THE RESTRICTED SHALEV CONJECTURE FOR SYMPLECTIC GROUPS IN THE VERY STABLE RANGE

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1. Introduction

One of the most important steps to studying a finite group is understanding the behavior of its character table. The characters of a finite group not only fully describe its representation theory, but they also capture interesting data about the group itself. One especially nice example of this is the way characters can be used to study commutators.

Consider a finite group G, and let us write $\widehat{G}$ for the set of its irreducible representations (over $\mathbb{C}$). For an element $g \in G$, we may consider the sum over every non-trivial irreducible G-representation of its corresponding character value at g divided by its dimension:

$$\sum_{1\neq\rho\in\widehat{G}} \frac{\chi_\rho(g)}{dim(\rho)} \tag{1}$$

This is called the *character ratio sum* of G at g. One of the most famous appearances of these sums was their role played in the proof of M.W. Liebeck, E.A. O'Brien, A. Shalev, and P.H. Tiep [7, 9] of the *Ore conjecture* which states that in a non-abelian simple group G, every non-trivial element is expressible as a commutator $xyx^{-1}y^{-1}$ for some $x, y \in G$. Specifically, for any $g \neq 1$, the number of pairs $(x, y) \in G\times G$ for which $g = xyx^{-1}y^{-1}$ can be expressed in terms of the order of G and the sum (1) as

$$|G| \cdot \sum_{\rho\in\widehat{G}} \frac{\chi_\rho(g)}{dim(\rho)} = |G| + |G| \cdot \sum_{1\neq\rho\in\widehat{G}} \frac{\chi_\rho(g)}{dim(\rho)}, \tag{2}$$

according to a formula of Frobenius. The estimation of certain character ratio sums was a key techincal step in proving the Ore conjecture [7, 9] by veryifying (2) is always non-zero.

This proof naturally suggests that the distribution of commutators can also be studied using character ratio sums. In [11], A. Shalev conjectured that for non-abelian finite simple groups G, commutators are approximately uniformly distributed; in the case of a finite non-abelian group of Lie type $G(\mathbb{F}_q)$, this can be concretely stated as the claim that, for a non-trivial element $1 \neq g \in G(\mathbb{F}_q)$, its character ratio sum vanishes as q goes to infinity:

$$\lim_{q\to\infty} \sum_{1\neq\rho\in\widehat{G(\mathbb{F}_q)}} \frac{\chi_\rho(g)}{dim(\rho)} = 0. \tag{3}$$

More specifically, given the q for which $g \in G(\mathbb{F}_q)$, we consider the limit over powers of q going to infinity, i.e.

$$\lim_{n\to\infty} \sum_{1\neq\rho\in\widehat{G(\mathbb{F}_{q^n})}} \frac{\chi_\rho(g)}{dim(\rho)}.$$

This is always what we shall mean when we write $\lim_{q\to\infty}$ in this note.

From now on, we specifically consider symplectic groups $Sp_{2N}(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ for q an odd prime power. In [2, 3], S. Gurevich and R. Howe computationally observed two prominent effects in character ratio sums for symplectic groups:

(A) the largest terms in (3) seemed to be contributed by "small representations,"
(B) and terms corresponding to representations of "similar size" exhibited cancellation.

They define the notion of "size" based on their concept of *U-rank* which they conjecture, in the small representations, is the same as *tensor rank*, i.e. where a representation arises in the *η-correspondence*. This conjecture was proved in [6]. They provide substantial computational evidence for (A) and (B).

(A) suggest that one needs to prove that the sum of the terms of (1) coming from the *stable range*, i.e. the terms corresponding to representations ρ in the image of the eta correspondence from $O(W, B)$ to $Sp_{2N}(\mathbb{F}_q)$ with $n = dim_{\mathbb{F}_q}(W) \leq N$, is 0. The purpose of the present paper is to prove this in the *very stable range*, i.e. for $n << N$.

To be more specific, it is well known that the two oscillator representations ω_a, ω_b of $Sp_{2N}(\mathbb{F}_q)$ (we fix an identification of $\mathbb{F}_q$ with its Pontrjagin dual) decompose into the lowest-dimensional and second-dimensional representations of $Sp_{2N}(\mathbb{F}_q)$:

$$\omega_a = \omega_a^- \oplus \omega_a^+, \quad \omega_b = \omega_b^- \oplus \omega_b^+,$$

with $dim(\omega_a^\pm) = dim(\omega_b^\pm) = (q^N \pm 1)/2$. The oscillator representations not only have miraculously small dimension, but they and their summands are also the only $Sp_{2N}(\mathbb{F}_q)$-representations of rank 1. Since their concept of rank is additive with respect to tensor product, Gurevich and Howe studied tensor powers of the oscillator representations to construct small representations. These explicit computations were approached by emulating the story of Howe duality over a finite field.

