FAST-GROWING $\mathbb{C}$-LINEAR ADDITIVE LOCALLY FINITE CATEGORIES WITH ACU TENSOR PRODUCT AND STRONG DUALITY

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The purpose of this note is to describe a class of symmetric, rigid, locally finite tensor categories whose growth is aribtarily large. The categories we construct do not have a tensor abelian envelope. Write $[[n]] = \{1, \dots, n\} \times \{0, 1\}$, $[n]_\epsilon = \{1, \dots, n\} \times \{\epsilon\}$, for $\epsilon = 0, 1$.

1. T-Algebras

A graded symmetric, rigid, locally finite tensor category generated by an object X is determined by the strucutre on $End(X^{\otimes n})$ such that $End(1) = \mathbb{C}$ (graded means that for all n, m, k, ℓ such that $m+\ell \neq n+k$,

$$Hom(X^{\otimes m} \otimes (X^\vee)^{\otimes k}, X^{\otimes n} \otimes (X^\vee)^{\otimes \ell}) = 0).$$

The structure which determines such a category consists of describing the properties of the traces and products.

Definition 1. *Define a* T-algebra *over* $\mathbb{C}$ *as a collection of the following data:*

(1) *A sequence of complex* $\Sigma_n \times \Sigma_n$*-representations* V_n *indexed by* $n \in \mathbb{N}_0$

(2) *For every* $n \in \mathbb{N}$, $k \leq n$, *a* $\Sigma_k \times (\Sigma_{n-k})^2$*-equivariant map*

$$\sigma_k : V_n \to V_{n-k}$$

(where we interpret $\Sigma_k \times (\Sigma_{n-k})^2 \subset (\Sigma_k \times \Sigma_{n-k})^2$ *by embedding* Σ_k *diagonally) satisfying*

$$\sigma_{k+\ell} = \sigma_k \circ \sigma_\ell,$$

as maps over $\Sigma_k \times \Sigma_\ell \times (\Sigma_{n-k} \times \Sigma_{n-k-\ell})^2 \subset \Sigma_{k+\ell} \times (\Sigma_{n-k-\ell})^2$.

(3) *A product map*

$$\pi : V_m \otimes V_n \to V_{n+m}$$

which is equivariant with respect to

$$(\Sigma_m \times \Sigma_m) \times (\Sigma_n \times \Sigma_n) \subset \Sigma_{m+n} \times \Sigma_{m+n}$$

that is compatible with all σ_k, σ_ℓ *for* $k \leq m$, $\ell \leq n$. *More specifically, the diagram*

$$\begin{array}{ccc} V_m \otimes V_n & \xrightarrow{\pi} & V_{m+n} \\ {\scriptstyle \sigma_k \otimes \sigma_\ell} \downarrow & & \downarrow {\scriptstyle \tau^{-1}\sigma_{k+\ell}\tau} \\ V_{m-k} \otimes V_{n-\ell} & \xrightarrow{\pi} & V_{m+n-k-\ell} \end{array}$$

where τ *denotes the permutation given by, for* $i \in \{1, \ldots, m+n\}$

$$\tau(i) = \begin{cases} i & \text{if } i \leq k \\ i + \ell & \text{if } k < i \leq m \\ i - n + k & \text{if } m < i \leq m + \ell \\ i & \text{if } m + \ell < i \leq m + n \end{cases}$$

(4) π *is commutative, associative, unital in the obvious sense. For example, commutativity means commutativity of the diagram*

$$\begin{array}{ccc} V_m \otimes V_n & \xrightarrow{\pi} & V_{m+n} \\ {\scriptstyle T} \downarrow & & \downarrow {\scriptstyle (\sigma,\sigma)} \\ V_n \otimes V_m & \xrightarrow{\pi} & V_{m+n} \end{array}$$

where T *denotes the switch of tensor factors, and* $\sigma \in \Sigma_{m+n}$ *denotes the permutation sending* $\{1, \ldots, m\}$ *to* $\{n+1, \ldots, m+n\}$ *and* $\{m+1, \ldots, m+n\}$ *to* $\{1, \ldots, n\}$ *in the order-preserving way.*

(5) *An element* $\iota \in V_1$ *such that*

$$\sigma_1((12) \times Id(\iota\pi x)) = x$$

It is also useful to consider the operations

$$\sigma_{i,j} = (\tau \times \tau')^{-1}\sigma_1(\tau \times \tau')$$

where we take permutations $\tau = (12\ldots i)$, $\tau' = (12\ldots j)$.

