs, multiplied by suitable coefficients.

The coefficient is again 1 in the case of (2.6). In the case of (2.7), the number of equivalence classes becomes $\ell - 1$, but we should divide by the ℓ choices of the number in $\{1,\dots,\ell\}$ we omit. This suggests the coefficient

$$\frac{t-\ell+1}{\ell}. \tag{2.16}$$

However, compatibility with product (which is the only non-trivial axiom to verify) turns out to force (2.16) to be independent of ℓ, which occurs for $t=-1$ (when the value of (2.16) is -1).

The product is described by ordered gluings, which are surjections (2.10) that are strictly increasing on the discrete summands $\{1,\dots,\ell\}$, $\{1,\dots,\ell'\}$ in the source. To glue $(U_1,\dots,U_\ell)$ to $(V_1,\dots,V_{\ell'})$, for $1\leq j\leq \ell+\ell'-h$, take, again, the union of U_s and/or $V_{s'}$ whenever $\kappa(s)=j$ resp. $\kappa(s')=j$ (whichever applies).

Proving the compatibility of trace with product is non-trivial only in the case of (2.7). Assume we take the trace $\tau_{i,j}$ in the case (2.7), to produce the ordered set of equivalence classes (2.15). When taking product of (2.15) with

$$(V_1,\dots,V_\ell), \tag{2.17}$$

consider separately each case of $\kappa|_{\{1,\dots,\ell-1\}}$. Let $r=\kappa(j-1)$, $r'=\kappa(j)$. In the case when we take the product with (2.17) first, there are $2(r'-r)-1$ choices of $\kappa(j)$ (note that j becomes $j+1$ and r may stay the same or increase by one depending on whether $\kappa(j)$ is equal to $\kappa(s)$ for $s\in\{1,\dots,\ell'\}$ or not). There are $r'-r-1$ of the former cases ("clashes") and $r'-r$ of the latter cases ("non-clashes"). Thus, when taking the product first and then the trace, we obtain a coefficient of

$$(r'-r)\cdot(-1)+(r'-r+1)=-1,$$

i.e. same as taking the trace first and then the product, as required.

Since the ordering eliminates products of the form (2.11), the Delannoy category exists in any characteristic.

To prove semisimplicity, the analogue of (2.12) becomes the sum of its copies over all orderings i.e.

$$\sum_{\sigma\in S_n}(\{(\sigma(1))_1,(\sigma(1))_2\},\dots,\{(\sigma(n))_1,(\sigma(n))_2\}). \tag{2.18}$$

Call the image of this idempotent, again, U_n. The analogue of Proposition 4.2 of [3] is describing the trace pairing matrix

$$Hom(U_0,U_n)\otimes Hom(U_n,U_0)\to k. \tag{2.19}$$

One has $dim(Hom(U_0, U_n)) = n!$ (corresponding to permutations on $[n] = \{1, \dots, n\}$) and the matrix is diagonal with diagonal entries $(-1)^n$, so it is non-singular.

The analogue of Lemma 5.2 of [3] can be stated as the following

Lemma 2.5. *The "top part" of $End_{\mathscr{D}}(U_n)$ (i.e. the part which has 0 morphisms to the simple summands of $X^{\otimes r}$ for $r < n$, see* [3]*, 3.7-3.9) is freely generated by elements of S_n composed with elements ϕ_R, for all choices of subsets*

$$R = \{r_1 < \dots < r_m\} \subseteq \{1, \dots, n\},$$

given by

(2.20)

$$\begin{gathered}\phi_R = (\{1_1, 1_2\}, \dots \\ \dots \{(r_1-1)_1, (r_1-1)_2\}, \{(r_1)_1\}, \{(r_1)_2\}, \{(r_1+1)_1, (r_1+1)_2\}, \dots \\ \dots \\ \dots \{(r_m-1)_1, (r_m-1)_2\}, \{(r_m)_1\}, \{(r_m)_2\}, \{(r_m+1)_1, (r_m+1)_2\}, \dots \\ \dots, \{n_1, n_2\}).\end{gathered}$$

Proof. One notes that (2.18) splits as a sum of disjoint idempotents corresponding to its summands for each individual $\sigma \in S_n$. Let us consider the idempotent summand ι corresponding to $\sigma = Id_{[n]}$. Denote its image by $U_{n,1}$. Then $End_{\mathscr{D}}(U_{n,1})$ is a free k-vector space on generators of the form

$$[n] \amalg [n] \xrightarrow{\phi \amalg \psi} [m]$$

where ϕ and ψ are injective, order-preserving and $\phi \amalg \psi$ is onto, and, for each $i \in [n]$, the classes of each i_1 and i_2 (where, again, the subscript refers to which copy of $[n]$ in $[n] \amalg [n]$ the element i is considered to be in) must be either equal or next to each other in the ordering. (In particular,

$$dim(End_{\mathscr{D}}(U_{n,1})) = 3^n.)$$

Now let $I_{n,1}$ be the ideal generated by $\mathscr{D}$-morphism of the form

$$U_{n,1} \to X^{\otimes r} \to U_{n,1}$$

for $r < n$. Such morphisms are compositions of morphisms corresponding to

$$\begin{array}{ccc} [n] \amalg [r] & & [r] \amalg [n] \\ {\scriptstyle \phi_1 \amalg \psi_1}\downarrow & \text{and} & \downarrow{\scriptstyle \phi_2 \amalg \psi_2} \\ [m_1] & & [m_2] \end{array}$$

where $[m_1]$ and $[m_2]$ are glued via the images of $[r]$, and that image is then removed, thus giving the map

$$\begin{array}{c} [n] \amalg [n] \\ \downarrow \\ [m_1 + m_2 - r], \end{array} \tag{2.21}$$

summed over all compatible orderings on $[m_1 + m_2 - r]$. Since $r < n$ and $m_1, m_2 \geq n$, we have $m_1 + m_2 - r > n$. Thus, interpreting (2.21) as an equivalence relation on $[n] \amalg [n]$ with ordered equivalence classes, at least one $i_1 \in [n]_1$ and one $i'_2 \in [n]_2$ (subscript indicating, again, which copy of $[n]$ in $[n] \amalg [n]$ we are considering) must be in a class by itself. After completing the ordering, for a generator x where $\iota \circ x \circ \iota \neq 0$, one must have $i = i'$, and the classes of $i_1 \in [n]_1$ and $i_2 \in [n]_2$ must be either equal or adjacent in the order. So, there are three choices of the ordering between the classes of i_1, i_2 (we shall denote by $<$, $=$, and $>$). Thus, the image of ι is generated by sums of ordered equivalence classes where for some $i \in [n]$, i_1 and i_2 are in these possible relations, i.e. they are equal, the equivalence class containing i_1 is less than and adjacent to the equivalence class containing i_2, or the equivalence class contianing i_1 is more than and adjacent to the equivalence class containing i_2, respectively.

Now we may write

$$End_{\mathscr{D}}(U_{n,1}) = \bigoplus_{S \subseteq [n]} \bigotimes_{i \in [n] \smallsetminus S} k\{<_i, =_i, >_i\}$$

the symbols $<_i$, $=_i$, and $>_i$ denote the three possible relationships between the equivalence class containing i_1 and the equivalence class containing i_2. We proved that the generators of $I_{n,1}$ always contained a sum $(<_i) + (=_i) + (>_i)$ tensored with the same factors. This implies that we can eliminate $>_i$ and in fact write

$$End_{\mathscr{D}}(U_{n,1}) = \bigoplus_{S \subseteq [n]} \bigotimes_{i \in [n] \smallsetminus S} k\{<_i, =_i\},$$

which corresponds to (2.20).

□

For a finite group G, consider the groupoid Γ of G acting on itself by translation. The *"Drinfeld double" of* G (note: terminology may vary) is $k[Mor(\Gamma)]$ where composition of morphisms is defined as composition when morphisms are composable and as 0 otherwise. Then the Drinfeld double is isomorphic to the matrix algebra $M_{|G|}(k)$ of $|G| \times |G|$ matrices with entries in k. Using this fact, the "top part" of $End_{\mathscr{D}}(U_n)$ is isomorphic to the tensor product of $M_{n!}(k)$ with the algebra of endomorphisms of U_n generated by elements (2.20), which is

$$\prod_n k[\mathbb{Z}/2, \cdot] = \prod_{2^n} k.$$

Thus, the top part of $End_{\mathscr{D}}(U_n)$ is isomorphic to

$$\prod_{2^n} M_{n!}(k).$$

The formulas (2.20) are simpler than the formulas in Subsection 5.1 of [8]. This is due to the fact that we are suppressing terms going through U_m for $m < n$.

2.6. **The interpolated category of representations of the general linear group of a finite field.** We now describe the category $\underline{Rep}(GL_t(\mathbb{F}_q))$ for $q = p^m$ for a prime p and any t not a non-negative integer (not all non-negative integers need to be excluded, see [7, 10, 11]) using T-algebras. Here the basic object can be thought of as interpolating the permutation representation of $GL_n(\mathbb{F}_q)$ on its vector representation $(\mathbb{F}_q)^n$.

For finite sets S, T, take the representation

$$\underline{Rep}(GL_t(\mathbb{F}_q))_{S,T} = Hom_{\underline{Rep}(GL_t(\mathbb{F}_q))}(X^{\otimes S}, X^{\otimes T})$$

to be the free k-module (where k is a field of characteristic 0) on the set of equivalence classes of vector space surjections

$$f : \mathbb{F}_q^{S \amalg T} = \mathbb{F}_q^S \times \mathbb{F}_q^T = \mathbb{F}_q\{x_i \mid i \in S\} \times \mathbb{F}_q\{y_j \mid j \in T\} \to V \tag{2.22}$$

where the equivalence relation is

$$f \sim g \circ f$$

for $g \in GL(V)$.

For $i \in S$, $j \in T$, the partial trace $\tau_{i,j}(f)$ is defined to be 0 if

$$f(x_i) \neq f(y_j). \tag{2.23}$$

If

$$f(x_i) = f(y_j), \tag{2.24}$$

to define the partial trace, let $f_{i,j}$ be the restriction of f to $\mathbb{F}_q^{S\smallsetminus\{i\}} \times \mathbb{F}_q^{T\smallsetminus\{j\}}$, considered as a map onto its image. Then in the case (2.24), the trace $\tau_{i,j}(f)$ is a multiple of $f_{i,j}$. The coefficient is 1 if

$$V_{i,j} = V. \tag{2.25}$$

If

$$dim(V_{i,j}) < dim(V),$$

then necessarily

$$dim(V_{i,j}) = dim(V) - 1. \tag{2.26}$$

In the case (2.26), if $t = N$ were a large integer (so we are working with $Rep(GL_N(\mathbb{F}_q)))$, the coefficient would be the number of choices how $f_{i,j}$ can arise in this fashion. Thus, the coefficient is

$$q^t - q^{n-1}. \tag{2.27}$$

The product of (2.22) with

$$f' : \mathbb{F}_q^{S'\amalg T'} \to V'$$

is again given by the sum of maps obtained by composing $f \oplus f'$ with a surjection

$$\mu : V \oplus V' \to W \tag{2.28}$$

which is injective on each of the summands in the source. Again, two choices of μ are considered equal if they are in the same orbit of the left action of $GL(W)$.

