NOETHER'S PROBLEM FOR ORIENTATION p-SUBGROUPS OF SYMMETRIC GROUPS

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Abstract. We give a positive solution to Noether's rationality problem for certain index p subgroups of the p-Sylow subgoups of symmetric groups.

1. Introduction

Let G be a finite group acting linearly on an affine space over $\mathbb{C}$. Emmy Noether [20] asked whether the quotient variety is always rational. A counterexample was found by Saltman [22] (see also Bogomolov [2]), using an invariant called the unramified Brauer group. In a different form, this invariant was first discovered by Artin and Mumford [1] who used it to give a different example of unirational varieties which are not rational. (For a smooth projective variety over $\mathbb{C}$, the unramified Brauer group is isomorphic to the torsion in its third singular cohomology group.) Colliot-Thélène and Ojanguren [12] and Peyre [21] found more examples of unirational non-rational varieties using a more general invariant called unramified cohomology. The question for which groups G Noether's problem has a positive solution is still widely open except in some special cases. For abelian groups, a positive solution was given by Fischer [13]. For groups of order p^n, $n \leq 4$, it was solved positively by Chu and Kang [7]. It is proved in [17], Theorem 1.4, that for p-Sylow subgroups of symmetric groups, Noether's problem also has a positive solution (see [5, 4]), and some other cases are also known [19, 8, 9, 10, 11]. The most famous case of Noether's problem is $G = A_n$ (the rationality of the discriminant variety). It is known to be true for $n \leq 5$ (the most interesting case is Maeda's Theorem [19] for $n = 5$), but is still open for $n \geq 6$. It was proved by Bogomolov and Petrov [6] that unramified cohomology is 0 in this case.

In this note, we consider Noether's problem for a class of index p subgroups of p-Sylow subgroups of symmetric groups. For our purposes, it is convenient to choose certain particular generators of those subgroups. A p-Sylow subgroup of the symmetric group Σ_n on $\{1, \dots, n\}$ has generators $\sigma_{i,s}$ which cyclically permute p consecutive blocks of p^i

consecutive numbers $1, \dots, n$, starting with a number congruent to 1 mod p^{i+1}. The group $H_{n,p}$ is the index p subgroup generated by $\sigma_{0,s}\sigma_{0,s'}^{-1}$ and $\sigma_{i,s}$ for $i > 0$. For more detail on these groups, see Section 2 below.

The group $H_{8,3}$ is the 3-Sylow subgroup of the group of positions of the corners of Rubik's cube. Because of this, we call $H_{n,p}$ the *orientation p-subgroup of* Σ_n.

The following is our main theorem.

Theorem 1. *For a field F of characteristic 0 containing p'th roots of unity, let $H_{n,p}$ act on*

$$F(x_1, \dots, x_n)$$

by permuting the x_i's. Then for every prime $p \geq 2$, the field of fixed points $F(x_1, \dots, x_n)^{H_{n,p}}$ is rational over F.

Note that $H_{n,2}$ is a 2-Sylow subgroup of the alternating group A_n. Therefore, one obtains as a corollary a simple proof of the result of Bogomolov and Petrov [6]:

Theorem 2. *([6]) We have $H^k_{nr}(\mathbb{C}(a_1, \dots, a_n)^{A_n}, \mathbb{Z}/\ell) = 0$.*

For $p = 2$, there is an elementary proof of Theorem 1 based on the fact by letting the commuting elements $\sigma_{1,s}$ act on the (twisted) torus corresponding to the lattice D_m^* (dual to the root lattice of type D_m algebraic groups), the quotient variety is birationally equivalent to the product of an affine space with another such (twisted) torus equivariantly with respect to the permutations $\sigma_{i,s}$ for $i > 1$. This, in turn, is basically due to the simple fact that $\Sigma_2 = \mathbb{Z}/2$. (See Comment below the proof of Lemma 8 in Section 3 below.) Since this proof is short and easy, we present it first in Section 3 below.

For $p > 2$, the proof of Theorem 1 is given in Section 4. It is based on the following result (used in the proof of Lemma 11 in Section 4 below):

Lemma 3. *If F contains p'th roots of unity, and on $F(x_1, \dots, x_p)$, $\mathbb{Z}/p$ acts by*

$$x_1 \mapsto x_2 \mapsto \dots x_p \mapsto x_1, \tag{1}$$

then $F(x_1, \dots, x_p)^{\mathbb{Z}/p}$ is rational over $F(w)$ where w denotes the product $x_1 \cdot \dots \cdot x_p$.

This Lemma is a corollary of the following result of Hajja:

Lemma 4. *([15], Lemma 2 (iv), p.244) Let p be an odd prime number, k a field of characteristic not equal to p. Assume that k contains a* *primitive p'th root of unity. If $a \in k^\times$, and σ acts on the rational function field $k(x_1, \dots, x_{p-1})$ by*

$$\sigma : x_1 \mapsto x_2 \mapsto \cdots \mapsto x_{p-1} \mapsto a/(x_1 \dots x_{p-1}),$$

then the fixed field $k(x_1, \dots, x_{p-1})^{\langle\sigma\rangle}$ is rational over k.

