ON WEIL RECIPROCITY IN MOTIVIC COHOMOLOGY

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Abstract. Using Voevodsky's derived category of motives, we prove a reciprocity law in motivic cohomology of a smooth projective morphism of dimension 1 over a smooth scheme over a perfect field.

1. Introduction

The norm map in Milnor K-theory was defined by Bass-Tate [3] and Kato [10] by a reciprocity law stating that the sum of the norms of the residues of a given element of the Milnor K-theory of the function field of $\mathbb{P}^1_k$ at closed points is 0 where k is a given field. This reciprocity also holds for any smooth projective curve C over k.

To be precise, Milnor K-theory of a field F is defined as

$$K_M^*(F) = T(F^\times)/(a \otimes (1-a), a \in F \smallsetminus \{0,1\}),$$

where T denotes the tensor algebra. We have residue homomorphisms

$$\partial_R : K_M^n(F) \to K_M^{n-1}(k_R)$$

for a discrete valuation ring R with residue field $k_R = R/m_R$. As explained in Bass-Tate [3], Ch I. §5 and Kato [10], the homomorphism ∂_R is uniquely determined by

$$\partial_R(\alpha_1, \dots, \alpha_n) = v_R(\alpha_n)(\alpha_1 \bmod m_R, \dots, \alpha_{n-1} \bmod m_R) \tag{1}$$

for $\alpha_1, \dots, \alpha_{n-1} \in \mathcal{O}_R^\times, \alpha_n \in F^\times$, and we write $(\alpha_1, \dots, \alpha_n)$ for the image $\alpha_1 \otimes \dots \otimes \alpha_n$ in $K_M^n(F)$. The papers [3, 10] also defined, for a finite extension E of a field k, a norm homomorphism

$$N_{E/k} : K_M^n(E) \to K_M^n(k),$$

which is uniquely determined by

$$\sum_{p \in \mathbb{P}^1_k} N_{k_p/k} \partial_{R_p}(\alpha) = 0 \in K_M^{n-1}(k) \tag{2}$$

for all $\alpha \in K_M^n(k(x))$ where p runs through closed points of $\mathbb{P}^1_k$.

Theorem 1. *([8], Proposition 7.4.4) Let k be a field, C be a smooth projective curve over k, and $L = K(C)$ be the function field of C. Let $\alpha \in K_M^n(L)$. Then*

$$\sum_{p \in C} N_{k_p/k} \partial_{R_p}(\alpha) = 0 \in K_M^{n-1}(k),$$

where R_p is the discrete valuation ring with field of fractions L associated with a closed point $p \in C$.

This can be proved by choosing a finite morphism $C \to \mathbb{P}^1_k$ and using the functoriality of the norm homomorphism with respect to finite field extensions.

Kato [10] (Proof of Corollaries in §2.4, p. 647) remarked that a similar statement is also true in Quillen K-theory. In a recent paper [13], E. Musicantov and A. Yom Din proved reciprocity laws in Quillen K-theory for varieties of higher-dimension.

The purpose of this note is to generalize Theorem 1 to a statement about motivic cohomology of smooth schemes over a field. This work originated from a suggestion of E. Musicantov and A. Yom Din [12] to interpret Theorem 1 in terms of motivic cohomology.

Suppose k is a perfect field. Let $f : C \to S$ be a smooth projective morphism of dimension 1 where C, S are smooth varieties over $Spec(k)$. Let $T \subseteq C$ be a reduced closed subscheme of codimension 1 which is generic over S (i.e. $f(Spec(K(Q))) = Spec(K(S))$ for every irreducible component Q of T). When T is smooth over S, we can denote the composition

$$H^{p,q}(C \smallsetminus T) \xrightarrow{\Delta} H^{p+1,q}(C, C \smallsetminus T) \xrightarrow{\gamma} H^{p-1,q-1}(T)$$

where Δ is the connecting map of the long exact sequence in motivic cohomology of the pair $(C, C \smallsetminus T)$ and γ is the Gysin map ([7] Ch. 5, also studied by Déglise [6]), by

$$\partial_T : H^{p,q}(C \smallsetminus T) \to H^{p-1,q-1}(T).$$

In Definition 15 below, we will define, for a finite morphism $g : T \to S$ of smooth schemes over $Spec(k)$, a norm map

$$N_g = N_{T/S} : H^{p,q}(T) \to H^{p,q}(S).$$

(We note the these norms are not directly related to the norms considered by T. Bachmann and M. Hoyois in [1]. Our norms are additive and hence are group homomorphisms, while the multiplicative norms considered in [1] are not.)