Verification of compatibility of partial traces with product in the case of (2.26) (which is the only non-trivial case of the axioms) again comes down to a polynomial identity in q^t which, in each degree, can be deduced from the case $t = N$ for a large integer N, where it follows from the fact that we just described the structure arising in the case of $Rep(GL_N(\mathbb{F}_q))$.

To prove semisimplicity, we again use the method of [3], Sections 3-5. The object U_n is defined to be the image of the idempotent on $End_{\underline{Rep}(GL_t(\mathbb{F}_q))}(X^{\otimes n})$ given by the codiagonal

$$\begin{array}{rl} \mathbb{F}_q^n \times \mathbb{F}_q^n & \xrightarrow{\nabla} \mathbb{F}_q^n \\ (x, y) & \mapsto x + y \end{array} \tag{2.29}$$

The analogue of Lemma 5.2 says that the top part is $k[GL_n(\mathbb{F}_q)]$, so $End_{\underline{Rep}(GL_t(\mathbb{F}_q))}(U_n)$ is semisimple because k is of characteristic 0.

2.7. **The Borel quantum Delannoy category $\mathscr{B}_{q,k}$.** We shall now define the first main new example of the presen paper, the Borel quantum Delannoy category $\mathscr{B}_{q,k}$ by combining, in a sense, the above constructions of $\underline{Rep}(GL_t(\mathbb{F}_q))$ and the Delannoy category.

We shall define $\mathscr{B}_{q,k}$ by taking the spaces $\mathscr{B}_{S,T}$ to be the free k-modules on vector space surjections

$$f : \mathbb{F}_q^{S \amalg T} \to V \tag{2.30}$$

and choices of a maximal flag

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{dim(V)} = V \tag{2.31}$$

(i.e. where for each $k = 1, \ldots, dim(V)$,

$$dim(V_k) - dim(V_{k-1}) = 1).$$

We identify the quotient maps (2.30) under the action of the Borel subgroup on the target.

We define $\tau_{i,j}(f)$ again to be 0 in the case of (2.23). In the case of (2.24), we define $\tau_{i,j}(f)$ to be a multiple of $f_{i,j}$ (with the induced flag on the image in the case of (2.26)). The coefficient is 1 in the case of (2.25). In the case of (2.26), the coefficient should be (2.27) divided by the number of ways the restricted flag could arise, which is

$$\frac{q^t - q^{n-1}}{1 + q + \cdots + q^{n-1}}. \tag{2.32}$$

However, again, we will only get compatibility with products when (2.32) is independent of n, which occurs for

$$t = -1,$$

in which case the coefficient (2.32) becomes

$$q^{-1} - 1. \tag{2.33}$$

The product of (2.30) with

$$f' : \mathbb{F}_q^{S' \amalg T'} \to V' \tag{2.34}$$

with a maximal flag

$$0 = V_0' \subset V_1' \subset \cdots \subset V_{dim(V')}' = V' \tag{2.35}$$

is obtained by choosing an ordered surjection (2.10) (taking $\ell = dim(V)$, $\ell' = dim(V')$) and a surjection (2.28) which sends the V_i to $W_{\kappa(i)}$ and

V_j' to $W_{\kappa(j)}$. The product is then defined as a sum over all such choices together with a choice of a maximal flag on W compatible with the flags (2.31), (2.35).

To verify the compatibility of trace and product, the non-trivial case again is (2.26). Consider a case of (2.30) where (2.26) arises for a partial trace $\tau_{i,j}$, and consider a product with (2.34). Performing $\tau_{i,j}$ on (2.30) first and then taking the product with (2.34), consider again separately each case of $\kappa|_{\{1,\dots,\ell-1\}}$. Let $r = \kappa(s-1)$, $r' = \kappa(s)$. Then, again, in the case where we take the product first, there are $2(r'-r)-1$ choices of $\kappa(s)$. Note that s becomes $s+1$ and r may stay the same or increase by one depending on whether $\kappa(s)$ is equal to $\kappa(u)$ for $u \in \{1,\dots,\ell'\}$ or not. There are still $r'-r-1$ of the former cases ("clashes") and $r'-r$ of the latter cases ("non-clashes").

In the clashing choice preceding a non-clashing choice, the number of choices of flags will be multiplied by

$$\frac{q-1}{q}.$$

Thus, the coefficient of the two choices (a clashing one preceding a non-clashing one) will be equal with opposite signs (see (2.33)). Hence, after tracing out the pair corresponding to the original (i,j)-pair after performing the product, all of the choices of $\kappa(s)$ will again cancel out except the lowest choice, which is non-clashing and corresponds to what we get if we do $\tau_{i,j}$ first, followed by the product. Another proof of this fact will folow from the description of $\mathscr{B}_{q,k}$ in terms of measures on oligomorphic groups [7], which we shall discuss in Section 3, and which will include a proof of the semisimplity of the category $\mathscr{B}_{q,k}$. A different proof of semisimplicity, following the methods of [3], will be given in the Appendix.

2.8. **The unipotent quantum Delannoy category $\mathscr{U}_{q,k}$.** We now define our second main example, which is a variant of the above category $\mathscr{B}_{q,k}$, replacing, in a sense, the maximal Borel subgroup of the general linear group by a maximal unipotent subgroup.

We denote this category by $\mathscr{U}_{q,k}$ (again, we omit the q,k subscript if q and k are fixed). Its T-algebra is defined as follows:

For finite sets S, T, we take the space $\mathscr{U}_{S,T}$ to be the free k-vector space on equivalence classes of the data of a quotient map

$$f : \mathbb{F}_q^{S \amalg T} \twoheadrightarrow V$$

and a choice of ordered basis

$$(v_1, v_2, \dots v_{dim(V)})$$

generating V, over the equivalence relation that

$$(f, (v_1, \dots, v_{dim(V)})) \sim (g \circ f, (g(v_1), \dots, g(v_{dim(V)})))$$

for any isomorphism

$$g : V \to V$$

which is "unipotent" in $GL(V)$, i.e. such that we can express

$$g(v_i) = v_i + a_{i,i-1} \cdot v_{i-1} + \dots + a_{i,1} \cdot v_i$$

for some coefficients $a_{i,j} \in \mathbb{F}_q$.

The partial trace is defined by taking

$$\tau_{i,j}(f, (v_1, \dots, v_{dim(V)}))$$

for $i \in S$, $j \in T$ to be

(1) 0 if $f(x_i) \neq f(y_j)$ (denoting the free generators of $\mathbb{F}_q^{S \amalg T}$ corresponding to the elements of S and T by x_s and y_t, respectively, for $s \in S$, $t \in T$).

(2)
$$f_{i,j} := f|_{\mathbb{F}_q^{S \smallsetminus \{i\}} \times \mathbb{F}_q^{T \smallsetminus \{j\}}} : \mathbb{F}_q^{S \smallsetminus \{i\}} \times \mathbb{F}_q^{T \smallsetminus \{j\}} \twoheadrightarrow Im(f_{i,j})$$
if $dim(V_{i,j}) = dim(V)$, with the same basis as V.

(3) an $-q^{-1} \cdot f_{i,j}$ is $dim(V_{i,j}) < dim(V_{i,j})$, with the induced basis on $V_{i,j}$.

Product is defined analogously as for $\mathscr{B}$, and the compatibility of trace and product and semisimplicity in $\mathscr{U}$ proceed similarly as above.

In the unipotent case, analogously to formula (5.4) of the Appendix, if we denote by Θ_n^0 the algebra of endomorphisms of the image of the idempotent

$$(\nabla : \mathbb{F}_q^n \times \mathbb{F}_q^n \twoheadrightarrow \mathbb{F}_q^n, \ (e_1, \dots, e_n))$$

(for $(e_1, \dots, e_n)$ denoting the standard basis of $\mathbb{F}_q^n$), we obtain

$$dim(\Theta_n^0) = q^{\binom{n}{2}} \, (q+1)^n. \tag{2.36}$$

In particular, we have

Theorem 2.9. *Let $q = p^m$ for a prime p and k be a field of characteristic not p. Then the category $\mathscr{U}_{q,k}$ is semisimple.*

□

3. Quantum Delannoy Categories via Oligomorphic Groups

In this section, we give alternate descriptions of $\mathscr{B}_{q,k}$ and $\mathscr{U}_{q,k}$ using the theory of oligomorphic groups developed by N. Harman and A. Snowden in [7]. Recall that a permutation Γ acting on a set A is called *oligomorphic* where each Cartesian power A^n (with the diagonal action) is a union of finitely many Γ-orbits.

3.1. **The oligomorphic group for $\mathscr{B}_{q,k}$.** Let us consider the free $\mathbb{F}_q$-vector space $V = \mathbb{F}_q[\mathbb{R}]$ (i.e. a sum of copies of $\mathbb{F}_q$, indexed by $s \in \mathbb{R}$), and let us put

$$V^* = V \smallsetminus \{0\}.$$

The vector space V has a natural $\mathbb{R}$-indexed filtration given by

$$F_r(V) = \mathbb{F}_q\{(s) \mid s \leq r \in \mathbb{R}\}. \tag{3.1}$$

We may consider the group

$$\Gamma = B \rtimes Aut(\mathbb{R}, <)$$

where $Aut(\mathbb{R}, <)$ denotes the group of order-preserving bijections of $\mathbb{R}$ and $B \subset GL(V)$ denotes the subgroup of all linear isomorphisms from V to itself preserving the filtration (i.e. Γ is the group of automorphisms of V which preserve $F_s(V)$ for all $s \in \mathbb{R}$). $Aut(\mathbb{R}, <)$ acts on B by ordered permutation of the basis elements, which preserves the flag. One then sees that Γ acts on V^*, and in fact, forms an oligomorphic group. To obtain a semisimple pre-Tannakian category, we construct a measure in the sense of [7]. Specifically, we define the measure of an orbit of Γ consisting of tuples of vectors generating an n-dimensional vector subspace of V to be

$$(q^{-1} - 1)^n. \tag{3.2}$$

Write $\mu(\Gamma/H)$ for the measure of such an orbit Γ/H. An attractive feature of this approach is that N. Harman and A. Snowden [7], Theorem 13.2, prove that the condition that

$$\mu(\Gamma/H) \neq 0 \in k$$

for all open subgroups $H \subseteq \Gamma$ implies semisimplicity of $\mathscr{B}_{q,k}$ under the condition (2.1).

Now recall that an open subgroup (in the sense of [7]) of Γ is one which contains the stabilizer of some finite sequence of elements of V^*.

However, we only specified measures of orbits Γ/H when there exists a finite-dimensional subspaces $W \subset V$ such that

$$H = H_W = \{g \in \Gamma \mid \forall w \in W,\ g(w) = w\}. \tag{3.3}$$

Denote the class of such subgroup $\mathcal{E}$. Now the difficulty is that for an open subgroup $K \subseteq \Gamma$, there may not exist an $H \in \mathcal{E}$ with

$$[K : H] < \infty.$$

For example, consider the subgroup

$$K = \{g \in \Gamma \mid g(F_0(V)) = F_0(V)\}.$$

This subgroup is open since $K \supset H_{\langle(0)\rangle}$, but has no subgroup H_W of finite index.