Proposition 2. *Given a* T*-algebra* $V = (V_n)$, *there exists a* $\mathbb{C}$*-linear pre-additive category* $\mathcal{C}(V)$ *with an ACU tensor product and strong duality such that for a certain* $X \in Obj(V)$,

$$Obj(\mathcal{C}(V)) = \{X^{\otimes m} \otimes (X^\vee)^{\otimes n} | m, n \in \mathbb{N}_0\}$$

$$Mor(X^{\otimes m_1} \otimes (X^\vee)^{\otimes n_1}, X^{\otimes m_2} \otimes (X^\vee)^{\otimes n_2}) =$$
$$= \begin{cases} V_{m_1+n_2} & \text{if } m_1 + n_2 = m_2 + n_1 \\ 0 & \text{else} \end{cases}$$

Proof. The defining axioms of a T-algebra are translations of the corresponding categorical ones.

□

In the remaining sections, we shall construct a T-algebra $\mathbb{T}$ such that $\mathbb{T}_0 = \mathbb{C}$ and $dim(\mathbb{T}_n)$ is finite but grows faster than any given function in n. This gives a symmetric, rigid, locally finite category of arbitrarily high growth.

2. Representation Structure

Definition 3. *Fix a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$. Define $\mathbb{T}_n$ as the free $\mathbb{C}$-vector space on the set $\mathbb{S}_n$ of choices of the following data:*

(1) *For $1 \le k \le n$, a subset*
$$\mathscr{T}(k) \subseteq \mathscr{P}([[n]]),$$
letting $\mathscr{P}$ denote the power set.

(2) *A function*
$$\chi : \mathscr{T}(k) \to \{1, \dots, n_k\}.$$

(3) *Subsets $W_0 \subseteq [n]_0$ and $W_1 \subseteq [n]_1$ with*
$$|W_0| = |W_1|,$$
and a bijection
$$\beta : W_0 \to W_1$$

(4) *A bijection $b : Z_0 \to Z_1$ where*
$$Z_\epsilon := [n]_\epsilon \setminus \left(W_\epsilon \cup \bigcup_{k=1}^{n} \bigcup_{T\in\mathscr{T}(k)} T \right)$$

satisfying the following conditions:

(1) *For each $T \in \mathscr{T}(k)$, for both $\epsilon = 0, 1$,*
$$|T \cap [n]_\epsilon| = k.$$

(2) *For all distinct $T \in \mathscr{T}(k)$, $T' \in \mathscr{T}(\ell)$,*
$$T \cap T' = \emptyset$$
and for $\epsilon = 0, 1$,
$$T \cap W_\epsilon = \emptyset.$$

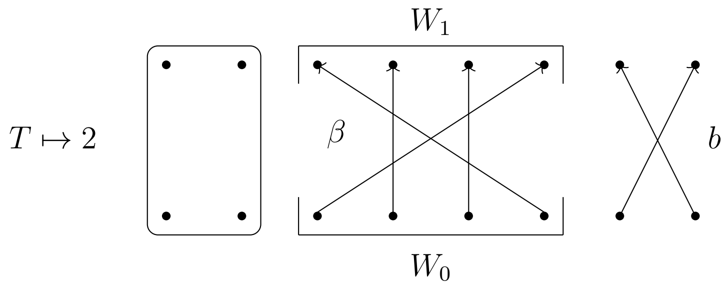
(Note that conditions (1), (2) imply

$$|Z_0| = |Z_1|.)$$

For example, the following diagram is a visulaization of an element of $\mathbb{S}_8$ corresponding to taking $T = \{(1,0),(2,0),(1,1),(2,1)\}$,

$$\mathscr{T}(2) = \{T\},\ \chi(T) = 2,$$

$\mathscr{T}(k) = \emptyset$ for all $k \neq 2$, and $W_\epsilon = \{3,4,5,6\} \times \{\epsilon\}$:



The $\mathbb{C}$-vector space $\mathbb{T}_n$ also has a $\Sigma_n \times \Sigma_n$-action induced by the $\Sigma_n \times \Sigma_n$-action on $[[n]]$ given by letting the first and second factors act on $[n]_0$ and $[n]_1$, respectively.