Denote $\Phi = N_{T/S} \circ \partial_T$. When we do not assume that T is smooth over k, we can generalize this construction using higher Chow groups as follows: [11] Part 5 prove that we have

$$H^{p,q}(X) = CH_{dim(X)-q}(X, 2q-p)$$

(see Definition 16 below), where the right hand side denotes higher Chow groups. Now Bloch [4] defines a long exact sequence

$$\begin{array}{ccc} \ldots \longrightarrow CH_r(T,m) \longrightarrow & CH_r(C,m) & \\ & \downarrow & \\ & CH_r(C \smallsetminus T, m) & \\ & \downarrow \gamma & \\ & CH_r(T, m-1) & \longrightarrow \ldots \end{array} \tag{3}$$

Also, Chow groups are covariantly functorial with respect to finite morphisms. Thus, we can define Φ as the composition

$$\begin{array}{ccc} H^{p,q}(C \smallsetminus T) & \xrightarrow{=} & CH_{dim(S)+1-q}(C \smallsetminus T, 2q-p) \\ \Big\downarrow \Phi & & \Big\downarrow \gamma \\ & & CH_{dim(S)+1-q}(T, 2q-p-1) \\ & & \Big\downarrow g_* \\ H^{p-1,q-1}(S) & \xleftarrow{=} & CH_{dim(S)+1-q}(S, 2q-p-1). \end{array}$$

The main result of this paper is the following

Theorem 2. *We have*

$$0 = \Phi : H^{p,q}(C \smallsetminus T) \to H^{p-1,q-1}(S).$$

In the case when $S = Spec(k)$, $L = K(C)$, an element of $K^n_M(L) = H^{n,n}(Spec(L))$, by definition, has non-zero residues at only finitely many closed points $p_1, \ldots, p_m \in C$, and Theorem 1 is a special case by letting $T = \{p_1, \ldots, p_m\} \subseteq C$ for $p = q = n$. I prove at the end of Section 3 that the direct definitions of ∂_p, $N_{E/k}$ agree with the definitions using motivic cohomology.

Note also that for C as in Theorem 2, there in general does not exist a finite morphism over S into $\mathbb{P}^1_S$ (although it exists locally), and thus the proof of [8] for the case of a point does not immediately generalize to the present setting. For completeness, we give a simple example in the Appendix.

In Section 2 below, I review the relevant foundations of the derived category of motives, mainly following the book of Mazza, Voevodsky, and Weibel [11]. The texts by Voevodsky [15] and Friedlander, Suslin, Voevodsky [7] are also used. In Section 3, I prove the main result.

2. Preliminaries: Motives and Higher Chow Groups

In this section, we will give some definitions of concepts relating to motives that will be used in the proof of Theorem 1. See [7, 11, 15] for additional background information.

First, we will define the additive category $SmCor_S$ for S a smooth scheme (we always mean of finite type) over a field k.

Definition 3. *The objects of $SmCor_S$ are smooth schemes over S. For X, Y smooth schemes over S, $SmCor_S(X, Y)$ is defined to be the group of algebraic cycles on $X \times_S Y$ whose support is finite over X (i.e. proper over X of relative dimension 0). Composition is defined in (*[11]*, Appendix* 1*A).*

Now we will define a notion that will be crucial for the remainder of this paper.

Definition 4. *For S a smooth scheme over a field k, define a* presheaf with transfers on S*, or a PST, to be an additive functor*

$$F : SmCor_S^{Op} \to Ab$$

(where Ab is the category of abelian groups and C^{Op} denotes the opposite category of a category C).