N. Harman and A. Snowden give a method for dealing with this situation by working *relative* to a class of open subgroups of Γ. The difficulty of working with the class $\mathcal{E}$ directly is that in the relative case, the analogue of Theorem 13.2, guaranteeing semisimplicity, requires a technical condition on the class $\mathcal{E}$ (Condition $(*)$ of Remark 5.5 of [7]), which asserts that the stabilizers of finite subsets of an orbit Γ/H with $H \in \mathcal{E}$ be in $\mathcal{E}$. This is false for the class $\mathcal{E}$ defined by (3.3). We can solve this issue by passing to the class $\overline{\mathcal{E}}$ consisting of subgroups of Γ of the form $H_{W,K}$ where for $W \subset V$, $K \subseteq B(W)$ (where $B(W) \subseteq GL(W)$ is the set of elements preserving the induced flag on W), we set

$$H_{W,K} = \{\gamma \in \Gamma \mid \gamma|_W \in K\}.$$

By assumption (2.1), for all choices of W, K, the index $[H_{W,K} : H_W] \in k^\times$, so we can put

$$\mu(\Gamma/H_{W,K}) = \frac{\mu(\Gamma/H_W)}{[H_{W,K} : H_W]}.$$

The axioms of [7] in the case of field-valued measures which are non-zero on orbits require that for every pullback diagram

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z, \end{array} \tag{3.4}$$

where X and Y are finite disjoint unions of Γ-orbits with stabilizers in $\mathcal{E}$, the maps (3.4) preserve Γ-action, and Z is a single Γ-orbit, that

$$\mu(X \times_Z Y) = \frac{\mu(X)\mu(Y)}{\mu(Z)}. \tag{3.5}$$

3.2. **A key Lemma.** To prove (3.5), one notes that there is an equivalence from the category whose objects are pairs of finite dimensional vector spaces A with a choice of a complete flag F_A and where a morphism

$$(A, F_A) \to (B, F_B)$$

is a pair of an inclusion $f : A \hookrightarrow B$ such that intersecting the flag F_B with A gives F_A to the opposite category of Γ-orbits where all stabilizers are in $\mathcal{E}$. This equivalence of categories is the functor defined by taking an object (A, F_A) to the Γ-orbit $O(A)$ of inclusions

$$\iota : A \to V$$

where the flag F_A is induced by the filtration (3.1), with Γ acting by composition (for any element ι, the stabilizer is then the subgroup of Γ fixing $\iota(A)$, forming an element of the class $\mathcal{E}$). This functor takes a morphism

$$f : (A, F_A) \to (B, F_B)$$

to a map of Γ-sets

$$O(B) \to O(A)$$

given by precomposing an element of $O(B)$ with f.

Given this equivalence, a fiber product $O(B) \times_{O(A)} O(B')$ (as in (3.4)) can be expressed as the Γ-set given as the disjoint union of $O(C)$ for all choices of (C, F_C) which form a gluing, meaning a diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ B' & \longrightarrow & C \end{array} \tag{3.6}$$

where each arrow is an injection and the flag on the target induces the flag on the source, and the maps from B, B' to C are jointly surjective.

Thus, to show (3.5), it suffices to show the following

Lemma 3.3. *For finite dimensional vector spaces A, B, and B' with flags*

$$F_A : \ 0 = A_0 \subset A_1 \subset \cdots \subset A_{\ell-1} \subset A_\ell = A$$

$$F_B : \ 0 = B_0 \subset B_1 \subset \cdots \subset B_{n-1} \subset B_n = B$$

$$F_{B'} : \ 0 = B'_0 \subset B'_1 \subset \cdots \subset B'_{m-1} \subset B'_m = B'$$

and flag-preserving inclusions

$$B \hookleftarrow A \hookrightarrow B', \tag{3.7}$$

letting $G_{B\hookleftarrow A\hookrightarrow B'}$ denote the set of (C, F_C) which form a gluing of (3.7) as in (3.6), the following formula holds:

$$(q^{-1}-1)^{dim(B)+dim(B')-dim(A)} = \sum_{(C,F_C)\in G_{B\hookleftarrow A\hookrightarrow B'}} (q^{-1}-1)^{dim(C)} \tag{3.8}$$

([5]*).*

We may rewrite the formula (3.8) as

$$\sum_{(C,F_C)\in G_{B\hookleftarrow A\hookrightarrow B'}} (\frac{q}{1-q})^{dim(B)+dim(B')-dim(A)-dim(C)} = 1. \tag{3.9}$$

Proof. (following P. Deligne [5]) By induction on the numbers n, m, it suffices to prove Lemma 3.3 when

$$dim(B) = dim(B') = dim(A) + 1.$$

It is beneficial to rewrite the statement in basis notation. Note that a finite totally ordered set $(S, \leq)$ corresponds to the finite dimensional vector space $\mathbb{F}_q^S$, with the S-indexed filtration given by, for $s \in S$,

$$F_s(\mathbb{F}_q^S) = \mathbb{F}_q^{\{t\in S | t\leq s\}}.$$

An order-preserving injection

$$(S, \leq) \to (T, \leq)$$

induces a morphism

$$\mathbb{F}_q^S \hookrightarrow \mathbb{F}_q^T \tag{3.10}$$

preserving these filtrations. In fact, for every finite dimensional vector spaces A and B with complete flags F_A and F_B (resp.), every flag-preserving inclusion

$$A \hookrightarrow B \tag{3.11}$$

will be the same, up to isomorphism, as an order-preserving map (3.10). Since any flag-preserving automorphism on A in (3.11) will extend to a flag-preserving morphism on B, any diagram (3.7) is isomorphic to one coming from a diagram of order-preserving injections

$$\begin{array}{ccc} (T, \leq) & & \\ \uparrow & & \\ (S, \leq) & \longrightarrow & (T', \leq). \end{array} \tag{3.12}$$

It suffices, then, to prove (3.8) for a diagram (3.7) arising from a diagram (3.12) for sets S, T, T' with

$$|T| = |T'| = |S| + 1.$$

Without loss of generality, $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$, with the standard ordering. Note that we may then write

$$T = S \amalg \{t\}$$

$$T' = S \amalg \{t'\}$$

where t is inserted in the ith place, i.e. $i - 1 < t < i$ in the total ordering of T, and t' is inserted in the jth place, i.e. $j - 1 < t' < j$ in the total ordering of T'.

Case 1: $i \neq j$. It suffices to prove that there is only a single gluing which is

$$\begin{array}{ccc} (T, \leq) & \longrightarrow & (R, \leq) \\ \uparrow & & \uparrow \\ (S, \leq) & \longrightarrow & (T', \leq) \end{array}$$

of (3.12), with

$$R = S \amalg \{t\} \amalg \{t'\} \tag{3.13}$$

inserting t and t' so that $i - 1 < t < i$, $j - 1 < t < j$, since then

$$|R| = |T| + |T'| - |S|.$$

For a gluing diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \uparrow & & \uparrow \\ A & \longrightarrow & B' \end{array} \tag{3.14}$$

of flag-preserving inclusions of (3.7), we must have $C = B \oplus_A B'$ (since, otherwise, $C \cong B \cong B'$, and the flag-preserving inclusions

$$A \hookrightarrow B$$

$$A \hookrightarrow B'$$

would be isomorphic, which contradicts the assumption of this case).

Without loss of generality, let us assume that $i < j$. We can choose an ordered basis

$$e_1, \ldots, e_i, \ldots, e_{j+1}, \ldots, e_{n+2} \tag{3.15}$$

of $B \oplus_A B'$ such that for $0 \leq k \leq n + 2$, the first k elements

$$A = \langle e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_{j+1}}, \ldots, e_{n+2} \rangle$$

$$B = \langle e_1, \ldots, \widehat{e_{j+1}}, \ldots, e_{n+2} \rangle$$

$$B' = \langle e_1, \ldots, \widehat{e_i}, \ldots, e_{n+2} \rangle$$

To prove that the only case we need to consider is (3.13), it suffices to show that the only complete flag

$$0 = F_0(B \oplus_A B') \subset \cdots \subset F_{n+2}(B \oplus_A B') = B \oplus_A B' \tag{3.16}$$

inducing the given flags on B and B' is the one given by the ordered basis (3.15). For a flag (3.16), we must have

$$F_1(B \oplus_A B') \nsubseteq B \text{ or } F_1(B \oplus_A B') \nsubseteq B' \tag{3.17}$$

(both are not possible, by dimension). Without loss of generality, $i = 1$. The possibilities (3.17) then imply that

$$F_k(B \oplus_A B') = F_1(B \oplus_A B') \oplus F_{k-1}(B) \tag{3.18}$$

or

$$F_k(B \oplus_A B') = F_1(B \oplus_A B') \oplus F_{k-1}(B'), \tag{3.19}$$

respectively. We have

$$B = \langle e_1, \ldots, \widehat{e_j}, \ldots, e_{n+2}\rangle$$

$$B' = \langle e_2, \ldots, e_{n+2}\rangle.$$

If $F_1(B \oplus_A B') \subseteq B$, then $F_1 = \langle e_1\rangle$ and $F_1(B \oplus_A B') \nsubseteq B'$, so

$$F_k(B \oplus_A B') = F_1(B \oplus_A B') \oplus F_{k-1}(B') = \langle e_1\rangle \oplus \langle e_2, \ldots, e_k\rangle,$$

giving the claim. If $F_1(B \oplus_A B') \nsubseteq B$, then $F_1(B \oplus_A B') \subseteq B'$, so

$$F_1(B \oplus_A B') = \langle e_2\rangle,$$

which is impossible since then it can not induce the given filtration on B.

Case 2: $i = j$. Let us choose an ordered basis for B

$$e_1, \ldots, e_{n+1}$$

such that

$$F_k(B) = \langle e_1, \ldots, e_k\rangle$$

and

$$A = \langle e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\rangle$$

Gluings of B and B' along A will be of dimension $n+1$ or $n+2$. For a gluing of dimension $n+1$, the exponent of $\frac{q}{1-q}$ in the corresponding summand of (3.9) is 1. Each such gluing corresponds to a flag-preserving automorphism

$$B \to B$$

which is identity when restricted to A, and must therefore send e_k to itself for all $k \neq i$ and send e_i to a linear combination

$$a_1 \cdot e_1 + \cdots + a_i \cdot e_i$$

for coefficients $a_1, \ldots, a_{i-1} \in \mathbb{F}_q$, $a_i \in \mathbb{F}_q^\times$. Thus, there are $q^{i-1}(q-1)$ choices of gluings of dimension $n+1$, contributing

$$\frac{q}{1-q} \cdot (q^{i-1}(q-1)) = -q^i. \tag{3.20}$$

For the gluings of dimension $n+2$, again a gluing diagram (3.14) will have $C = B \oplus_A B'$, and we can choose ordered bases

$$B = \langle e_1, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_{n+1} \rangle$$

$$B' = \langle e_1, \ldots, e_{i-1}, e_i', e_{i+1}, \ldots, e_{n+1} \rangle$$

$$A = \langle e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1} \rangle,$$

(inducing the given flags on A, B, B'). Gluings C then are given by a choice of complete flag inducing the given flags on B and B', each of which will contribute a term of $(\frac{q}{1-q})^0 = 1$ in (3.9).