This implies Lemma 3 by setting $k = F(w)$ (considered as a subfield of $F(x_1, \dots, x_p)$), $a = w$.

For example for $p = 3$, consider the cubic equation

$$x^3 + bx^2 + cx + w = 0.$$

If we denote by D the discriminant

$$D = 18bcw - 4b^3w + b^2c^2 - 4c^3 - 27w^2,$$

Lemma 3 says that $F(b, c, w)(\sqrt{D})$ is rational over $F(w)$. To make the proof of Theorem 1 self-contained, we give a proof of Lemma 3 in Section 4 below.

We also note that there is an easy analogue of Theorem 1 for a larger group. Consider the homomorphism

$$\phi : \Sigma_n \wr \mathbb{Z}/p \to \mathbb{Z}/p$$

given on the normal subgroup $(\mathbb{Z}/p)^n$ by adding the coordinates of each element, and trivial on the subgroup Σ_n which acts on $(\mathbb{Z}/p)^n$ by permutation of factors. Explicitly, representing an element of $\Sigma_n \wr \mathbb{Z}/p$ as a tuple $(a_1, \dots, a_n; \tau)$ where $a_i \in \mathbb{Z}/p$, $\tau \in \Sigma_n$, we put

$$\phi(a_1, \dots, a_n; \tau) := \sum_{i=1}^{n} a_i.$$

Let $K_{n,p} = Ker(\phi)$. Then $H_{np,p}$ is a p-Sylow subgroup of $K_{n,p}$.

Theorem 5. *For all n and p, the field $\mathbb{C}(x_1, \dots, x_{np})^{K_{n,p}}$ where $K_{n,p}$ acts by permutation of variables is rational.*

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2. Conventions and notation

The main purpose of this section is to explain in detail the construction of the groups $G_{n,p}$, $H_{n,p}$, and our notation for the generators.

Lemma 6. *A p-Sylow subgroup $G_{n,p} \subset \Sigma_n$ (the group of all permutations on $\{1, \dots, n\}$) is generated by permutations $\sigma_{i,s}$ consisting of the cycles*

$$(s-1)p^{i+1}+j \mapsto (s-1)p^{i+1}+p^i+j \mapsto \cdots \mapsto (s-1)p^{i+1}+(p-1)p^i+j \mapsto (s-1)p^{i+1}+j$$

for $1 \leq j \leq p^i$. (The remaining elements are fixed.) The parameters i, s range over integers satisfying

$$1 \leq s \leq \lfloor \frac{n}{p^{i+1}} \rfloor,\ 0 \leq i \leq \lfloor \log_p(n) \rfloor - 1. \tag{2}$$

Proof. It is well known that for $n = \ell_0 + \ell_1 p + \cdots + \ell_m p^m$, where $0 \leq \ell_i \leq p-1$ are integers and we have $m = \lfloor \log_p(n) \rfloor$ (this implies $\ell_m > 0$), a p-Sylow subgroup of Σ_n is isomorphic to

$$\prod_{i=0}^{m-1} (\underbrace{\mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p}_{i+1})^{\ell_{i+1}}.$$

Now, by definition, the generators

$$\sigma_{m-k,1}, \dots, \sigma_{m-k,p^{k-1}},\ k = 1, \dots, m \tag{3}$$

generate a group isomorphic to $G_{p^m,p}$ and commute with all the other generators. We will show that

$$G_{p^m,p} \cong \underbrace{\mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p}_{m}. \tag{4}$$

Therefore, we obtain an isomorphism

$$G_{n,p} \cong (\underbrace{\mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p}_{m}) \times G_{n-p^m,p},$$

by sending the generators (3) to the generators of the first factor, and the remaining generators by

$$\sigma_{m-k,s} \mapsto \sigma_{m-k,s-p^{k-1}},$$

for $s > p^{k-1}$. Then our statement follows by induction on n.