Now, define $PST(S)$ to be the category which has presheaves with transfers as objects, and natural transformations between them as morphisms. We now give the definition of a particularly useful example of a presheaf with transfers:

Definition 5. *Let S be a smooth scheme over k, and X be a smooth scheme over S. The* presheaf with transfers $\mathbb{Z}_{tr}(X)$ represented by X *is defined by, for $U \in SmCor_S$,*

$$\mathbb{Z}_{tr}(X)(U) = SmCor_S(U, X).$$

As noted in [15], Section 2, $\mathbb{Z}_{tr}(?)$ extends canonically to a functor from presheaves of sets to presheaves with transfers, which is left adjoint to the forgetful functor.

Let us define, for a presheaf with transfers F over S, $Lres(F)$ as the canonical left resolution (in the abelian category of presheaves with transfers) of F by direct sums of presheaves with transfers over S of the form $\mathbb{Z}_{tr}(X)$. Then we get a natural quasi-isomorphism

$$Lres(F) \to F.$$

We have

Lemma 6. *([15], Lemma 3.1) Let F, G be presheaves of sets over S. Then there is a natural isomorphism*

$$\mathbb{Z}_{tr}(F \times_S G) = \mathbb{Z}_{tr}(F) \otimes \mathbb{Z}_{tr}(G).$$

□

Lemma 7. *([15], Lemma 3.3) Let K, K', L be complexes of presheaves with transfers of the form $\bigoplus_i \mathbb{Z}_{tr}(X_i)$ and $K \to K'$ be a quasi-isomorphism. Then $K \otimes L \to K' \otimes L$ is a quasi-isomorphism.*

□

Let K, L be complexes of presheaves with transfers. Then define

$$K \boxtimes L := Lres(K) \otimes Lres(L).$$

By Lemma 7, $\boxtimes$ is the left derived functor of $\otimes$ in the abelian category $PST(S)$.

Following [11], define

$$\Delta^n = Spec(k[x_0, \ldots, x_n]/(\sum_{i=0}^{n} x_i = 1).$$

The jth face map $\partial_j : \Delta^n \to \Delta^{n+1}$ is given by the equation $x_j = 0$.

Definition 8. *([11] Definition 2.14) Suppose F is a presheaf with transfers on S. Define C_*F to be the chain complex of presheaves with transfers on S that takes U to*

$$\ldots \longrightarrow F(U \times \Delta^2) \longrightarrow F(U \times \Delta^1) \longrightarrow F(U) \longrightarrow 0 \ ,$$

where the arrows are alternating sums of face maps.

Definition 9. *([11] Definition 3.1) Define $\mathbb{Z}(1)$ to be the totalization of the double chain complex of presheaves with transfers over S*

$$C_*\mathbb{Z}_{tr}(S) \to C_*\mathbb{Z}_{tr}((\mathbb{G}_m)_S)$$

(where $\mathbb{Z}_{tr}(S)$ is in bidegree $(0,0)$). Define

$$\mathbb{Z}(n) = \underbrace{\mathbb{Z}(1) \boxtimes \cdots \boxtimes \mathbb{Z}(1)}_{n \text{ times}}. \tag{4}$$

(In fact, we could use $\otimes$ instead of $\boxtimes$ in (4), but often, $\mathbb{Z}(n)$ is considered as an object of the derived category of presheaves with transfers, in which case we must keep in mind that $\otimes$ does not preserve equivalences.)

Now we will discuss motives:

Definition 10. *A presheaf with transfers F over S is called a Nisnevich sheaf with transfers when its underlying presheaf is a sheaf on the category of smooth schemes over S with the Nisnevich topology (see* [11]*, Lecture 13).*

Now let $Sh_{Nis}(SmCor_S)$ be the category of Nisnevich sheaves with transfers over S. Let D^- be the derived category of bounded below (as chain complexes) complexes of Nisnevich sheaves with transfers with respect to quasi-isomorphisms.