First note that the number of choices of such flags

$$0 = F_0(B \oplus_A B') \subset \cdots \subset F_{n+2}(B \oplus_A B') = B \oplus_A B' \tag{3.21}$$

depends only on i. There must be exactly one $0 \leq k_0 \leq n+1$ such that

$$F_{k_0}(B \oplus_A B') \cap B = F_{k_0+1}(B \oplus_A B') \cap B \tag{3.22}$$

(and similarly, there exists exactly one $0 \leq k_0' \leq n+1$ for which this holds with B replaced by B'). Thus, for all $k > i$, by (3.22),

$$e_1, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_{k-1} \in F_k(B \oplus_A B')$$

and, by the analogue for B',

$$e_1, \ldots, e_{i-1}, e_i', e_{i+1}, \ldots, e_{k-1} \in F_k(B \oplus_A B').$$

Thus, by dimension,

$$\langle e_1, \ldots, e_{i-1}, e_i, e_i', e_{i+1}, \ldots, e_{j-1} \rangle = F_j(B \oplus_A B').$$

For $k \leq i$,

$$F_k(B \oplus_A B') \subseteq F_i(B \oplus_A B'),$$

and hence the data of the flag (3.21) is in bijective correspondence with a flag on

$$\langle e_1, \ldots, e_{i-1}, e_i, e_{i'} \rangle \tag{3.23}$$

giving the induced flags on

$$\langle e_1, \ldots, e_{i-1}, e_i \rangle \tag{3.24}$$

$$\langle e_1, \ldots, e_{i-1}, e_i' \rangle \tag{3.25}$$

from the flags on B and B'.

In fact, the number of such flags on (3.23) is

$$q^{i-1}(q-1) + q^{i-2}(q-1) + \cdots + q(q-1) + (q+1) \tag{3.26}$$

(interpreted to be $q+1$ if $i = 1$), which we prove by induction on i:

Without loss of generality, by the above argument, let us suppose

$$A = \langle e_1, \ldots, e_{i-1} \rangle$$

and B and B' are (3.24) and (3.25), respectively. For $i = 1$, $dim(A) = 0$ and $dim(B) = dim(B') = 1$, so any complete flag on $B \oplus B'$, of which there are $q+1$, will work.

For $i > 1$, if $F_1(B \oplus_A B') \nsubseteq B, B'$, then there exists a vector v of the form

$$a_1 \cdot e_1 + \cdots + a_i \cdot e_i + e_i' \tag{3.27}$$

with $a_1, \ldots, a_{i-1} \in \mathbb{F}_q$, $a_i \in \mathbb{F}_q^\times$, with

$$F_1(B \oplus_A B') = \langle v \rangle$$

and for $k \geq 1$

$$F_k(B \oplus_A B') = \langle v, e_1, \ldots, e_{k-1} \rangle.$$

There are $q^{i-1}(q-1)$ choices of (3.27).

If $F_1(B \oplus_A B')$ is in B or B', then

$$F_1(B \oplus_A B') = \langle e_1 \rangle \subseteq B, B'.$$

Quotienting out $F_1(B \oplus_A B')$ then gives a flag on

$$\langle e_2, \ldots, e_{i-1}, e_i, e_{i'} \rangle$$

of which the number of choices is

$$q^{i-2}(q-1) + q^{i-3}(q-1) + \cdots + q(q-1) + (q+1)$$

by the induction hypothesis. Summing these two cases together, we get (3.26).

Summing (3.20) with (3.26) gives that the left hand side of (3.9) is

$$-q^i + q^{i-1}(q-1) + q^{i-2}(q-1) + \cdots + q(q-1) + (q+1) = 1.$$

□

3.4. **A Matrix Description of Orbits of Γ and the Unipotent Analogue.** The orbits of $(V^*)^n$ with respect to the action of Γ can be described as equivalence classes of matrices with n non-zero columns with entries in $\mathbb{F}_q$ under the equivalence relation $\sim_b$ generated by operations

(1) Addition of an $\mathbb{F}_q$-multiple of the ith row to the jth row for $i < j$
(2) Omission of zero rows
(3) Multiplication of a row by an element of $\mathbb{F}_q^\times$

Equivalence classes have unique representatives in *semi-echelon form* which means that the matrix contains no zero rows and if we call the left-most non-zero entry of each row a *pivot*, then we require that

(1) All entries below a pivot are 0
(2) Every pivot is equal to 1

The measure of an orbit corresponding to a matrix in semi-echelon form is $(q^{-1}-1)^\ell$ where ℓ is the number of pivots.

A variant of the construction where we use V instead of V^* for the oligomorphic group action gives the same characterization of orbits by matrices except that we allow matrices with 0 columns.

Symmetric group action on orbits is given by permuting the columns of a matrix and putting the resulting matrix in semi-echelon form. Product of a matrix with rows $\{1,\dots,i\}$ and a matrix with rows $\{1,\dots,j\}$ is a sum indexed by "ordered gluings of rows," i.e. surjections

$$\phi : \{1,\dots,i\} \amalg \{1,\dots,j\} \to \{1,\dots i+j-h\}$$

which are order-preserving injections on each disjoint summand. We arrange the matrices side by side while moving their rows according to ϕ. If follows that the resulting matrix is in semi-echelon form.

Partial trace of a matrix M with respect to the ith and jth columns (without loss of generality, assume $i < j$) is defined by summing over all matrices M' that are equivalent to M where the ith and jth columns coincide (this does not depend on the representative) such that after deleting the jth column, M' is in semi-echelon form. The corresponding summand to M' is the matrix in semi-echelon form which is equivalent to M' after deleting the ith and jth columns, with coefficient equal to the sum of measures of the orbits of the omitted columns.

To construct the category $\mathscr{U}_{q,k}$, let us first note that $B \subset GL(V)$ also can be replaced by a "finitary" variant

$$B_{fin} = \bigcup_{S \subset \mathbb{R},\ |S| < \infty} B_S$$

where $B_S \subseteq GL(\mathbb{F}_q[S])$ is the Borel subgroup of lower triangular matrices. The resulting group

$$\Gamma_{fin} = B_{fin} \rtimes Aut(\mathbb{R}, <)$$

is also oligomorphic and in fact the Γ_{fin}-orbits of $(V^*)^n$ (or V^n) are the same as the orbits of Γ.

The advantage of the "finitary" approach is that it also has a *unipotent version*

$$U_{fin} = \bigcup_{S \subset \mathbb{R},\ |S| < \infty} U_S$$

where $U_S \subseteq GL(\mathbb{F}_q[S])$ is the maximal unipotent subgroup of the Borel subgroup B_S. One can then from the group

$$\Gamma_u = U_{fin} \rtimes Aut(\mathbb{R}, <),$$

which also acts oligomorphically on V and V^*.

Let us now describe the orbits of $(V^*)^n$ with respect to Γ_u and a non-zero measure, which gives the semisimple pre-Tannakian category $\mathscr{U}_{q,k}$.

The description of the orbits is by an equivalence of matrices, which we denote by $\sim_u$, analogous to that for Γ except that we drop the operation (3). Accordingly, the representatives of equivalence classes are described by matrices in *weak semi-echelon form*, which satisfy the conditions (1), (2) (dropping condition (3)).

The symmetric group action on orbits and product can be described directly analogously as for $\mathscr{B}_{q,k}$. Again, we take the partial trace of a matrix M with respect to the ith and jth columns (without loss of generality, assume $i < j$) to be the sum over all matrices M' equivalent (now only using operations (1) and (2)) to M where the ith and jth columns coincide (this does not depend on the representative) such that after deleting the jth column, the matrix is in weak semi-echelon form. The corresponding summand in the partial trace of M is the matrix in weak semi-echelon form which is equivalent to M' with the ith and jth columns deleted, with coefficient equal to the sum of measures of the orbits of the omitted columns.

The measure of an orbit corresponding to a matrix in weak semi-echelon form with ℓ pivots is defined to be $(-q)^{-\ell}$. This is a measure in the sense of [7] relative to the class of subgroups $\mathcal{E}_u$ consisting, for finite subsets $S \subset \mathbb{R}$, of subgroups $G \subset \Gamma_n$ which send

$$\mathbb{F}_q[S] \to \mathbb{F}_q[S]$$

by a transformation in U_S.

It then suffices to prove multiplicativity of the measure on orbits given by stabilizers of finite sequences. The proof is analogous as for Γ, with the product summand given by a term containing a clash canceling with the summand obtained by shifting the lowest clashing term of the first factor to a neighboring non-clashing term to the right. This again leaves only one summand whose measure is the required product by definition.

Since the measures of Γ_u/H with $H \in \mathcal{E}_u$ have denominators given by powers of q, the category $\mathscr{U}_{q,k}$ is defined and semisimple over any field k of characteristic $\neq p$.

4. Structural Results

In this section, we will investigate the structure of the categories $\mathscr{B}_{q,k}$, $\mathscr{U}_{q,k}$. First, we study the decompositions of the "basic objects" into simple summands, which is surprisingly subtle. Also, it is helpful to consider further variants of the basic object. Eventually, it is possible to use the same method to actually characterize all the isomorphism classes of the simple objects of $\mathscr{U}_{q,k}$.

In Subsection 4.8, we discuss comparison functors between the various categories considered. In particular, we construct a functor

$$\mathscr{D} \to \mathscr{B}_{q,k}$$

from a remarkable universal property of the Delannoy category, which we prove.

4.1. **The Decomposition of the Basic Object of $\mathscr{B}_{q,k}$ into Simple Objects.** Let k be a field satisfying (2.1) and containing $(q-1)$th roots of unity. In this section, we shall give the semisimple decompositions of the basic objects of $\mathscr{B}_{q,k}$ (Proposition 4.2 below).

We have $q-1$ multiplicative characters

$$\psi : \mathbb{F}_q^\times \to k^\times.$$

Let $\Omega = [V^*] \in Obj(\mathscr{B}_{q,k})$ in the notation of [7]. Then the action of $\mathbb{F}_q^\times$ on V^* by multiplication induces a decomposition

$$\Omega = \bigoplus_\psi \Omega_\psi$$

into pieces on which $\mathbb{F}_q^\times$ acts by ψ. The corresponding idempotent on Ω is

$$\iota_\psi = \frac{1}{q-1} \sum_{\psi \in \mathbb{F}_q^\times} \psi(a) \cdot (a), \tag{4.1}$$

where (a) denotes the action of $a \in \mathbb{F}_q^\times$ by multiplication. The action of $a \neq 1 \in \mathbb{F}_q^\times$ has trace 0, and the trace of 1 is

$$dim(\Omega) = (q-1)\left(-\frac{1}{q}\right).$$

Hence,

$$tr(\iota_\psi) = -\frac{1}{q},$$

The orbits of $(V^*)^2$ have the following representatives in semi-echelon form:

$$\begin{aligned} (>) &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (<) &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ (a) &:= \begin{pmatrix} 1 & a \end{pmatrix} \\ (a)^\sim &:= \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{4.2}$$

where $a \in \mathbb{F}_q^\times$. One has

$$(<)(a) = (a)(<) = (<)$$

$$(>)(a) = (a)(>) = (>).$$

It follows that $(>)$, $(>)$ map Ω_1 to itself and acts by 0 on Ω_ψ, for all non-trivial characters $\psi \neq 1$. We have

$$(a)^\sim(b) = (b)(a)^\sim = (a \cdot b)^\sim. \tag{4.3}$$

It follows that $(a)^\sim$ maps Ω_ψ to itself and in Ω_ψ,

$$(a)^\sim = \psi(a) \cdot (1)^\sim.$$

One concludes that

$$Hom(\Omega_\psi, \Omega_\phi) = 0 \text{ for } \psi \neq \phi$$

$$(4.4) \qquad Hom(\Omega_\psi, \Omega_\psi) = \begin{cases} \langle Id_{\Omega_\psi}, (1)^\sim|_{\Omega_\psi}\rangle & \text{if } \psi \neq 1 \\ \langle Id_{\Omega_\psi}, (<), (>), (1)^\sim|_{\Omega_1}\rangle & \text{if } \psi = 1. \end{cases}$$