To show (4), recall that for a permutation group G on ℓ elements and any group H, the wreath product $G \wr H$ is defined as the semidirect product $G \ltimes (H^\ell)$ where the action of G on H^ℓ is by permutation of factors. We can prove (4) by induction on m. The statement is obvious for $m = 1$. Now by definition, the generators

$$\sigma_{m-k,1}, \dots, \sigma_{m-k,p^{k-1}},\ k = 2, \dots, m \tag{5}$$

generate a group isomorphic to

$$(G_{p^{m-1},p})^p. \tag{6}$$

Now for an element $(\alpha_1, \dots, \alpha_p) \in (G_{p^{m-1},p})^p$ presented in the generators (5), we have

$$\sigma_{m-1,1} \cdot (\alpha_1, \dots, \alpha_p) \cdot \sigma_{m-1,1}^{-1} = (\alpha_p, \alpha_1 \dots, \alpha_{p-1}).$$

Additionally, for $i, j \in \{0, \dots, p-1\}$, $(\alpha_1, \dots, \alpha_p), (\beta_1, \dots, \beta_p) \in G_{p^{m-1},p}$, we clearly have

$$\sigma_{m-1,1}^i \cdot (\alpha_1, \dots, \alpha_p) = \sigma_{m-1,1}^j \cdot (\beta_1, \dots, \beta_p)$$

if and only if $i = j$ and $(\alpha_1, \dots, \alpha_p) = (\beta_1, \dots, \beta_p)$, thus gving an isomorphism

$$G_{p^m,p} \cong \mathbb{Z}/p \wr G_{p^{m-1},p}.$$

If (4) is true with m replaced by $m-1$, then by the induction hypothesis, $G_{p^{m-1},p}$ is isomorphic to

$$(\underbrace{\mathbb{Z}/p \wr \cdots \wr \mathbb{Z}/p}_{m-1}),$$

thus giving our statement. □

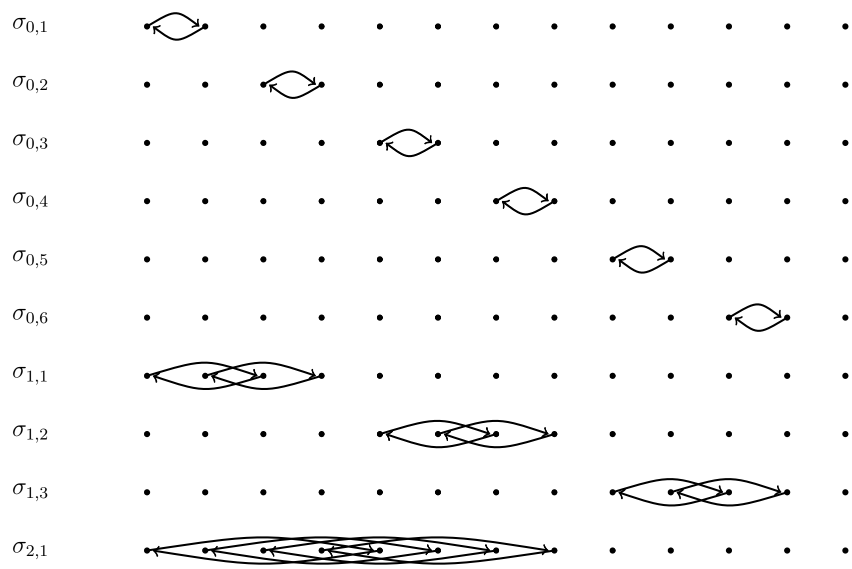
Example: Let $n = 1 + 2^2 + 2^3 = 13$, $p = 2$. Then

$$0 \le i \le 2 = \lfloor \log_2(13) \rfloor - 1.$$

For $i = 0$, we have $1 \le s \le 6 = \lfloor \frac{13}{2} \rfloor$. For $i = 1$, we have $1 \le s \le 3 = \lfloor \frac{13}{4} \rfloor$, and for $i = 2$, we have $1 \le s \le 1 = \lfloor \frac{13}{8} \rfloor$. Hence, there are $6 + 3 + 1 = 10$ generators of the form $\sigma_{i,s}$. For example, for $\sigma_{0,1}$, there is only $j = 1$, and the only cycle is

$$1 \mapsto 2 \mapsto 1.$$

The generators of $G_{13,2}$ can be visualised as follows.



Here $\sigma_{0,1}, \sigma_{0,2}, \sigma_{0,3}, \sigma_{0,4}, \sigma_{1,1}, \sigma_{1,2}, \sigma_{2,1}$ generate a copy of $\mathbb{Z}/2 \wr \mathbb{Z}/2 \wr \mathbb{Z}/2$ and $\sigma_{0,5}, \sigma_{0,6}, \sigma_{1,3}$ generate a copy of $\mathbb{Z}/2 \wr \mathbb{Z}/2$.

Now consider the normal subgroup $H_{n,p} \subset G_{n,p}$ generated by all the elements $\sigma_{0,s}\sigma_{0,s'}^{-1}$ where $1 \leq s, s' \leq \lfloor \frac{n}{p} \rfloor$, and $\sigma_{i,s}$ where (2) holds and $i \geq 1$.