Definition 11. *Let $\mathcal{E}_{\mathbb{A}}$ be the smallest full additive subcategory of D^-*

(1) *which contains the cone of all morphisms of the form*
$$\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}(X),$$

(2) *for any distinguished triangle*
$$A \to B \to C \to A[1]$$
in D^-, if two out of the three of the objects A, B, C are in $\mathcal{E}_{\mathbb{A}}$, so is the third,

(3) *if $A \oplus B \in \mathcal{E}_A$, then $A, B \in Obj(\mathcal{E}_{\mathbb{A}})$, and*

(4) *if $A_i \in Obj(\mathcal{E}_{\mathbb{A}})$, $i \in I$, and $\oplus A_i \in Obj(D^-)$, then $\oplus A_i \in Obj(\mathcal{E}_{\mathbb{A}})$.*

Let $W_{\mathbb{A}}$ denote the class of morphisms in D^- whose cone is in $\mathcal{E}_{\mathbb{A}}$. A morphism in $W_{\mathbb{A}}$ is called an $\mathbb{A}^1$-weak equivalence.

Definition 12. *(*[11]*, Lecture 14) The triangulated category of motives $DM^{eff,-}_{Nis}(S) = D^-[W_{\mathbb{A}}^{-1}]$ over S is defined to be the localization (i.e. derived category) of D^- with respect to $\mathbb{A}^1$-weak-equivalences.*

Remark 13. *By Remark 14.7 of* [11]*, we may equivalently define*
$$DM^{eff,-}_{Nis}(S)$$
as the full subcategory of D^- on objects of the form $C_*(K)$ for a bounded below chain complex K of Nisnevich sheaves with transfers. (Strictly speaking, $C_*(K)$ is a double chain complex; we mean its totalization.)

If X is a smooth scheme over S, we write $M(X)$ for the class of $\mathbb{Z}_{tr}(X)$ in $DM_{nis}^{eff,-}(S)$, and call it the *motive* of X.

Definition 14. *([11] Lecture 14) The category $DM^{-}(S)$ is obtained from $DM_{nis}^{eff,-}(S)$, by inverting the Tate operation*

$$T : M \mapsto M(1) = M \boxtimes \mathbb{Z}(1).$$

If we choose a model of T which is injective on objects, this may be taken to be the colimit, in the category of categories, of the sequence

$$DM_{nis}^{eff,-}(S) \xrightarrow{T} DM_{nis}^{eff,-}(S) \xrightarrow{T} \dots$$

There are also unbounded versions of these categories which we will denote by D, $DM_{Nis}^{eff}(S) = D[W_{\mathbb{A}}^{-1}]$, $DM(S)$ where D is the derived category of unbounded chain complexes of Nisnevich sheaves with transfers with respect to quasisomorphisms. (For the construction of an unbounded derived category of an abelian category with enough injectives see [9].) Remark 13 also remains true if we replace "bounded below" by "cell". A *cell object* is a chain complex of Nisnevich sheaves with transfers of the form

$$C = \operatorname{colim} C_{(m)}$$

where $C_{(0)} = 0$ and $C_{(m+1)}$ is the mapping cone of a morphism

$$C_*(K_m) \to C_{(m)}$$

where K_m is a bounded below chain complex of Nisnevich sheaves with transfers. The category $DM_{Nis}^{eff}(S)$ is a full subcategory of the category $DM(S)$.

One defines motivic cohomology of a smooth scheme X over a ground field k by

$$H^{n,m}(X) = H^n(X, \mathbb{Z}(m))$$

where the right hand side denotes sheaf cohomology in the Nisnevich (or, equivalently, Zariski) topology on X. (Strictly speaking, then, we are restricting the sheaf $\mathbb{Z}(m)$ to X; see [11] Lecture 3). One can also define $A(m)$ for any abelian group of coefficients A. In this paper, the coefficients will be $\mathbb{Z}$ unless specified otherwise. Equivalently, this is

$$RHom^n(\mathbb{Z}, \mathbb{Z}(m))$$

in the category $DM^{-}(X)$ (see [15]). For $M \in Obj(DM(X))$, we may put, more generally,

$$H^n_X(M) = RHom^n(\mathbb{Z}, M).$$

Now we will define the norm map

$$N_g = N_{T/S} : H^p(T, \mathbb{Z}(q)) \to H^p(S, \mathbb{Z}(q)),$$

for $g : T \to S$ a finite morphism of smooth schemes over $Spec(k)$. By [11] Lecture 2, page 13, we have a canonical transfer morphism

$$Cor(U \times_S T, \mathbb{G}_m^k \times \mathbb{A}^s) \to Cor(U, \mathbb{G}_m^k \times \mathbb{A}^s).$$

Thus, we get a morphism of complexes of Nisnevich sheaves

$$g_*\mathbb{Z}(q)_T \to \mathbb{Z}(q)_S.$$

It is known that in the Nisnevich topology, the pushforward along finite morphisms is exact. Also, by [11], Lecture 13, Zariski and Nisnevich cohomology of smooth schemes with coefficients in $\mathbb{Z}(q)$ agree.