We next calculate the coefficients $s, t \in k$ such that

$$((1)^\sim)^2 = s \cdot (1)^\sim + t \cdot Id.$$

To calculate t, we note that

$$(4.5) \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so the fiber above $(v, v) \in V^* \times V^*$ in $(V^*)^3$ is isomorphic to V^* and hence has measure $q^{-1} - 1$, so

$$t = q^{-1} - 1.$$

To calculate s, one notes that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 1 & x \\ 0 & 1 \end{pmatrix}, \text{ for } x \in \mathbb{F}_q$$

which gives q orbits of measure $q^{-1} - 1$ in $(V^*)^3$ above a representative with semi-echelon form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and also

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which gives 1 additional orbit of measure $q^{-1} - 1$ in $(V^*)^3$ over a representative with semi-echelon form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Also, using

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim_b \begin{pmatrix} 1 & 1 \\ 1 & x \end{pmatrix} \quad \text{for } x \neq 1, 0$$

(we exclude 0 to avoid the case (4.5)) gives additional $q - 2$ orbits of measure 1 in $(V^*)^3$ over a representative with semi-echelon form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus,

$$s = (q + 1)(q^{-1} - 1) + (q - 2) = q^{-1} - 2,$$

and hence,

$$(4.6) \qquad ((1)^\sim)^2 = (q^{-1} - 2) \cdot (1)^\sim + (q^{-1} - 1) \cdot Id.$$

Now solving the equation (4.6), we get roots $(1)^{\sim} = -1, q^{-1} - 1$. Thus, 1 decomposes into idempotent multiples of

$$(1)^{\sim} + 1,\ (1)^{\sim} + 1 - q^{-1}.$$

Noting that

$$1 = q \cdot ((1)^{\sim} + 1) - q \cdot ((1)^{\sim} + 1 - q^{-1}),$$

we get that the idempotents are

$$\iota^{+} := q((1)^{\sim} + 1),\ \iota^{-} := -q((1)^{\sim} + 1 - q^{-1}), \tag{4.7}$$

giving a decomposition

$$\Omega_{\psi}^{+} \oplus \Omega_{\psi}^{-} = \Omega_{\psi}$$

for objects Ω_{ψ}^{+}, Ω_{ψ}^{-}, of dimensions -1, $1-q^{-1}$, respectively, corresponding to the idempotents

$$\iota_{\psi}^{+} := \iota_{\psi} \circ \iota^{+} = \iota^{+} \circ \iota_{\psi}$$

$$\iota_{\psi}^{-} := \iota_{\psi} \circ \iota^{-} = \iota^{-} \circ \iota_{\psi}$$

(recalling that ι^{+} and ι^{-} commute with ι_{ψ} since $(1)^{\sim}$ does, by (4.3)).

It follows from (4.4) that Ω_{ψ}^{+}, Ω_{ψ}^{-} are simple for $\psi \neq 1$. For $\psi = 1$, we additionally have the idempotents

$$\overline{e}_0 = \frac{(<)}{q^{-1} - 1},\ \overline{e}_{\infty} = \frac{(>)}{q^{-1} - 1}$$

of trace 0. One notes that $\overline{e}_0$, $\overline{e}_{\infty}$ commute, have trace 0, and

$$tr(\overline{e}_0\, \overline{e}_{\infty}) = 1.$$

Thus, the idempotents

$$e_0 := \overline{e}_0 - \overline{e}_0\, \overline{e}_{\infty},\quad e_{\infty} := \overline{e}_{\infty} - \overline{e}_0\, \overline{e}_{\infty},\quad \text{and}\quad \overline{e}_0\, \overline{e}_{\infty}$$

are disjoint and give a decomposition

$$\Omega_1^{+} \cong \Omega_{1,0}^{+} \oplus \Omega_{1,\infty}^{+} \oplus 1$$

where the summands correspond to the idempotents respectively and are of dimensions -1, -1, and 1.

By the description (4.2) of the orbits of $(V^*)^2$, we have

$$dim(End_{\mathscr{B}_{q,k}}(\Omega) = 2q,$$

and thus Ω_{ψ}^{+}, Ω_{ψ}^{-}, Ω_1^{-}, $\Omega_{1,0}^{+}$, $\Omega_{1,\infty}^{+}$ and 1 are all simple and non-isomorphic objects.

In fact, we have

$$[V] = \Omega \oplus (0)$$

and we know that

$$dim(End_{\mathscr{B}_{q,k}}([V]) = 2q + 3.$$

By counting dimensions of the simple summands, one can therefore deduce

$$1 \cong (0)$$

(which can also be checked directly by noting that

$$(0,0),\ (1,0),\ (0,1),\ (1,0)(0,1)$$

form a two-by-two matrix algebra).

Thus, we have proved the following

Proposition 4.2. *The decomposition of* Ω *into simple objects is*

$$\Omega = \left(\bigoplus_{\psi\neq 1}(\Omega_\psi^- \oplus \Omega_\psi^+)\right) \oplus \Omega_1^- \oplus \Omega_{1,0}^+ \oplus \Omega_{1,\infty}^+ \oplus 1$$

$$[V] = \left(\bigoplus_{\psi\neq 1}(\Omega_\psi^- \oplus \Omega_\psi^+)\right) \oplus \Omega_1^- \oplus \Omega_{1,0}^+ \oplus \Omega_{1,\infty}^+ \oplus 2\cdot 1$$

where all the summands are non-isomorphic and

$$dim(\Omega_\psi^-) = 1 - q^{-1}$$

$$dim(\Omega_\psi^+) = dim(\Omega_{1,0}^+) = dim(\Omega_{1,\infty}^+) = -1$$

$$dim(1) = 1.$$

□

4.3. **The Simple Objects of $\mathscr{U}_{q,k}$.** To investigate the case of $\mathscr{U}_{q,k}$, we begin by noting that we have a Γ_u-invariant map

$$\theta : V^* \to \mathbb{F}_q^\times,$$

given by sending a vector to the coefficient of its coordinate which has the highest index in $\mathbb{R}$, i.e. sending

$$a_1\cdot(r_1) + \cdots + a_n\cdot(r_n) \in \mathbb{F}_q[\mathbb{R}] \smallsetminus \{0\}$$

for $a_1,\dots,a_n \in \mathbb{F}_q^\times$, $r_1 > \cdots > r_n \in \mathbb{R}$, to $a_1 \in \mathbb{F}_q^\times$. Each fiber $\theta^{-1}(i)$ gives a suborbit of Ω, which is isomorphic to the projective space

$$P := \mathbb{P}(\mathbb{F}_q[\mathbb{R}] \smallsetminus \{0\}) = (\mathbb{F}_q[\mathbb{R}] \smallsetminus \{0\})/\mathbb{F}_q^\times.$$

The basic objects Ω, $[V]$ then splits as

$$\Omega \cong (q-1)[P],$$

$$[V] \cong (q-1)[P] \oplus (0).$$

Similarly to Ω, we can use $[P]$ as the basic object for $\mathscr{U}_{q,k}$, since it generates $[V]$ (and Ω) and therefore generates an equivalent category.

Proposition 4.4. *For a field k of characteristic not equal to p containing pth roots of unity, we have decompositions*

$$[P] \cong 1 \oplus \Omega^+_{1,0} \oplus \Omega_{1,\infty} \oplus \bigoplus_{\alpha \neq 1 : \mathbb{F}_q \to k^\times} \widetilde{\Omega}_\alpha$$

$$\Omega \cong (q-1) \cdot \left(1 \oplus \Omega^+_{1,0} \oplus \Omega^+_{1,\infty} \oplus \bigoplus_{\alpha \neq 1 : \mathbb{F}_q \to k^\times} \widetilde{\Omega}_\alpha \right),$$

$$[V] \cong q \cdot 1 \oplus (q-1) \cdot \left(\Omega^+_{1,0} \oplus \Omega^+_{1,\infty} \oplus \bigoplus_{\alpha \neq 1 : \mathbb{F}_q \to k^\times} \widetilde{\Omega}_\alpha \right)$$

where the direct sum runs over non-trivial additive characters

$$\alpha : \mathbb{F}_q \to k^\times$$

for simple non-isomorphic objects $\widetilde{\Omega}_\alpha$, with

$$dim(\widetilde{\Omega}_\alpha) = q^{-1}$$
$$dim(\Omega^+_{1,0}) = dim(\Omega^+_{1,\infty}) = -1$$
$$dim(1) = 1.$$

Proof. First note that the orbits of P^2 have the following representatives in (weak) semi-echelon form:

$$(4.8)\qquad \begin{aligned} (>) &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (<) &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ (1) &:= \begin{pmatrix} 1 & 1 \end{pmatrix} \\ [a] &:= \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \end{aligned}$$

for $a \in \mathbb{F}_q^\times$ (note that, for every $a \in \mathbb{F}_q^\times$, we have $[a] \sim_b (1)^\sim$).

First, let us compute $[a][b]$ for $a, b \in \mathbb{F}_q^\times$. Note that for any choice of $a, b \in \mathbb{F}_q^\times$, we have

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ 0 & a \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix} \sim_u \begin{pmatrix} 1 & 1 \\ a & a \\ 0 & b \end{pmatrix},$$

giving $-q^{-1}[a]$ as a summand of $[a][b]$ (since the partial trace of the product with respect to the middle columns is

$$\begin{pmatrix}1 & 1\\ 0 & a\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ 0 & a\\ 0 & b\end{pmatrix}$$

with coefficient $-q^{-1}$ because the dimension of the corresponding fiber is 1, i.e. we delete one row to put it into weak semi-echelon form). Similarly, the fact that

$$\begin{pmatrix}1 & 1\\ 0 & a\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ 0 & 0\\ 0 & a\end{pmatrix}$$

and

$$\begin{pmatrix}1 & 1\\ 0 & b\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ 0 & b\\ a & 0\end{pmatrix}$$

gives a multiple of $[b]$, since

$$\begin{pmatrix}1 & 1\\ 0 & b\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ 0 & b\\ 0 & 0\end{pmatrix},$$

as a summand of $[a][b]$. Again, the coefficient of $[b]$ will be $-q^{-1}$ since the dimension of the corresponding fiber is 1 (again, we delete one row to put the product matrix into weak semi-echelon form after deleting the traced columns). So, for any choice of $a, b \in \mathbb{F}_q^\times$, the composition $[a][b]$ will have summands $-q^{-1}[a]$ and $-q^{-1}[b]$.