Lemma 7. *Letting $G_{\lfloor n/p \rfloor,p}$ act on $(\mathbb{Z}/p)^{\lfloor \frac{n}{p} \rfloor}$ by permutation of factors, the group $H_{n,p}$ is isomorphic to a semidirect product of $G_{\lfloor \frac{n}{p} \rfloor,p}$ with the subgroup of $(\mathbb{Z}/p)^{\lfloor \frac{n}{p} \rfloor}$ of elements whose coordinates add up to 0. Additionally, $H_{n,p}$ is an index p subgroup of $G_{n,p}$.*

Proof. The subgroup A of $G_{n,p}$ generated by $\sigma_{0,i}$ is a normal subgroup isomorphic to $(\mathbb{Z}/p)^{\lfloor n/p \rfloor}$. Further, the quotient is isomorphic to $G_{\lfloor n/p \rfloor,p}$ (acting, instead of on single elements, on p-tuples of consecutive elements of $\{1, \ldots, p\lfloor n/p \rfloor\}$, each starting with a number 1 mod p). Thus, the short exact sequence

$$1 \to A \lhd G_{n,p} \to G_{\lfloor n/p \rfloor,p} \to 1$$

splits, and hence is a semidirect product. Now define a homomorphism $h : A = (\mathbb{Z}/p)^{\lfloor n/p \rfloor} \to \mathbb{Z}/p$ by adding the coordinates (in the category of abelian groups, this is the "codiagonal"). If we denote, for $\alpha \in G_{\lfloor n/p \rfloor,p}$ and $q \in A$, by q^α the image of q under the automorphism of A given by α, then we have

$$h(q^\alpha) = h(q).$$

Therefore, the homomorphism h extends uniquely to a homomorphism $\widetilde{h} : G_{n,p} \to \mathbb{Z}/p$ which is trivial on the subgroup $G_{\lfloor n/p \rfloor, p}$. The group $H_{n,p}$ is, by definition, the kernel of this homomorphism, which gives us a diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & Ker(h) & \longrightarrow & Ker(\widetilde{h}) & \longrightarrow & G_{\lfloor n/p \rfloor, p} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \cong & & \\
1 & \longrightarrow & A & \longrightarrow & G_{n,p} & \longrightarrow & G_{\lfloor n/p \rfloor, p} & \longrightarrow & 1 \\
 & & \downarrow h & & \downarrow \widetilde{h} & & & & \\
 & & \mathbb{Z}/p & \xrightarrow{\cong} & \mathbb{Z}/p & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

The columns are exact and the middle row is exact, hence so is the top row. Further, the middle row splits, and the composition of $\widetilde{h}$ with the splitting is 0. Thus, the splitting lifts to the top row. $\square$

Comment: It follows from our proof of Lemma 6 that an element of $G_{n,p}$ can be uniquely written as a product (in any fixed order) of powers $\sigma_{i,s}^{r(i,s)}$ with $r(i,s) \in \{0, \dots, p-1\}$, where i, s are as in (2). Therefore, the group order of $G_{n,p}$ is p to the power equal to the number of pairs of integers (i,s) satisfying (2), which is easily checked (by an induction mimicking that in the proof of Lemma 6) to be the maximum power of p dividing $n!$. Similarly, it follows from our proof of Lemma 7 that an element of the group $H_{n,p}$ can be written uniquely as a product (in any fixed order) of powers $\sigma_{i,s}^{r(i,s)}$ with $r(i,s) \in \{0, \dots, p-1\}$, where i, s are as in (2) such that

$$\sum_{s=1}^{\lfloor n/p \rfloor} r(0,s) \equiv 0 \mod p.$$

This also shows that $|H_{n,p}| = |G_{n,p}|/p$.

3. The case $p = 2$.

In this section, we prove Theorem 1 for $p = 2$, and Theorem 2.

Proof of Theorem 1 for $p = 2$: When n is odd, the variable x_n is fixed by $H_{n,2}$, so we can replace F by $F(x_n)$ and n by $n-1$. Thus, we may assume $n = 2m$ is even. We have $F(x_1, \dots, x_n) = F(z_1, \dots, z_m, t_1, \dots, t_m)$

$$z_i = x_{2i-1} + x_{2i}$$
$$t_i = x_{2i-1} - x_{2i}$$
$$1 \leq i \leq m.$$

Let $J_n \triangleleft H_{n,2}$ be the subgroup generated by the pairs of elements $\sigma_{0,s}\sigma_{0,s'}$, $1 \leq s, s' \leq m$. Then

$$F(x_1, \dots, x_n)^{J_n} = F(z_1, \dots, z_m, t_1^2, \dots, t_{m-1}^2, t_1 t_2 \cdots t_m).$$

The factor group $H_{n,2}/J_n$ is generated by $\sigma_{i,s}$ where $i \geq 1$ and formula (2) holds. Hence, it is isomorphic to the 2-Sylow subgroup $G_{m,2} \subseteq \Sigma_m$ by the isomorphism $\sigma_{i,s} \mapsto \sigma_{i-1,s} \in G_{m,2}$. The group $G_{m,2}$ acts on the z_i's and t_i's by permutation. Therefore we have reduced the proof to the following statement. □