Definition 15. *The norm map $N_g = N_{T/S}$ is defined to be the composition*

$$\begin{array}{ccccc} H^p(T, \mathbb{Z}(q)) & \xrightarrow{=} & H^p_{Nis}(T, \mathbb{Z}(q)) & \xrightarrow{=} & H^p_{Nis}(S, g_*\mathbb{Z}(q)) \\ & & & & \downarrow \\ & & & & H^p_{Nis}(S, \mathbb{Z}(q)) \xrightarrow{=} H^p(S, \mathbb{Z}(q)). \end{array}$$

Definition 16. ([5]) *Let X be a separated, reduced scheme of finite type over $Spec(k)$ (not necessarily smooth). Let $Z_i(X, m)$ be the free abelian group on dimension $i+m$ subvarieties of $X \times \Delta^m$ which intersect with all faces $X \times \Delta^j$ properly for $j < m$ (properly means in a subscheme of dimension $\leq i + j$).*

Let $\partial_k : Z_i(X, m) \to Z_i(X, m-1)$ be the map given by intersection with the ith degree (which is well defined, since the faces are effective Cartier divisors). Define

$$d : Z_i(X, m) \to Z_i(X, m-1)$$

by $d = \sum_{k=0}^m (-1)^k \partial_k$. This defines a chain complex

$$\mathcal{Z}_i(X) : \ldots \xrightarrow{d} Z_i(X, m) \xrightarrow{d} Z_i(X, m-1) \xrightarrow{d} \ldots \xrightarrow{d} Z_i(X, 0)$$

Define $CH_i(X, m) := H_m(\mathcal{Z}_i(X))$.

Theorem 17. ([11], *Part 5): If X is a smooth scheme over k, then we have a canonical isomorphism*

$$H^{p,q}(X) \cong CH_{dim(X)-q}(X, 2q - p).$$

□

3. Proof of the Main Result

Recall that k is a perfect field, $f : C \to S$ is a smooth projective morphism of dimension 1 where $C,\ S$ are smooth varieties over $Spec(k)$, and $T \subseteq C$ is a reduced closed subscheme of codimension 1 which is generic over S (i.e. $f(Spec(K(Q))) = Spec(K(S))$ for every irreducible component Q of T).

Proof of Theorem 2. We will be using the diagram

$$\begin{array}{ccc} C \smallsetminus T & \xrightarrow{\iota} & C \\ & \searrow^{\widetilde{f}} & \downarrow f \\ & & S. \end{array} \tag{5}$$

We have the derived functors

$$\widetilde{f}_* : DM_{C\smallsetminus T} \to DM_S$$

and

$$\widetilde{f}^* : DM_S \to DM_{C\smallsetminus T}.$$

The functor $\widetilde{f}^*$ preserves $\boxtimes$, and we have a projection formula for $\mathcal{F} \in Obj(DM_S),\ \mathcal{G} \in Obj(DM_{C\smallsetminus T})$

$$\widetilde{f}_*(\widetilde{f}^*(\mathcal{F}) \boxtimes \mathcal{G}) = \mathcal{F} \boxtimes \widetilde{f}_*(\mathcal{G}).$$

(In a very general context, this is proved in [2].) We will use this for $\mathcal{F} = \mathbb{Z}(q-1), \mathcal{G} = \mathbb{Z}(1)$. Thus, we have

$$H^{p,q}(C \smallsetminus T) = H^p_S(\mathbb{Z}(q-1) \boxtimes \widetilde{f}_*(\mathbb{Z}(1))).$$