If $b \neq -a$, then the only other possible summand is a multiple of $[a+b]$. The term $[a+b]$ can arise from considering

$$\begin{pmatrix}1 & 1\\ 0 & b\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ a & a+b\end{pmatrix},$$

giving a copy of $[a+b]$ with coefficient 1 since the dimension of a corresponding fiber will be 0 (the product matrix is already in weak semi-echelon form after deleting the traced columns). We also get an additional $q-1$ copies of $[a+b]$ with coefficient $-q^{-1}$ arising from

$$\begin{pmatrix}1 & 1\\ 0 & a\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ 0 & a\\ 0 & 0\end{pmatrix}$$

and

$$\begin{pmatrix}1 & 1\\ 0 & b\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ a & a+b\\ 0 & d\end{pmatrix}$$

for any choice of $d \in \mathbb{F}_q^\times$. Thus, if $a \neq -b$,

$$[a][b] = q^{-1}[a+b] - q^{-1}[a] - q^{-1}[b]. \tag{4.9}$$

If $b = -a$, then the only possible summands of $[a][b]$ other than the terms $-q^{-1}[a]$ and $-q^{-1}[-a]$ which we described above, are (1) and $[d]$ for $d \in \mathbb{F}_q^\times$. By considering

$$\begin{pmatrix}1 & 1\\ 0 & -a\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ a & 0\end{pmatrix},$$

we obtain the term (1) with coefficient $-q^{-1}$ since the dimension of the corresponding is 1 (we delete one row when putting the product matrix into weak semi-echelon form after deleting the traced columns). For $d \in \mathbb{F}_q^\times$, we also obtain a copy of $[d]$ with coefficient $-q^{-1}$ by considering

$$\begin{pmatrix}1 & 1\\ 0 & a\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ 0 & a\\ 0 & 0\end{pmatrix}$$

and

$$\begin{pmatrix}1 & 1\\ 0 & b\end{pmatrix} \sim_u \begin{pmatrix}1 & 1\\ a & 0\\ 0 & d\end{pmatrix}$$

(the coefficient again arises since we lose one row when putting the combined matrix into weak semi-echelon form). Thus,

$$[a][-a] = -q^{-1}(1) - q^{-1}[a] - q^{-1}[-a] - q^{-1}\sum_{d \in \mathbb{F}_q^\times}[d]. \tag{4.10}$$

These multiplication formulas give that, if we write, for $a \in \mathbb{F}_q^\times$,

$$[a]' := q \cdot [a] + (1),$$

then we have, for $a, b \in \mathbb{F}_q^\times$ with $b \neq -a$,

$$[a]'[b]' = [a+b]' \tag{4.11}$$

and

$$[a]'[-a]' = -\left(\sum_{d \in \mathbb{F}_q^\times}[d]'\right). \tag{4.12}$$

Let us write

$$[0]' := -\left(\sum_{d\in\mathbb{F}_q^\times}[d]'\right).$$

Note that for every $a\in\mathbb{F}_q^\times$,

$$tr([a]') = -q^{-1}$$

and

$$tr([0]') = -(q-1)\cdot(-q^{-1}) = 1-q^{-1}.$$

In fact, the formulas (4.11) and (4.12) imply that $[0]'$ is an idempotent, and moreover, for every $a\in\mathbb{F}_q^\times$,

$$[a]'[0]' = [0]'[a]' = [a]'.$$

For a non-trivial additive character

$$\alpha:\mathbb{F}_q\to k^\times$$

(of which there are $q-1$), we may then write

$$z_\alpha := \sum_{a\in\mathbb{F}_q^\times}\alpha(a)\cdot[a]',$$

and get that composing z_α with itself gives

$$z_\alpha^2 = (q-2)\cdot z_\alpha + (q-1)\cdot[0]'.$$

Solving this equation for idempotents gives an idempotent

$$e_\alpha := q^{-1}\cdot z_\alpha + q^{-1}\cdot[0]' = q^{-1}\cdot\sum_{a\in\mathbb{F}_q}\alpha(a)\cdot[a]',$$

and its complement $[0]'-e_\alpha$, for every choice of α. It follows elementarily from the independence of characters that for two distinct choices $\alpha\neq\beta$,

$$e_\alpha e_\beta = 0.$$

Take

$$\widetilde{\Omega}_\alpha = Im(e_\alpha),$$

which then has dimension equal to the trace of e_α, which is q^{-1}.

Similarly as in the proof of Proposition 4.2, we may also consider idempotents

$$\overline{e}_0 = -\frac{(<)}{q^{-1}},\ \ \overline{e}_\infty = -\frac{(>)}{q^{-1}}$$

of trace 0, which, again, commute and satisfy

$$tr(\overline{e}_0\,\overline{e}_\infty) = 1.$$

Thus we have disjoint idempotents

$$e_0 := \overline{e}_0 - \overline{e}_0\overline{e}_\infty,\ e_\infty := \overline{e}_\infty - \overline{e}_0\overline{e}_\infty, \text{ and } \overline{e}_0\overline{e}_\infty$$

which therefore give summands

$$\Omega^+_{1,0} \oplus \Omega^+_{1,\infty} \oplus 1 \subseteq [P],$$

of dimensions -1, -1, and 1, respectively. Since these summands and the $\widetilde{\Omega}_\alpha$ are all disjoint, and (4.8) implies that

$$dim(End_{\mathscr{U}_{q,k}}([P])) = q + 2,$$

they must be simple, giving the stated simple decomposition of $[P]$, and thus, also, the simple decomposition of Ω.

The number of orbits of $V \times V$ with respect to the Γ_u-action is

$$q^3 - q + 1$$

(the orbits being

$$(0,0),\ (a,0),\ (0,b),\ (a,b),\ \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix},\ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$$

with $a, b \in \mathbb{F}_q^\times$, $x \in \mathbb{F}_q$, giving

$$1 + 2(q-1) + 2(q-1)^2 + (q-1)^2 q = q^3 - q + 1$$

orbits). Hence,

$$\begin{gathered} dim(End_{\mathscr{U}_{q,k}}([V])) = q^3 - q + 1 = \\ (q-1)^2(q+2) + 2q - 1 = \\ dim(End_{\mathscr{U}_{q,k}}(\Omega)) + q^2 - (q-1)^2, \end{gathered}$$

and thus, by counting dimensions, we must have

$$1 \cong (0).$$

(Again, this can also be reasoned directly since the elements

$$(0,0),\ (a,0),\ (0,b),\ (a,0)(0,b), \text{ for } a, b \in \mathbb{F}_q^\times$$

form a q-by-q matrix algebra).

$\square$

4.5. **The Simple Objects of $\mathscr{U}_{q,k}$.** Let us now assume that k is an algebraically closed field of characteristic not p. Let us also write Q_n for the object of $\mathscr{U}_{q,k}$ corresponding to the Γ_u-orbit in P^n generated by elements

$$(1\cdot(r_1),\dots,1\cdot(r_n))\in(\mathbb{F}_q[\mathbb{R}])^n$$

for $r_1>\dots>r_n$.

There are $(q+1)^n$ disjoint idempotents in $End_{\mathscr{U}_{q,k}}(Q_n)$ corresponding to a linear combination of orbits of

$$(1\cdot(r_1),\dots,1\cdot(r_n),1\cdot(s_1),\dots,1\cdot(s_n))$$

such that

$$r_1,s_1>r_2,s_2>\dots>r_n,s_n,$$

which are described as follows:

The idempotents $e_{(\alpha_1,\dots,\alpha_n)}$ are indexed by choosing

$$\alpha_i\in\{0,\infty\}\cup(Hom(\mathbb{F}_q,k^\times)\smallsetminus\{1\}) \tag{4.13}$$

(where, as above, 1 denotes the trivial additive character). By extending linearly direct sums of the matrices (4.8), we may write direct sums of idempotents. We then put

$$e_{(\alpha_1,\dots,\alpha_n)}=e_{\alpha_1}\oplus\dots\oplus e_{\alpha_n} \tag{4.14}$$

where the columns of the matrices correspond to

$$r_1,s_1,\dots,r_n,s_n.$$

The simple central idempotents of the group algebra $k[U_n]$ (where U_n denotes the unipotent subgroup of the Borel subgroup $B_n\subseteq GL_n(\mathbb{F}_q)$) commute with (4.14). Therefore, by (2.36), these idempotents correspond to all the non-isomorphic simple summands of Θ_n^0.

They are non-trivial because they have non-zero trace. The trace of an element $\sum a_g\cdot g$ of the group algebra is

$$a_1\cdot tr(1).$$

In the present setting,

$$tr(1)=tr(e_{(\alpha_1,\dots,\alpha_n)})=(-1)^{|\{i|\alpha_i\in\{0,\infty\}\}|}\cdot q^{-|\{j|\alpha_j\notin\{0,\infty\}\}|}.$$

Using the results of I. M. Isaacs [9], we know that the dimensions of the simple representation of U_n are powers of q. Thus, we have proved the following

Theorem 4.6. *Let $q = p^m$ and let k be an algebraically closed field of characteristic not equal to p. The non-isomorphic simple objects of $\mathscr{U}_{q,k}$ are obtained by taking a simple U_n-summand of $e_{(\alpha_1,\dots,\alpha_n)}$ with α_i as in (4.13). The dimensions of these objects are all of the form $\pm q^{-\ell}$ for $\ell \in \mathbb{N}_0$.*

□

4.7. **Comparison of the simple objects of $\mathscr{B}_{q,k}$ and $\mathscr{U}_{q,k}$.** Note that, despite the similarity of the statements, Proposition 4.2 assumes the target field k has $(q-1)$th roots of unity, while Proposition 4.4 assumes it has pth roots of unity. If we assume that the field k has charateristic not dividing $q(q-1)$, then there is a canonical tensor functor

$$\mathscr{B}_{q,k} \to \mathscr{U}_{q,k}$$

(given by morphisms of T-algebras). If, in addition, both $(q-1)$th and pth roots of unity are present in k, we can compare the summands in Proposition 4.2 and Proposition 4.4.

Since the elements

$$(0,0),\ (a,0),\ (0,b),\ (a,0)(0,b),\quad \text{for } a,b \in \mathbb{F}_q^\times$$

form a matrix algebra $M_q(k)$, corresponding to the q copies of 1 in $[V]$, some of the summands in Proposition 4.2 must have a summand of 1 in $\mathscr{U}_{q,k}$. We have that

$$(0,1)\iota^+ \neq 0$$

$$(0,1)\iota^- = 0,$$

so we can conclude that each of the summands Ω_ψ^+, for a multiplicative character

$$\psi : \mathbb{F}_q^\times \to k^\times,$$

has a summand of 1 in $\mathscr{U}_{q,k}$. In fact, since $[V]$ has a commutative algebra structure as in [3], Section 8, we have

$$\Omega_\psi^+ \cong 1 \oplus \Omega_{1,0}^+ \oplus \Omega_{1,\infty}^+$$

for every multiplicative character

$$\psi : \mathbb{F}_q^\times \to k^\times.$$

The remaining summands described in Proposition 4.4 give a decomposition

$$\Omega_\psi^- \cong \bigoplus_{\alpha \neq 1 : \mathbb{F}_q \to k^\times} \widetilde{\Omega}_\alpha,$$

where the direct sum is over non-trivial additive characters

$$\alpha : \mathbb{F}_q \to k^\times.$$

In particular, for all multiplicaive characters $\psi, \phi : \mathbb{F}_q^\times \to k^\times$, as objects of $\mathscr{U}_{q,k}$,

$$\Omega_\psi^- \cong \Omega_\phi^-.$$

4.8. **Comparison Functors and the Universality of the Delannoy Category.** One has a commutative diagram of tensor functors of the following form, where all categories are over a field k of characteristic 0:

$$\begin{array}{ccccccc} & & \underline{Rep}(S_{-1}) & \xrightarrow{\alpha} & \mathscr{D} & & \\ & & & & \downarrow & & \\ \underline{Rep}(S_{q^{-1}}) & \xrightarrow{\epsilon} & \underline{Rep}(GL_{-1}(\mathbb{F}_q)) & \xrightarrow{\gamma} & \mathscr{B}_{q,k} & \xrightarrow{\delta} & \mathscr{U}_{q,k}. \end{array}$$

The functors α and ϵ send the basic objects of their sources to the basic objects of their targets, and their existence follows from Proposition 8.3 of [3]. The functors γ, δ follow directly from the construction of the T-algebra, and also from the oligomorphic group method. The existence of the functor β follows from the universality of the Delannoy category of [8], which we state and prove in this section.