3. Product Structure

For all n, m, we further give a homomorphism over $(\Sigma_n)^2 \times (\Sigma_m)^2 \subseteq (\Sigma_{n+m})^2$ mapping

$$\pi_{n,m} : \mathbb{T}_n \times \mathbb{T}_m \to \mathbb{T}_{n+m}.$$

In diagrams, we will take this operation to be placing diagram side by side, i.e. using disjoint union. More precisely, let us fix elements

$$\Phi = (\mathscr{T}(1), \dots, \mathscr{T}(n), \chi, \beta, W_0, W_1, b) \in \mathbb{T}_n$$

$$\Phi' = (\mathscr{T}'(1), \dots, \mathscr{T}'(m), \chi', \beta', W_0', W_1', b') \in \mathbb{T}_m$$

Take, then,

$$\widetilde{\mathscr{T}(k)} = \mathscr{T}(k) \amalg \mathscr{T}'(k)$$

(taking undefined sets to be empty and identifying $\{1,\dots,n\} \amalg \{1,\dots,m\} \cong \{1,\dots,n+m\}$ by sending $j \mapsto j+n$ for $j \in \{1,\dots,m\}$),

$$\widetilde{\chi} = \chi \amalg \chi' : \widetilde{\mathscr{T}(k)} \to \{1,\dots,n_k\},$$

for $\epsilon = 0, 1$

$$\widetilde{W_\epsilon} = W_\epsilon \amalg W_\epsilon',$$

$\widetilde{\beta} = \beta \amalg \beta'$ and $\widetilde{b} = b \amalg b'$. Then we put

$$\pi_{n,m}((\Phi, \Phi')) = (\widetilde{\mathscr{T}(1)}, \ldots, \widetilde{\mathscr{T}(n)}, \widetilde{\chi}, \widetilde{\beta}, \widetilde{W_0}, \widetilde{W_1}, \widetilde{b}),$$

inducing such a product map $\pi_{n,m}$.

4. Trace Structure

To give the sequence of $\Sigma_n \times \Sigma_n$-representations $(\mathbb{T}_n)$ the structure of a T-algebra, we must also describe trace. We define $\Sigma_{n-i} \times \Sigma_{n-i} \times \Sigma_i$-equivariant maps

$$tr_\sigma : \mathbb{T}_n \to \mathbb{T}_{n-i}$$

(embedding $\Sigma_{n-i} \times \Sigma_{n-i} \times \Sigma_i$ diagonally into

$$\Sigma_{n-i} \times \Sigma_{n-i} \times \Sigma_i \times \Sigma_i \subseteq \Sigma_n \times \Sigma_n$$

for the left hand side) after being given a bijection σ between two i-element subsets of $[n]_0$ and $[n]_1$.

Suppose we are given two such subsets $R_0 \subseteq [n]_0$, $R_1 \subseteq [n]_1$ with

$$|R_0| = |R_1| = i$$

and a bijection

$$\sigma : R_0 \to R_1.$$

Our convention is to use the order-preserving bijections

$$[n-i]_\epsilon \to [n]_\epsilon \smallsetminus R_\epsilon \tag{1}$$

for the definition of tr_σ.

Consider the graph Γ with vertices $[[n]]$ and edges $\{i, \sigma(i)\}$, $\{j, b(j)\}$. The vertices of Γ have degree ≤ 2, so components can be individual vertices, (connected) cycles, or paths. First of all, we eliminate all (connected) cycles and replace each with a factor c (where $c \in \mathbb{C} \smallsetminus \mathbb{Z}$ is a number fixed throughout). Let s be the number of such cycles.

Paths from $[n]_0$ to $[n]_1$ can be identified with the data of subsets $\widehat{R_0} \subseteq [n]_0$, $\widehat{R_1} \subseteq [n]_1$ and a bijection $\widehat{\sigma} : \widehat{R_0} \to \widehat{R_1}$.