Now, by [11], Thm. 4.1, we have a canonical equivalence between

$$\mathbb{Z}(1) \xrightarrow{\sim} \mathbb{G}_m[-1]\ . \tag{6}$$

More precisely, on the right hand side, we mean the motive associated with the PST $\mathbb{G}_m$ shifted by -1. By Diagram (5), (and the fact that $\mathbb{Z}(n)$, $\mathbb{G}_m$ are preserved by pullbacks), we have the canonical morphisms of motives

$$f_*(\mathbb{Z}(1)) \longrightarrow \widetilde{f}_*(\mathbb{Z}(1))\ ,$$

$$f_*(\mathbb{G}_m) \longrightarrow \widetilde{f}_*(\mathbb{G}_m)\ .$$

As PST's, we have

$$f_*(\mathbb{G}_m)(U) = \{U \times_S C \longrightarrow \mathbb{G}_m\}$$

$$\widetilde{f}_*(\mathbb{G}_m)(U) = \{U \times_S (C \smallsetminus T) \xrightarrow{h} \mathbb{G}_m\}\ . \tag{7}$$

From Diagram (5), we get a morphism of motives over C

$$f_*(\mathbb{G}_m) \longrightarrow \widetilde{f}_*(\mathbb{G}_m).\ ,$$

which on the level of PST's, is given by restricting an invertible regular function on $U \times_S C$ to $U \times_S (C \smallsetminus T)$. Thus, we have a distinguished triangle

$$F \longrightarrow f_*\mathbb{Z}(1) \longrightarrow \widetilde{f}_*(\mathbb{Z}(1)) \xrightarrow{\delta} F[1]\ . \tag{8}$$

In other words, F is the mapping co-cone (or "derived kernel") of $f_*\mathbb{Z}(1) \to \widetilde{f}_*\mathbb{Z}(1)$. Then the connecting map

$$\Delta : H^{p,q}(C \smallsetminus T) \longrightarrow H^{p+1,q}(C, C \smallsetminus T)$$

is induced by $\mathbb{Z}(q-1)\boxtimes\delta$. Now, we will give a geometric interpretation of the map δ in (8): In (7), $h \in K(U \times_S C)$, where $K(U \times_S C)$ is the field of rational functions on $U \times_S C$. Thus, we have the divisor

$$Div(h) = \sum_D v_d(h)D \in \mathcal{Z}^1(U \times_S C) = f_*(\mathcal{Z}^1)(U), \tag{9}$$

where $\mathcal{Z}^1(U \times_S C)$ is the free abelian group on the set of closed codimension 1 subvarieties of $U \times_S C$. If $Div(h)$ is a section of

$$ker(\mathbb{Z}_{tr}(C) \twoheadrightarrow \mathbb{Z}_{tr}(C \smallsetminus T)),$$

(and thus represents a section of $F[1]$), since it vanishes when restricted to $U \times_S (C \smallsetminus T)$, it only contains summands of the form $U \times_S Q$ where Q is an irreducible component of T. Adding their coefficients, multiplied by the degrees of the finite morphisms $Q \to S$, defines a degree map

$$ker(\mathbb{Z}_{tr}(C) \twoheadrightarrow \mathbb{Z}_{tr}(C \smallsetminus T)) \xrightarrow{deg} \underline{\mathbb{Z}}_S. \tag{10}$$

Recall (Definition 5) that $\underline{\mathbb{Z}}_S = \mathbb{Z}_{tr}(S)$. Now, (10) induces a map (see (8))

$$F \longrightarrow \underline{\mathbb{Z}}[-1]\ , \tag{11}$$

and (recalling that tensoring with $\underline{\mathbb{Z}}[-1]$ shifts cohomological degrees by $(-2,-1)$), by applying $\mathbb{Z}(q-1)\boxtimes ?$ in the category DM_S to (11), we get a map

$$\psi : H^{p+1,q}(C, C \smallsetminus T) \to H^{p-1,q-1}(S).$$

Lemma 18. *We have*

$$\psi \circ \Delta = \Phi : H^{p,q}(C \smallsetminus T) \to H^{p-1,q-1}(S).$$

Lemma 18 easily implies our claim. For $h \in K(U \times_S C)$, we have

$$deg(Div(h)) = 0,$$

(looking, for example, at the fiber of a regular value of the projection $T \to S$), and thus, by the fact that ψ (and Δ) can be obtained by doing the construction on $\mathbb{Z}(1)$ and applying $\mathbb{Z}(q-1)\boxtimes ?$,

$$\psi \circ \Delta = 0,$$

thus proving our assertion.