Lemma 8. *Let F be a field of characteristic 0 and let $a^2 \in F$ (in other words, a is a square root of an element of F). Let E be the subfield of*

$$F[a](t_1, \dots, t_k, z_{1,1}, \dots, z_{k,1}, \dots, z_{1,\ell}, \dots, z_{k,\ell}) \tag{7}$$

generated by the (algebraically independent) elements

$$\begin{array}{l} t_1 \cdots t_k a, t_1^2, t_2^2, \dots, t_{k-1}^2, \\ z_{1,1}, z_{2,1}, \dots, z_{k,1}, \\ \vdots \\ z_{1,\ell}, z_{2,\ell}, \dots, z_{k,\ell} \end{array}$$

and let $G_{k,2}$ act on E by restriction of an action of Σ_k on (7) where Σ_k acts trivially on $F[a]$, and acts by permuting the indexes i in the t_i's and the $z_{i,j}$'s for a fixed j. Then

$$E^{G_{k,2}}$$

is rational over $F[a]$.

Comment: The additional variables $z_{i,j}$, and the element a, are introduced in order to enable a proof by induction.

Proof. Induction on k. If $k = 1$, $E^{G_{k,2}} = E$. If $k > 1$ is odd, $G_{k,2} = G_{k-1,2}$. Let $F' = F(t_k^2, z_{k,1}, \dots, z_{k,\ell})$, $a' = t_k a$, $t_i' = t_i$ for $i < k$, and $z_{i,j}$ for $i < k$, $j = 1, \dots, \ell$. (Note that $F'[a']$ is rational over $F[a]$.) This reduces the statement for k to the statement for $k - 1$. Thus, we may assume $k = 2m$ is even. Let $\Gamma_k \lhd G_{k,2}$ be the subgroup generated by the elements, $\sigma_{0,s}$, $1 \leq s \leq k$. Then

$$G_{k,2}/\Gamma_k \cong G_{m,2}.$$

We have

$$E^{G_{k,2}} = (E^{\Gamma_k})^{G_{k,2}/\Gamma_k}.$$

Further,

$$\begin{aligned} E^{\Gamma_k} = F[a](u_1u_2\cdots u_m a, u_1^2, u_2^2, \ldots, u_{m-1}^2, \\ v_1, v_2, \ldots, v_m, w_{1,j}, w_{2,j}, \ldots, w_{m,j}, \\ s_{1,j}, s_{2,j}, \ldots, s_{m,j} \mid j = 1, \ldots, \ell) \end{aligned} \tag{8}$$

for

$$u_1 = t_1t_2, u_2 = t_3t_4, \ldots, u_m = t_{2m-1}t_{2m},$$
$$v_1 = t_1^2 + t_2^2, v_2 = t_3^2 + t_4^2, \ldots, v_m = t_{2m-1}^2 + t_{2m}^2,$$
$$w_{1,j} = z_{1,j} + z_{2,j}, w_{2,j} = z_{3,j} + z_{4,j}, \ldots, w_{m,j} = z_{2m-1,j} + z_{2m,j},$$
$$s_{1,j} = (z_{1,j} - z_{2,j})(t_1^2 - t_2^2), \ldots, s_{m,j} = (z_{2m-1,j} - z_{2m,j})(t_{2m-1}^2 - t_{2m}^2).$$

In fact, clearly we have $\supseteq$ in (8). On the other hand, we see that the field E is an extension of the right hand side of (8) of degree $\leq 2^m$. (This is because we may use quadratic equations to solve for t_{2i-1}^2, t_{2i}^2 using u_i, v_i, $i = 1, \ldots m$, and then the $z_{i,j}$'s are recovered from linear equations.) Thus, equality in (8) follows from Galois theory.

Setting

$$t'_i = u_i$$
$$z'_{i,j} = w_{i,j}$$
$$z'_{i,j+n} = s_{i,j}$$
$$z'_{i,2n+1} = v_i,$$

$i = 1, \ldots, \ell$ reduces the statement to the induction hypothesis with k replaced by m.

□

Comment: The above proof came from the following idea: The D_k lattice (which is the root lattice of the type D_k Lie algebra, but that fact is of no consequence of us) consists of k-tuples of integers with an even sum. Note that $V = Spec((t_1 \cdots t_k)^{-1}R)$ where R is the subring of k-tuples of integers whose sum is even. $F[t_1, \ldots, t_k]$ generated by $t_1 \cdots t_k, t_1^2, t_2^2, \ldots, t_{k-1}^2$ can be identified with the torus over F corresponding to the dual lattice D_k^* of D_k. The variety $V_a = Spec((t_1 \cdots t_k)^{-1}R_a)$ where R_a is the subring of $F[a][t_1, \ldots, t_k]$ generated by $t_1 \cdots t_k a, t_1^2, t_2^2, \ldots, t_{k-1}^2$ is a principal homogeneous space of V. The induction is based on the fact that taking the quotient of V_a under the action of the abelian group Γ_k generated by $\sigma_{0,s}$, $1 \leq s \leq \lfloor k/2 \rfloor$, with the generators acting by permutation of coordinates (and trivially on $Spec(F[a])$), is birationally equivalent to a product of an affine space over F with a variety of the same kind with k replaced by $\lfloor k/2 \rfloor$. While we are not primarily interested in the case when $a \notin F$, considering this case is forced by the induction in the case of numbers k other than powers of 2.