A path from $[n]_\epsilon$ to $[n]_\epsilon$ ends with a σ-edge on one side and a b-edge on the other side. Thus, from these paths, we can extract sets $\overline{R_\epsilon} \subseteq [n]_\epsilon$, $\overline{R_\epsilon} \cap \widehat{R_\epsilon} = \emptyset$ and injections

$$\rho_\epsilon : \overline{R_\epsilon} \to [n]_\epsilon \smallsetminus \widehat{R_\epsilon}$$

which send the σ-end of the path to the b-end.

Definition 4. *Call an element of* $\mathbb{S}_n$*, i.e. a collection of data*

$$\Phi = (\mathscr{T}(1), \dots, \mathscr{T}(n), \chi, \beta, W_0, W_1, b),$$

matchable with resepct to σ *if for* $x \in W_0 \cap \widehat{R_0}$*,* $y \in W_1 \cap \widehat{R_1}$*,* $\widehat{\sigma}(x) = y$ *implies* $\beta(x) = y$*, and for all* $T \in \mathscr{T}(k)$ *one of the following is true:*

(1) *There exists* $T' \neq T \in \mathscr{T}(k)$ *such that* $T \cap \widehat{R_0} \neq \emptyset$ *or* $T \cap \widehat{R_1} \neq \emptyset$*,*

$$\widehat{\sigma}(T \cap \widehat{R_0}) \subseteq (T' \cap \widehat{R_1})$$

$$\widehat{\sigma}^{-1}(T \cap \widehat{R_1}) \subseteq (T' \cap \widehat{R_0}).$$

Note that the conditions imply that the above formulae must also then be true for T *and* T' *switched and that* T' *is unique.*

(2) *We have* $T \cap \widehat{R_0} = \emptyset$ *and* $T \cap \widehat{R_1} = \emptyset$*.*

If $\Phi \in \mathbb{S}_n$ is not matchable with respect to σ, put

$$tr_\sigma(\Phi) = 0.$$

We shall now define $tr_\sigma(\Phi)$ in the case when $\Phi \in \mathbb{S}_n$ is matchable.

Let $\widehat{W_\epsilon}$ be obtained from W_ϵ by deleting any source (resp. target) elements of $\widehat{\sigma}$ and replacing $x \in W_\epsilon \cap \overline{R_\epsilon}$ by $\widehat{x} = \rho_\epsilon(x)$ and define $\widehat{\beta}$ by taking β and replacing an element x of its source (resp. target) by $\widehat{X}$ when applicable. Similarly, for each $T \in \mathscr{T}(k)$, let $\widehat{T}$ be obtained by replacing each $x \in T \cap \overline{R_\epsilon}$ by $\rho_\epsilon(x)$.

Replace each $T \in \mathscr{T}(k)$ satisfying Case 2 of Definition 4 by $\widehat{T}$. Let $\widetilde{\mathscr{T}(k)}$ be the set of all such $\widehat{T}$, and put $\widetilde{\chi}(\widehat{T}) = \chi(T)$.

Now let $\widehat{\mathscr{T}(k)}$ be the set of all unordered pairs $\{T, T'\} \subseteq \mathscr{T}(k)$ satisfying Case 1 of Definition 4. For such a pair $\{T, T'\}$, define

$$\begin{aligned} \beta_{\{T,T'\}} = q(\textstyle\sum(\gamma : (\widehat{T} \cap [n]_0) \smallsetminus \widehat{R_0} \xrightarrow{\cong} (\widehat{T'} \cap [n]_1) \smallsetminus \widehat{R_1}))\cdot \\ (\textstyle\sum(\gamma' : (\widehat{T'} \cap [n]_0) \smallsetminus \widehat{R_0} \xrightarrow{\cong} (\widehat{T} \cap [n]_1) \smallsetminus \widehat{R_1})). \end{aligned} \tag{2}$$

(In (2), we consider a bijection as a "product" of its pairs, the product is distributive with respect to sums). Define, also,

$$W_\epsilon^{\{T,T'\}} = ((\widehat{T} \cup \widehat{T'}) \cap [n]_\epsilon) \smallsetminus \widehat{R_\epsilon}.$$