Proof of Lemma 18: We need to show that taking the map (11), composing with δ and applying $\mathbb{Z}(q-1)\boxtimes ?$ induces Φ on motivic cohomology. Now in identifying (11), we used (6), interpreting divisors of the appropriate kind as rational functions. When applying $\mathbb{Z}(q-1)\boxtimes ?$, we no longer have divisors, but algebraic cycles. From the point of view of (3), the starting data can be a higher algebraic cycle on T, of which both maps take the multiplicity over S (calculated by taking the degree of the extension in function fields). Thus, the statement is essentially a tautology. □

This completes the proof of Theorem 2.

□

By [11], Theorem 5.1, for a field F, we have a canonical isomorphism

$$K^n_M(F) \cong H^{n,n}(Spec(F))$$

where the right hand side denotes motivic cohomology. The following two lemmas link our setup to more classical constructions.

Lemma 19. *The following diagram commutes:*

$$(12) \qquad \begin{array}{ccc} H^{n,n}(Spec(L)) & \xrightarrow{\Delta} & H^{n+1,n}(C, Spec(L)) \\ & \searrow^{\partial_p} & \downarrow \\ & & H^{n+1,n}(Spec(\mathcal{O}_{C,p}), Spec(L)) \\ & & \downarrow \gamma \\ & & H^{n-1,n-1}(Spec(k_p)), \end{array}$$

where $H^{n,n}$ denotes motivic cohomology and the unlabeled vertical arrow is restriction, where ∂_p is defined by (1).

(Note: One extends the definition of motivic cohomology to $Spec(\mathcal{O}_{C,p})$ by taking colimits.)

Proof. Clearly, the composition of the top vertical and horizontal map of (12) can also be written as

$$\begin{array}{ccc} H^{n,n}(Spec(L)) & \xrightarrow{\Delta_p} & H^{n+1,n}(Spec(\mathcal{O}_{C,p}), Spec(L)) \\ & & \downarrow \gamma \\ & & H^{n-1,n-1}(Spec(k_p)), \end{array} \tag{13}$$

where Δ_p is the connecting map of the long exact sequence corresponding to the inclusion

$$Spec(L) \subset Spec(\mathcal{O}_{C,p}).$$

This is known to be a map of $H^*(Spec(\mathcal{O}_{C,p}))$-modules. One has ([11], Theorem 4.1)

$$H^{1,1}(Spec(\mathcal{O}_{C,p})) = H^0(Spec(\mathcal{O}_{C,p}), \mathbb{G}_m) = \mathcal{O}_{C,p}^{\times}.$$

By definition, ∂_p is also multiplicative with respect to $\mathcal{O}_{C,p}^{\times}$. Thus, it suffices to verify the commutativity of (12) for $n = 1$. This is a classical fact. □

Lemma 20. *The following diagram commutes:*

$$\begin{array}{ccc} H^{n+1,n}(C, Spec(L)) & & \\ \downarrow & \searrow^{\psi} & \\ \oplus_p H^{n+1,n}(Spec(\mathcal{O}_{C,p}), Spec(L)) & & \\ \downarrow \gamma & & \\ \oplus_p H^{n-1,n-1}(Spec(k_p)) & \xrightarrow[\oplus N_{k_p/k}]{} & H^{n-1,n-1}(Spec(k)) \end{array}$$

where the vertical arrows are as in (12), and $N_{k_p/k}$ is defined by (2).