Proof of Theorem 2: Let K be a function field over $\mathbb{C}$ and let $R \subset K$ be a DVR with field of fractions K. (Then we automatically have $\mathbb{C} \subseteq R$, since the valuation of any element of $\mathbb{C}^\times$ is infinitely divisible, and hence is 0 in $\mathbb{Z}$.) For any $\ell \in \mathbb{N}$, we have a *residue homomorphism*

$$\partial_R : H^n(K, \mathbb{Z}/\ell) \to H^{n-1}(k_R, \mathbb{Z}/\ell) \tag{9}$$

where k_R is the residue field of R. Colliot-Thélène and Ojanguren [12] define *unramified cohomology* by

$$H^n_{nr}(K, \mathbb{Z}/\ell) = \bigcap_R Ker(\partial_R).$$

They prove that unramified cohomology of rational fields vanishes in degrees ≥ 1.

For a finite extension $L \supseteq K$ and a DVR S with fraction field L containing R and residue field k', we have a commutative diagram (see [14], Section 8)

$$\begin{array}{ccc} H^n(K, \mathbb{Z}/\ell) & \xrightarrow{\partial_R} & H^{n-1}(k, \mathbb{Z}/\ell) \\ {\scriptstyle res}\downarrow & & \downarrow{\scriptstyle e\cdot res} \\ H^n(L, \mathbb{Z}/\ell) & \xrightarrow{\partial_S} & H^{n-1}(k', \mathbb{Z}/\ell) \end{array}$$

where e is the ramification index and res is the restriction map. This makes unramified cohomology functorial with respect to restriction:

$$res : H^n_{nr}(K, \mathbb{Z}/\ell) \to H^n_{nr}(L, \mathbb{Z}/\ell). \tag{10}$$

Now consider the norm (corestriction)

$$N : H^n(L, \mathbb{Z}/\ell) \to H^n(K, \mathbb{Z}/\ell).$$

We have

$$N \circ res = [L : K],$$

so we have proved the following

Lemma 9. *If $H^n_{nr}(L, \mathbb{Z}/\ell) = 0$, then $[L : K]$ annihilates $H^n_{nr}(K, \mathbb{Z}/\ell)$.*

Now consider $K = E^G$ where G is a finite group and E is a field of rational functions where G acts on the variables by permutations. If a group is annihilated by all primes, it is 0. Thus, we have proved

Lemma 10. *If $H^n_{nr}(E^{G_p}, \mathbb{Z}/\ell) = 0$ for all p-Sylow subgroups of G, then $H^n_{nr}(E^G, \mathbb{Z}/\ell) = 0$.*

In the case when $G = A_n$, the assumption of Lemma 10 for $p = 2$ is verified by Theorem 1. The p-Sylow subgroup of A_n for p odd is a p-Sylow subgroup $G_{n,p}$ of Σ_n. In this case, it is well known that $\mathbb{C}(x_1, \dots, x_n)^{G_{n,p}}$ is rational. (For a proof of this fact, see [17]. Briefly, taking the Fischer [13] generators of the fixed field of $\mathbb{C}(x_1, \dots, x_n)$ under the abelian subgroup generated by $\sigma_{0,s}$, the permutations $\sigma_{i,s}$ for $i \geq 1$ act on them also by permutation, while generating $G_{m,p}$ for some $m < n$. This gives a proof by induction.) This verifies the assumption of Lemma 10 for odd primes p, and hence concludes the proof of Theorem 2. □

4. The case $p > 2$.

In this section, we shall prove Theorem 1 for $p > 2$ and Theorem 5. We include a proof of Lemma 3, to make our proof of Theorem 1 self-contained.