Now let

$$\widetilde{W_\epsilon} = \widehat{W_\epsilon} \cup \bigcup_{k=1}^{n-i} \bigcup_{\{T,T\} \in \widehat{\mathscr{T}(k)}} W_\epsilon^{\{T,T'\}}$$

and

$$\widetilde{\beta} = \widehat{\beta} \cdot \prod_{k=1}^{n-i} \prod_{\{T,T'\} \in \widetilde{\mathscr{T}(k)}} \beta_{\{T,T'\}}.$$

Finally, let $\widetilde{b}$ be the restriction of $\widehat{\sigma}$ to

$$[n]_0 \smallsetminus \left(\bigcup_{k=1}^{n-i} \bigcup_{T \in \widetilde{\mathscr{T}(k)}} T \cup \widetilde{W_0} \right) \to [n]_1 \smallsetminus \left(\bigcup_{k=1}^{n-i} \bigcup_{T \in \widetilde{\mathscr{T}(k)}} T \cup \widetilde{W_1} \right).$$

Now we define

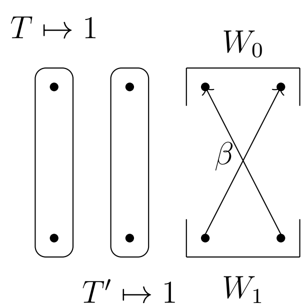
$$tr_\sigma(\Phi) = c^s \widetilde{\Phi}$$

where

$$\widetilde{\Phi} = (\widetilde{\mathscr{T}(1)}, \ldots, \widetilde{\mathscr{T}(n-i)}, \widetilde{\chi}, \widetilde{\beta}, \widetilde{W_0}, \widetilde{W_1}, \widetilde{b}),$$

using the idenitification (1).

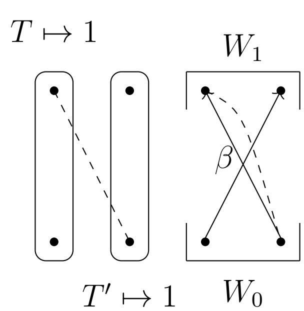
For example, the element Φ of $\mathbb{T}_4$ corresponding to the diagram



is matchable with respect to

$$\sigma : \{2,4\} \times \{0\} \to \{1,3\} \times \{1\}$$

given by $\sigma((2,0)) = (1,1)$, $\sigma((4,0)) = (3,1)$ (represented by the dotted lines):



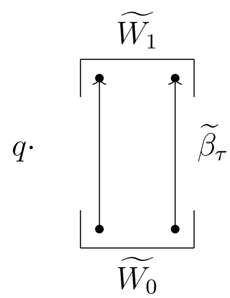
Then to find $tr_\sigma(\Phi)$ we delete the elements of the T's and W's connected by σ. In this case, Z_ϵ, $\overline{R_\epsilon}$ and $\widehat{R_\epsilon}$ are all empty. None of the $\widetilde{\mathscr{T}(k)}$'s will be non-empty, and all remaining points will belong to the new $\widetilde{W_\epsilon}$'s. There is only one unordered pair of sets $\{T, T'\} \in \widehat{\mathscr{T}(k)}$, and

$$W_0^{\{T,T'\}} = \{(1,0)\}$$
$$W_1^{\{T,T'\}} = \{(2,1)\},$$

with

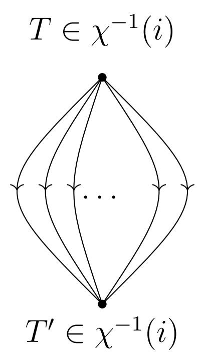
$$\beta_{\{T,T'\}} = q \cdot ((1,0) \mapsto (2,1))$$

Thus, $tr_\sigma(\Phi)$ can be visualized as



(the top row of points representing $\{(2,1),(4,1)\}$ and the bottom row of points representing $\{(1,0),(3,0)\}$).

Remark: The motivation of this definition comes from making traces of diagrams of the form



equal to q, and "introducing no other non-zero traces." The formalism of the set W is introduced to eliminate negligible elements that would arise from different values of i.