Suppose $\mathscr{C}$ is a pseudo-abelian category with ACU tensor product and strong duality with an object X such that there are multiplication and unit morphisms

$$\mu : X \otimes X \to X, \ \eta : 1 \to X \tag{4.15}$$

in $\mathscr{C}$ giving X the structure of an associative, commutative, unital (ACU) algebra. Recall that, following Section 8.1 of [3], we may construct a morphism $Tr : X \to 1$ as the composition

$$X \xrightarrow{Id_X \otimes coev_X} X \otimes X \otimes X^* \xrightarrow{\mu \otimes Id_{X^*}} X \otimes X^* \cong X^* \otimes X \xrightarrow{ev_X} 1$$

Theorem 4.9. *For a pseudo-abelian category $\mathscr{C}$ over a commutative ring R with ACU tensor product and strong duality, tensor functors*

$$\mathscr{D} \to \mathscr{C} \tag{4.16}$$

are in bijective correspondence with the following data:

(1) *An object $X \in Obj(\mathscr{C})$ which is an ACU algebra via morphisms (4.15) such that*

(4.17)
$$X \otimes X \xrightarrow{\mu} X \xrightarrow{Tr} 1$$

makes X its own dual.

(2) *A splitting*

(4.18)
$$X = X_+ \oplus 1 \oplus X_-$$

where

(a) $dim(X_+) = dim(X_-) = -1$

(b) $X_+ \oplus 1$, $X_- \oplus 1$ *are subalgebras of X with ideals X_+, X_-, respectively.*

(c) *The self-duality of X switches X_+ and X_- in the composition (4.18)*

If 2 *is not invertible in R, we also assume*

(d) *Let $\pi_+ : X \to X_+ \oplus 1$ be the morphism of $\mathscr{C}$ which is identity on $X_+ \oplus 1$ and* 0 *on X_-. Then the composition*

(4.19)
$$\begin{array}{ccc} X \longrightarrow X \otimes X \otimes X^\vee & \xrightarrow{\mu} & X \otimes X^\vee \\ & & \downarrow{\scriptstyle \pi_+ \otimes Id_{X^\vee}} \\ & & (X^+ \oplus 1) \otimes X^\vee \\ & & \downarrow{\scriptstyle \subseteq} \\ & & X \otimes X^\vee \\ & & \downarrow \\ & & 1 \end{array}$$

is 0 *(see Figure 1 below).*

Comment: Let also $\pi_- : X \to X_- \oplus 1$ denote the morphism which is identity on $X_- \oplus 1$ and 0 on X_+, and let $\pi_0 : X \to X$ denote the morphism which is identity on 1 and 0 on $X_+ \oplus X_-$. One can check that the assumption about $X_=$ and X_- being ideals implies that π_+ is dual to the inclusion $X_- \oplus 1 \subseteq X$ and similarly with $+$ and $-$ reversed.

This actually implies that the composition (4.19) is equal to its analogue where we replace π_+ by π_-. We also know that if we replace π_+

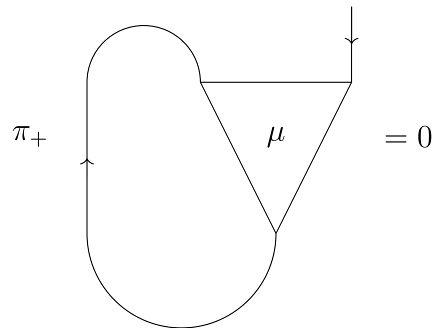

FIGURE 1. Condition (d)

(or π_-) by π_1, we get the same as when we replace it by Id_X, i.e. the composition (4.19) is Tr. This implies that (4.19) multiplied by 2 is 0.

P. Deligne [5] found a counterexample to Theorem 4.9 if we do not make the additional assumption (d) in characteristic 2. Let $\mathscr{C}$ be the category of finite dimensional vector spaces over a field k of characteristic 2 with the usual tensor product. Take $X = k^3$ with the product ring structure. Then $X_+ \oplus 1$, resp. $X_- \oplus 1$, is the subring given by the equation $x_1 = x_2$, resp. $x_2 = x_3$. As ideals, X_+, resp. X_-, is given by the equation $x_3 = 0$, resp. $x_1 = 0$. Then the composition (4.19) is not 0, so there exists no tensor functor

$$\mathscr{D} \to \mathscr{C}$$

sending the basic object of $\mathscr{D}$ to X, while $\mathscr{C}$ satsifies every assumption of Theorem 4.9 but (d).

Proof of Theorem 4.9. Recalling the description of the T-algebra corresponding to the Delannoy category described in Subsection 2.4, we must first construct elements of $Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T})$ that correspond to the ordered partitions

$$(U_1, \dots, U_\ell) \tag{4.20}$$

of $S \amalg T$ generating (2.13). We will begin by constructing an idempotent of $End_{\mathscr{C}}(X^{\otimes n})$ corresponding to the ordered partition

$$(\{1_1, 1_2\}, \{2_1, 2_2\}, \dots, \{n_1, n_2\}) \tag{4.21}$$

of $\{1, \dots, n\} \amalg \{1, \dots, n\}$ (where, again, the subscripts indicate which disjoint summand we are considering the element to be in).

First, we identify the ordered partition

$$(\{1_1\},\{1_2\}) \in \mathscr{D}_{\{1\},\{1\}} \tag{4.22}$$

of $\{1\} \amalg \{1\}$ with $-\pi_+ \in End_{\mathscr{C}}(X)$, and, similarly, we identify

$$(\{1_2\},\{1_1\}) \tag{4.23}$$

with $-\pi_-$.

Now the assumptions on X in particular imply those required in the universality property of $\underline{Rep}(S_{-1})$ (see Section 8.2, [3]), so we have a tensor functor

$$\underline{Rep}(S_{-1}) \to \mathscr{C}$$

(sending the basic object of $Rep(S_{-1})$ to X). In particular, we can construct elements

$$\mu \in Hom_{\mathscr{C}}(X^{\otimes 2}, X)$$
$$\nu \in Hom_{\mathscr{C}}(X, X^{\otimes 2})$$
$$\kappa \in Hom_{\mathscr{C}}(X^{\otimes 2}, X^{\otimes 2})$$

corresponding to (unordered) partitions

$$\{\{1_1, 2_1, 1_2\}\} \in \underline{Rep}(S_{-1})_{\{1,2\},\{1\}}$$
$$\{\{1_1, 1_2, 2_2\}\} \in \underline{Rep}(S_{-1})_{\{1\},\{1,2\}}$$
$$\{\{1_1, 2_1, 1_2, 2_2\}\} \in \underline{Rep}(S_{-1})_{\{1,2\},\{1,2\}},$$

respectively (recalling the description (2.5)). (Note that μ, ν correspond to the multiplication map and its dual, respectively.) We may consider the product

$$\underbrace{\nu \otimes \kappa \otimes \cdots \otimes \kappa \otimes \mu}_{n} \in End_{\mathscr{C}}(X^{\otimes(2n-1)}), \tag{4.24}$$

and trace it with

$$(-1)^{n-1} \cdot \underbrace{(\pi_+ \otimes \cdots \otimes \pi_+)}_{n-1} \in End_{\mathscr{C}}(X^{\otimes(n-1)}) \tag{4.25}$$

where the source of each π_+ is plugged in to the target of a tensor factor of (4.24) and its target is plugged into the source of the next term in (4.25). This trace we take to be the idempotent of $End_{\mathscr{C}}(X^{\otimes n})$ corresponding to (4.21). Given (4.21), we may compose with multiplication, the dual of multiplication, the unit, and the augmentation to get morphisms in $\mathscr{C}$ corresponding to all ordered partitions (4.20).

To verify the arising functor definition, write

$$A = Id_X - \pi_+ + \pi_1 = -\pi_-$$

$$B = Id_X - \pi_- + \pi_1 = -\pi_+$$

(the elements corresponding to (4.23) and (4.22), respectively). We then have

$$A^2 = -A,\ B^2 = -B,\ A \circ B = -A - B - Id_X = \pi_1.$$

Note that the categorical traces of A and B are 0.

By taking partial traces of morphisms in $\underline{Rep}(S_{-1})$ (see Subsection 2.3) with product of A or B repeatedly, we can order the equivalence classes defining a generating morphism of the T-algebra of $\underline{Rep}(S_{-1})$. Multiple sequences of partial traces with A resp. B can imply the same ordering, and we need to prove that the answers are indeed equal using our axioms. Similarly, some ordering definitions can be inconsistent, and we need to prove that the respective partial traces are 0.

This can be reduced to two specific statements which we will now describe. Denote for $f, g : X \to X$ the composition

$$X \xrightarrow{\nu} X \otimes X \xrightarrow{f \otimes g} X \otimes X \xrightarrow{\mu} X$$

by $f \odot g$. The we need to show

$$A \odot A = A \tag{4.26}$$

and

$$A \odot B = 0 \in End_{\mathscr{C}}(X). \tag{4.27}$$

In fact, it turns our that (4.27) implies (4.26).

To this end, note that we may express π_1 as the composition

$$X \xrightarrow{Tr} 1 \xrightarrow{\eta} X$$

(where, again, $\eta : 1 \to X$ denotes the unit of the algebra structure of X, which is dual to Tr). This, since X has dimension -1,

$$A \odot \pi_1 = -A.$$

On the other hand, since A has trace 0,

$$A \odot Id_X = 0.$$

Now, to prove (4.27), it suffices to show

$$(A \odot B) \circ A = 0,$$

since, by symmetry (and the commutativity of the algebra structure on X), then $(A \odot B) \circ B = 0$, and $(A \odot B) \circ \pi_1$ automatically. We may express $(A \odot B) \circ A$ as a partial trace of the composition

$$X \otimes X \xrightarrow{A \otimes A} X \otimes X \xrightarrow{\mu} X \xrightarrow{A} X \xrightarrow{\nu} X \otimes X$$

which is the same as the partial trace of

$$X \otimes X \xrightarrow{A \otimes A} X \otimes X \xrightarrow{\mu} X \xrightarrow{\nu} X \otimes X. \tag{4.28}$$

We may replace $\nu \circ \mu$ by the composition

$$X \otimes X \xrightarrow{Id_X \otimes \nu} X \otimes X \otimes X \xrightarrow{\mu \otimes Id_X} X \otimes X,$$

the partial trace of which, after composing with $A \otimes A$ as in (4.28), is

$$X = 1 \otimes X \xrightarrow{\Psi \otimes Id_X} X \otimes X \xrightarrow{\mu} X,$$

where $\Psi \in Hom_{\mathscr{C}}(1, X)$ is the partial trace of

$$X \xrightarrow{\nu} X \otimes X \xrightarrow{Id_X \otimes A} X \otimes X,$$

(in the second coordinate of the target) which is 0.

□

5. Appendix: The Top Algebra

In this Appendix, we give an alternative proof of the semisimplicity of the category $\mathscr{B}_{q,k}$, following the method of [3]. This involves studying the endomorphism algebra of the object U_n which is the image of the idempotent given by the sum of copies of (2.29) over all flags, modulo the ideal of morphisms which factor through U_m for some $m < n$.