Proof of Lemma 20: Fix a closed point $p \in C$. Analogs of (10) and (11), and hence also an analog ψ_p of ψ can be defined with C replaced by $Spec(\mathcal{O}_{C,p})$. Thus, it suffices to consider the diagram

$$\begin{array}{ccc} H^{n+1,n}(Spec(\mathcal{O}_{C,p}), Spec(L)) & & \\ \gamma \downarrow & \searrow^{\psi_p} & \\ H^{n-1,n-1}(Spec(k_p)) & \xrightarrow[N_{k_p/k}]{} & H^{n-1,n-1}(Spec(k)). \end{array} \tag{14}$$

By motivic purity (see e.g. [6]), up to equivalence, ($\cong$ in this derived category), the motive represented by the complex

$$\mathbb{Z}_{tr}(Spec(L)) \to \mathbb{Z}_{tr}(Spec(\mathcal{O}_{C,p}))$$

does not depend on C, in a way compatible with γ and ψ_p. Thus, it suffices to consider the Diagram (14) for $C = \mathbb{P}^1$, in which case it follows from the fact that Theorem 1 is by definition true for $C = \mathbb{P}^1$, and this characterizes $N_{k_p/k}$. $\square$

4. Appendix

It is easy to construct examples of schemes S, C as in Theorem 2 for which there does not exist a diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & S \times \mathbb{P}^1 \\ \downarrow & & \downarrow p_1 \\ S & \xrightarrow{=} & S \end{array} \tag{15}$$

with f finite, where p_1 is the projection to the first factor. Let $k = \mathbb{C}$. Let γ be a line bundle over S. Consider $P(\gamma \oplus 1)$, the associated projective bundle of $\gamma \oplus 1$ over S. Let $H_*(?)$, $H^*(?)$ denote singular homology and cohomology with coefficients in $\mathbb{Q}$. We have

$$H^*(P(\gamma \oplus 1)) = H^*(S)[u]/(u(u + c_1(\gamma)))$$

where $c_1(\gamma) \in H^2(S)$ is the first Chern class of γ. Let

$$H^*(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Q}[x]/(x^{n+1}).$$

Now, let $S = \mathbb{P}^2$. Denote by γ a line bundle over S with $c_1(\gamma) = x$. By definition,

$$H^*(P(\gamma \oplus 1)) = H^*(S)[u]/(u(u + c_1(\gamma))) = \mathbb{Q}[x, u]/(x^3, u(u + x)).$$

Now, assume we have a finite morphism

$$f : P(\gamma \oplus 1) \to \mathbb{P}^1 \times \mathbb{P}^2$$

over $\mathbb{P}^2$. Let β be a generator of the second homology of $\mathbb{P}_1 \times \{x\}$ where x is a closed point with coefficients in $\mathbb{Q}$. Let α be a generator of the second singular homology of the fiber Z of $P(\gamma \oplus 1)$ over x. Then f restricts to a finite morphism $Z \to \mathbb{P}^1 \times \{x\}$, and hence $f_*\alpha = n\beta$, $n \neq 0$.

We have

$$H^*(\mathbb{P}^2 \times \mathbb{P}^1) = \mathbb{Q}[x]/(x^3) \otimes_{\mathbb{Q}} \mathbb{Q}[v]/(v^2) = \mathbb{Q}[x, v]/(x^3, v^2).$$

Now

$$0 \neq \langle n\beta, v\rangle = \langle f_*\alpha, v\rangle = \langle \alpha, f^*v\rangle.$$

So, $f^*v \neq 0$. However, $v^2 = 0$, so $(f^*v)^2 = 0$. So, there exist an $m \in \mathbb{Q}$ and $k, \ell \in \mathbb{Q}$, not both zero, with

$$(kx + \ell u)^2 = mu(u + x). \tag{16}$$

Then

$$0 = (kx + \ell u)^2 - m \cdot u(u + x) =$$
$$= k^2x^2 + 2k\ell xu + \ell^2u^2 - mu^2 - mxu.$$

Note that on the right hand side of (16), there is no x^2, so $k^2x^2 = 0$. Thus $k^2 = 0$. Since $k \in \mathbb{Q}$, $k = 0$.

So, since

$$0 = \ell^2u^2 - mu^2 - mxu,$$

$$\ell^2u^2 = mu^2 + mxu. \tag{17}$$

Since there are no xu's on the left hand side of (17),

$$mxu = 0$$

$$m = 0.$$

So, $\ell^2u^2 = 0$. So, $\ell^2 = 0$. Since $\ell \in \mathbb{Q}$,

$$\ell = k = 0.$$

Contradiction.

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