Proof of Lemma 3: Denote the primitive pth root of unity in F by ζ_p. We will work in the field

$$L = F(x_1, \dots, x_p, w^{\frac{1}{p}}).$$

We have $\mathbb{Z}/p \times \mathbb{Z}/p = \mathbb{Z}/p\{g_1, g_2\}$ acting on L where g_1 acts by (1) (and trivially on $w^{\frac{1}{p}}$) and g_2 acts trivially on x_i, $i = 1, \dots, p$ and

$$g_2(w^{\frac{1}{p}}) = (\zeta_p)^{-1} w^{\frac{1}{p}}. \tag{11}$$

We have

$$K = L^{\mathbb{Z}/p\{g_1, g_2\}}.$$

Let

$$\bar{x_i} = \frac{x_i}{w^{\frac{1}{p}}}.$$

Then

$$L = F(\bar{x_1}, \dots, \bar{x_p}, w^{\frac{1}{p}})$$

and g_1 acts by

$$\bar{x_1} \mapsto \bar{x_2} \mapsto \dots \bar{x_p} \mapsto \bar{x_1},$$

while
$$\bar{x_1}\cdot\dots\cdot\bar{x_p}=1.$$
We will use the method of Chu and Kang [7] to describe $L^{\mathbb{Z}/p\{g_1\}}$. Let
$$u_i=1+\zeta_p^i\bar{x}_1+\zeta_p^{2i}\bar{x}_i\bar{x}_2+\dots+\zeta_p^{i(p-1)}\bar{x}_1\cdot\dots\cdot\bar{x}_{p-1},$$
$i=0,\dots,p-1$. Then
$$g_1(u_i)=\zeta_p^{-i}\frac{u_i}{\bar{x_1}}.$$
Therefore,
$$L^{\mathbb{Z}/p\{g_1\}}=F(w^{\frac{1}{p}})(u_0^pu_1^{-p},u_0^{-1}u_1^2u_2^{-1},u_1^{-1}u_2^2u_3^{-1},\dots,u_{p-3}^{-1}u_{p-2}^2u_{p-1}^{-1}).$$
Let
$$v_i=u_i^{-1}u_{i+1}^2u_{i+2}^{-1},\ i=0,\dots,p-2.$$
Note that
$$v_{p-2}^1v_{p-3}^2\cdot\dots\cdot v_0^{p-1}=u_0^{-p}u_1^p.$$
Therefore,
$$L^{\mathbb{Z}/p\{g_1\}}=F(w^{\frac{1}{p}})(v_0,\dots,v_{p-2})$$
where g_2 acts by (11) and
$$v_0\mapsto v_1\mapsto\dots\mapsto v_{p-2}\mapsto\frac{1}{v_0v_1\cdot\dots\cdot v_{p-2}}\mapsto v_0.$$
Let
$$z=1+v_0+v_0v_1+\dots+v_0\cdot\dots\cdot v_{p-2}.$$
Then
$$L^{\mathbb{Z}/p\{g_1\}}=F(w^{\frac{1}{p}})(\frac{1}{z},\frac{v_0}{z},\frac{v_0v_1}{z},\dots,\frac{v_0\cdot\dots\cdot v_{p-3}}{z})$$
while g_2 acts by (11) and
$$\frac{1}{z}\mapsto\frac{v_0}{z},\frac{v_0v_1}{z}\mapsto\dots$$
$$\mapsto\frac{v_0\cdot\dots\cdot v_{p-3}}{z}\mapsto 1-(\frac{1}{z}+\frac{v_0}{z}+\frac{v_0v_1}{z}+\dots+\frac{v_0\cdot\dots\cdot v_{p-3}}{z})\mapsto\frac{1}{z}.$$
Since $\mathbb{Z}/p\{g_2\}$ acts faithfully on $F(w^{\frac{1}{p}})$, by Theorem 1 of Hajja and Kang [16], there exist $\bar{z}_1,\dots,\bar{z}_{p-1}\in L^{\mathbb{Z}/p\{g_1\}}$ such that
$$L^{\mathbb{Z}/p\{g_1\}}=F(w^{\frac{1}{p}})(\bar{z}_1,\bar{z}_2,\dots,\bar{z}_{p-1})$$
and
$$g_2(\bar{z}_i)=\bar{z}_i.$$
Therefore,
$$K=F(w)(\bar{z}_1,\bar{z}_2,\dots,\bar{z}_{p-1}).$$
□

Using Lemma 3, we now prove the following

Lemma 11. *Let F be a field of characteristic 0 and let $\zeta_p \in F$, $a^p \in F$. Let $E = F(t_1 \cdot \cdots \cdot t_k a, t_1^p, t_2^p, \ldots, t_{k-1}^p)$ where $G_{k,p}$ acts on $t_1, \ldots, t_k$ by permutation. Then*

$$E^{G_{k,p}}$$

is rational over F.