5.1. **The basis of the top algebra.**

Our construction of $\mathscr{B}_{q,k}$ makes sense in any characteristic not dividing

$$q(q-1).$$

The object U_n is defined as the image of the idempotent given by a sum of copies of (2.29) over all possible choices of maximal flags, which are indexed by $GL_n(\mathbb{F}_q)/B_n$, where $B_n \subseteq GL_n(\mathbb{F}_q)$ is the Borel subgroup.

First note that we can again decompose the object U_n as

$$U_n = \bigoplus_{\mathcal{V}} U_{n,\mathcal{V}}$$

where the sum runs over maximal flags $\mathcal{V}$ on $\mathbb{F}_q^n$, where $U_{n,\mathcal{V}}$ is the image of the idempotent $\iota_{n,\mathcal{V}}$ is the morphism corresponding (2.29) with the flag $\mathcal{V}$ on the target. It suffices to describe

$$End_{\mathscr{B}_{q,k}}(U_{n,\mathcal{V}}), \tag{5.1}$$

since to get $End_{\mathscr{B}_{n,k}}(U_n)$, we can take the semidirect product of (5.1) with the groupoid whose objects are $GL_n(\mathbb{F}_q)/B_n$ and morphisms between two cosets are all elements of $GL_n(\mathbb{F}_q)$ going between them.

Without loss of generality, we may further assume that $\mathcal{V}$ is the standard flag.

Lemma 5.2. *The endomorphism algebra $End_{\mathscr{B}_{q,k}}(U_{n,\mathcal{V}})$ is a free $\mathbb{C}$-vector space on the following data: $\mathbb{F}_q$-vector space homomorphisms*

$$\begin{array}{c} \mathbb{F}_q^n \oplus \mathbb{F}_q^n \\ \Big\downarrow{\scriptstyle \phi\oplus\psi} \\ \mathbb{F}_q^m \end{array}$$

where the flag on $\mathbb{F}_q^m$ is the standard one, $\phi \oplus \psi$ is onto, and each ϕ, ψ are injective, and further we have sequences

$$1 \leq j_1 < \cdots < j_n \leq m \tag{5.2}$$

$$1 \leq \ell_1 < \cdots < \ell_n \leq m \tag{5.3}$$

where for each i, we have $j_{i-1} < \ell_i < j_{i+1}$ when applicable, and one of the following cases occurs:

- **Case 1:** *$\ell_i = j_i$ and $\phi(e_i) = \sum_{s\leq i} a_{i,s}e_{j_s}$ with $a_{i,s} \in \mathbb{F}_q$, and $a_{i,i} \neq 0$.*
- **Case 2:** *$\ell_i = j_i - 1$ and $\phi(e_i) = e_{\ell_i} + \sum_{s<i} a_{i,s}e_{j_s}$ for $a_{i,s} \in \mathbb{F}_q$*
- **Case 3:** *$\ell_i = j_i + 1$ and $\phi(e_i) = e_{\ell_i} + \sum_{s\leq i} a_{i,s}e_{j_s}$ for $a_{i,s} \in \mathbb{F}_q$.*

In particular,

$$dim(End_{\mathscr{B}_{q,k}}(U_{n,\mathcal{V}})) = 2^n q^{\binom{n+1}{2}}. \tag{5.4}$$

Proof. Analogous to the proof of Lemma 2.5 with the exception of the $a_{i,s}e_{j_s}$ summands. One notes that all the terms listed are non-equivalent.

Additionally, when $\ell_s \neq j_s$, $s < i$, adding a multiple of e_{ℓ_s} produced equivalent elements. One cannot add other $\phi(e_s)$ terms, since similarly as in the proof of Lemma 2.5, such a generator x would have

$$\iota_{n,\mathcal{V}} \circ x \circ \iota_{n,\mathcal{V}} = 0.$$

$\square$

Lemma 5.3. *The ideal* $I_{n,\mathcal{V}}$ *of* $End_{\mathscr{B}_{q,k}}(U_{n,\mathcal{V}})$ *generated by morphisms*

$$U_{n,\mathcal{V}} \to X^{\otimes r} \to U_{n,\mathcal{V}}$$

for $r < n$ *is generated by linear combinations of the form*

$$D + \sum_{a\in\mathbb{F}_q} D'_a \tag{5.5}$$

where D *is a data as in Lemma 5.2 where Case 2 occurs for a given* i *and in the data* D'_a*, renaming the sequences (5.2), (5.3) as* j'_s*,* ℓ'_s*, we have*

$$j'_s = j_s,\ \text{and}\ \ell'_s = \ell_s\ \text{for}\ s \neq i$$
$$j'_i = \ell_i,\ \text{and}\ \ell'_i = j_i$$

(i.e. Case 3 occurs for D'_a*), and writing the constants* $a_{i,s}$ *for* D'_a *as* $a'_{i,s}$*, we have*

$$a'_{s,t} = a_{s,t}\ \text{for}\ (s,t) \neq (i,i)$$
$$a'_{i,i} = a$$

(note that $a_{i,i}$ *is not defined).*

Proof. Consider a composition of two generator morphisms of the form

$$\begin{array}{c} \mathbb{F}_q^n \oplus \mathbb{F}_q^r \\ \downarrow{\scriptstyle \phi_1\oplus\psi_1} \\ \mathbb{F}_q^{m_1} \end{array} \quad \text{and} \quad \begin{array}{c} \mathbb{F}_q^r \oplus \mathbb{F}_q^n \\ \downarrow{\scriptstyle \phi_2\oplus\psi_2} \\ \mathbb{F}_q^{m_2} \end{array} \tag{5.6}$$

with the standard flags on $\mathbb{F}_q^{m_1}$, $\mathbb{F}_q^{m_2}$. The composition y is the sum of

$$\begin{array}{c} \mathbb{F}_q^n \oplus \mathbb{F}_q^n \\ \downarrow \\ V \end{array}$$

where V is the quotient of the pushout of the diagram

$$\begin{array}{ccc} \mathbb{F}_q^r & \xrightarrow{\phi_2} & \mathbb{F}_q^{m_2} \\ {\scriptstyle \psi_1}\downarrow & & \\ \mathbb{F}_q^{m_1} & & \end{array}$$

over $\mathbb{F}_q^r$ with all compatible flags. Since $m_1, m_2 \geq n$, we have

$$dim(V) > n.$$

Therefore, in generator summands x of y (i.e. summands satisfying $\iota_{n,\mathcal{V}} \circ x \circ \iota_{n,\mathcal{V}} \neq 0$), Cases 2 or 3 of Lemma 5.2 must occur for at least one choice of i, and different choices of flags always product sums of the form (5.5).

On the other hand, one also sees that all the generators listed in the statement occur for $r = n-1$.

□

Lemma 5.4. *The quotient $End_{\mathscr{B}_{q,k}}(U_{n,\mathcal{V}})/I_{n,\mathcal{V}}$ is a free $\mathbb{C}$-vector space on the same generators as in Lemma 5.2, where only Cases 1 and 3 are allowed. In fact,*

$$dim(End_{\mathscr{B}_{q,k}}(U_{n,\mathcal{V}})/I_{n,\mathcal{V}}) = (2q-1)^n q^{\binom{n}{2}}.$$

Proof. One can write the vector space $End_{\mathscr{B}_{q,k}}(U_{n,\mathcal{V}})$, by Lemma 5.2, as

$$\bigoplus_{S \subseteq [n]} \left(\bigotimes_{i \in S} k(\mathbb{F}_q^\times \{a_{i,i}\}) \otimes \bigotimes_{i \in [n] \smallsetminus S} k(\{\infty\} \amalg \mathbb{F}_q) \right) \tag{5.7}$$

where each ∞ corresponds to Case 2 and each $\mathbb{F}_q$ summand in the last factor corresponds to Case 3.

Now, by Lemma 5.3, each generator of $I_{n,\mathcal{V}}$ is of the form

$$z \otimes ((\infty) + \sum_{a \in \mathbb{F}_q} (a))$$

for some i-factor in the last tensor factor of (5.7), while z is some element of the tensor product of the other factors. This concludes our proof.

□

5.5. **The multiplicative structure.** For an arbitrary subset $S \subseteq [n]$, consider a groupoid $\mathbb{B}_S$ defined as follows:

Consider the set

$$\Phi_S = \{f : [n] \smallsetminus S \to \mathbb{F}_q\}$$

and the subgroups of the Borel subgroup

$$B_S = \{b \in B_n \mid b_{i,i} = 1 \text{ for } i \in [n] \smallsetminus S\}$$

$$B_{S,0} = \{b \in B_S \mid b_{i,j} = b_{j,i} = 0 \text{ for } i \in [n] \smallsetminus S,\ i \neq j\}.$$

Then $\mathbb{B}_S$ is the groupoid of the left action of $B_S \times \Phi_S$ on the set of left cosets $B_S/B_{S,0}$, where Φ_S acts trivially, and upon composition, elements of Φ_S are multiplied coordinate-wise.

In fact, the subalgebra $\Xi_n^0 \subset \Xi_n$ generated by the morphisms not changing the flag on $\mathbb{F}_q^n$ can be described as

$$k\left(\coprod_{S\subseteq[n]} Mor(\mathbb{B}_S)\right).$$

Even for Ξ_n^0, the composition formula can be complicated in general. However, we have a decreasing filtration F on Ξ_n^0 where

$$F^k\Xi_n^0 = k\left(\coprod_{S\subseteq[n],\ |S|\leq n-k} Mor(\mathbb{B}_S)\right).$$

The composition then has the form

$$F^k\Xi_n^0 \otimes F^\ell\Xi_n^0 \to F^m\Xi_n^0 \tag{5.8}$$

where $m \geq max(k, \ell)$. If the associated graded algebra is semisimple, then so is Ξ_n^0 (since it has 0 Jacobson radical). We can describe the part of (5.8) where $k = \ell = m$, which we will denote by $\Xi_n^{0,k}$.

We have

$$\Xi_n^{0,k} = k\left(\coprod_{S\subseteq[n],|S|=n-k} Mor(\mathbb{B}_S)\right),$$

where multiplication of two morphisms is given by composition when they are composable and is 0 otherwise. This algebra $\Xi_n^{0,k}$ is thus semisimple, the endomorphism algebra being $k[B_{S,0}]$.

To describe the full algebra Ξ_n, note that B_n acts on Ξ_n^0 by conjugation. The algebra Ξ_n then is the left Kan extension of Ξ_n^0 from the group B_n to the groupoid $GL_n(\mathbb{F}_q)$ acting on $GL_n(\mathbb{F}_q)/B_n$ (where product of non-composable morphisms is again declared to be 0). The algebra Ξ_n is semisimple because the endomorphism algebras Ξ_n^0 are semisimple. We see that

$$dim(\Xi_n^0) = q^{\binom{n}{2}}\ (2q-1)^n,$$

$$dim(\Xi_n) = q^{\binom{n}{2}}\ ([n]_q!)^2\ (2q-1)^n,$$

using the notation

$$[n]_q! = \frac{q^n - 1}{q-1} \cdot \dots \cdot \frac{q-1}{q-1} = |GL_n(\mathbb{F}_q)/B_n|.$$

Thus, we have proved

Theorem 5.6. *Let q be a prime power and k be a field of characteristic not dividing $q(q-1)$. Then the category $\mathscr{B}_{q,k}$ is semisimple.*

□

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