Proof. Induction on k. If $k < p$, $E^{G_{k,p}} = E$. If $k = mp + i$ where $m > 0$, $i = 1, \ldots, p-1$, $G_{k,p} = G_{mp,p}$. Let

$$F' = F(t_{mp+1}^p, \ldots, t_{mp+i}^p),$$

$a' = t_{mp+1} \cdot \cdots \cdot t_{mp+i} a, t'_i = t_i$ for $i = 1, \ldots, mp$. This reduces us from k to mp. Thus assume $k = mp$. Let $\Gamma_k \triangleleft G_{k,p}$ be the subgroup generated by the p-cycles $\sigma_{0,s}$. Then

$$G_{k,p}/\Gamma_k \cong G_{m,p}.$$

We have

$$E^{G_{k,p}} = (E^{\Gamma_k})^{G_{k,p}/\Gamma_k}.$$

For $i = 1, \ldots, m$, apply Lemma 3 to

$$x_1 = t_{(i-1)p+1}^p, \ldots, x_p = t_{ip}^p.$$

Let

$$u_i = t_{(i-1)p+1} \cdot \cdots \cdot t_{ip}.$$

By Lemma 3,

$$F(t_{(i-1)p+1}^p, \ldots, t_{ip}^p)^{\mathbb{Z}/p\{\sigma_{0,i}\}} = F(u_i^p, z_{i,1}, \ldots, z_{i,p-1}).$$

Therefore,

$$\begin{aligned} E^{\Gamma_k} = F(&u_1 u_2 \cdot \cdots \cdot u_m a,\\ &u_1^p, u_2^p, \ldots, u_{m-1}^p,\\ &z_{1,1}, \ldots, z_{1,p-1},\\ &\vdots\\ &z_{m,1}, \ldots, z_{m,p-1}). \end{aligned}$$

Now $G_{m,p}$ acts on the $z_{i,j}$'s by permutation of the i index and faithfully on

$$K = F(u_1 u_2 \cdot \cdots \cdot u_m a, u_1^p, \ldots, u_{m-1}^p).$$

By Theorem 1 of [16],

$$\begin{aligned} E^{\Gamma_k} = K(&z'_{1,1}, \ldots, z'_{1,p-1},\\ &\vdots\\ &z'_{m,1}, \ldots, z'_{m,p-1}) \end{aligned}$$

where $G_{m,p}$ fixes the generators $z'_{i,j}$. Thus, our statement follows from the induction hypothesis with F replaced by

$$F(z'_{1,1},\dots,z'_{1,p-1},\\ \vdots \\ z'_{m,1},\dots,z'_{m,p-1}).$$

$\square$

Proof of Theorem 1 for $p>2$: Let $n=mp+i$ where $m>0$, $i=0,\dots,p-1$. Let

$$t_j = x_{(j-1)p+1}+\zeta_p^{-1}x_{(j-1)p+2}+\dots+\zeta_p^{1-p}x_{jp},\ j=1,\dots m. \tag{12}$$

Then the action of $H_{n,p}$ on $F(x_1,\dots,x_n)$ restricts to an action on the subfield

$$L=F(t_1,t_2,\dots,t_m)$$

where $\sigma_{0,k}$ acts by $t_k\mapsto \zeta_p t_k$, and $t_j\mapsto t_j$ for $j\neq k,\ k=1,\dots,m$. The generators $\sigma_{\ell,s}$ for $\ell>0$ act by permutation on the t_k's.

Additionally,

$$F(x_1,\dots,x_{mp+i}) = L(x_j \mid j=1,\dots,n,\ p\nmid j) \tag{13}$$

and the action of $H_{n,p}$ on the generators on the right hand side of (13) is affine over L. By Theorem 1 of Hajja and Kang [16], $F(x_1,\dots,x_n)^{H_{n,p}}$ is rational over $L^{H_{n,p}}$. Thus, we may restrict attention to the action of $H_{n,p}$ on L. In particular, without loss of generality, $i=0$ and $n=mp$. Now if we denote by $J_n\lhd H_{n,p}$ the subgroup generated by $\sigma_{0,s}\sigma_{0,s'}^{-1}$ then

$$F(t_1,t_2,\dots,t_m)^{J_n} = F(t_1\cdot\dots\cdot t_m, t_1^p,\dots,t_{m-1}^p).$$

We have $H_{n,p}/J_n\cong G_{m,p}$, so our statement follows from Lemma 11 with $a=1$. $\square$

Proof of Theorem 5: Similarly as in our proof of Theorem 1 for $p>2$, by Theorem 1 of Hajja and Kang [16], we may again instead consider the action of $K_{n,p}$ on the subfield $L=\mathbb{C}(t_1,\dots,t_n)$ where the t_j's are defined by (12). Then Σ_n acts by permutation, and $\sigma_{0,i}$ acts by $t_i\mapsto\zeta_p t_i$, $t_j\mapsto t_j$ for $j\neq i$. Thus, we must prove that

$$\mathbb{C}(t_1^p,t_2^p,\dots,t_{n-1}^p,t_1\cdot\dots\cdot t_n)^{\Sigma_n} \tag{14}$$

is rational. However, (14) is the field of rational functions on the generators

$$\sigma_n(t_1,\dots,t_n),\\ \sigma_i(t_1^p,\dots,t_n^p),\ i=1,\dots,n-1$$

where σ_i are the elementary symmetric polynomials. □

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