For a symplectic space V and an orthogonal space (W,B), we may consider the inclusion

$$Sp(V)\times O(W,B)\hookrightarrow Sp(V\otimes W) \tag{4}$$

given by tensoring matrices. Restricting the oscillator representation $\omega_1[V\otimes W]$ (we denote $\omega=\omega_1$) along (4), and then further to $Sp(V)$, gives a tensor product of copies of $Sp(V)$'s oscillators representations corresponding to the eigenvalues of B. In [2, 3], Gurevich and Howe proved the existence of the *eta correspondence* for $dim(W)\leq dim(V)/2$, which is an injection

$$\eta^V_{W,B}:\widehat{O(W,B)}\hookrightarrow\widehat{Sp(V)}$$

such that for every irreducible $O(W,B)$-representation π, the tensor product $\pi\otimes\eta^V_{W,B}(\pi)$ appears as a "top term" in the restriction of $\omega[V\otimes W]$ along (4).

In [4, 5, 6], we fully calculated (and extended) the eta correspondence, obtaining an explicit decomposition of the restriction of $\omega[V\otimes W]$ along (4). In particular, we proved Gurevich and Howe's rank conjecture that, for any $r<N$, the irreducible representations of $Sp_{2N}(\mathbb{F}_q)$ of rank r are precisely those in an image of one of the two eta corespondences $\eta^V_{W,B}$ associated to the two non-equivalent choices of (W,B) with $dim(W)=n$ (differentiated by the discriminat of B if n is odd and by the dimension of the maximal isotropic subspace of W with respect to B if n is even).

Gurevich and Howe's computational evidence then suggests that to prove Shalev's conjecture, for example in the case of $Sp_{2N}(\mathbb{F}_q)$, it is useful to decompose a character ratio sum into pieces graded by the eta correspondence from orthogonal spaces of dimension less than N. This separates out each group of "small representations of similar size," where cancellation was experimentally observed. This can be stated as the following

Conjecture 1. *Fix an N and an $0<n<N$. For every $g\in Sp_{2N}(\mathbb{F}_q)$, the partial sum of character ratios*

$$\frac{\chi_\rho(g)}{dim(\rho)},$$

for ρ ranging over the disjoint union of the images of the two eta correspondences from the irreducible representations of orthogonal groups on $\mathbb{F}_q^n$, vanishes as q goes to infinity.

The purpose of this note is to prove this conjecture when n is very small compared to N.

First we fix some notation. For a given symplectic space dimension $2N$ and orthogonal space dimension n, let us write $\eta_{n,\pm}^{2N}$ for the two eta correspondences, where if n is odd, the sign corresponds to the discriminant of the symmetric bilinear form used in the orthogonal group and if n is even, it denotes whether the orthogonal group is split or non-split (with $+$ corresponding to the split case where the orthogonal space has a $n/2$-dimensional isotropic subspace and $-$ corresponding to the non-split cases where the maximal dimension of an isotropic subspace is $(n/2)-1$).

The main result of this note is

Theorem 2. *Fix a natural number m.*

1. *Fixing a sign $\sigma = \pm$, considering the odd eta correspondence from the orthogonal group on a space of dimension $2m+1$ with respect to a form with discriminant σ to the symplectic group on a space of dimension $2N$ for $N >> m$, for any non-trivial group element $g \in Sp_{2N}(\mathbb{F}_q)$,*

$$\lim_{q\to\infty} \sum_{\rho \in Im(\eta_{2m+1,\sigma}^{2N})} \frac{\chi_\rho(g)}{dim(\rho)} = 0. \tag{5}$$

2. *Considering the even eta correspondences from the orthogonal groups on a space of dimension $2m$ with respect to split or non-split forms to the symplectic group on a space of dimension $2N$ for $N >> m$, for any non-trivial group element $g \in Sp_{2N}(\mathbb{F}_q)$*

$$\lim_{q\to\infty} \sum_{\rho \in Im(\eta_{2m,+}^{2N})} \frac{\chi_\rho(g)}{dim(\rho)} + \sum_{\rho \in Im(\eta_{2m,-}^{2N})} \frac{\chi_\rho(g)}{dim(\rho)} = 0, \tag{6}$$

Remark: We note that in (5), we only sum over the image of one eta correspondence, which is sufficient in the case of odd orthogonal groups. In the even case, however, the character ratios arising from both eta correspondences $\eta_{2m,\pm}^{2N}$ (corresponding to split and non-split even orthogonal groups) need be summed in (6). Our proof actually gives that for elements $g \in Sp_{2N}(\mathbb{F}_q)$ which are not conjugate to a transvection, for both signs $\sigma = \pm$, the individual terms

$$\lim_{q\to\infty} \sum_{\rho \in Im(\eta^{2N}_{2m,\sigma})} \frac{\chi_\rho(g)}{dim(\rho)} = 0.$$

For a transvection $T \in Sp_{2N}(\mathbb{F}_q)$, however, we have computational evidence that

$$\lim_{q\to\infty} \sum_{\rho \in Im(\eta^{2N}_{2m,\sigma})} \frac{\chi_\rho(g)}{dim(\rho)} = \sigma \cdot \epsilon(-1)^m$$

where ϵ denotes the quadratic character of $\mathbb{F}_q^\times$ (the significance of $\sigma \cdot \epsilon(-1)^m$ is that it is the discriminant of a symmetric bilinear form whose orthogonal group is $O^\sigma_{2m}(\mathbb{F}_q)$). This effect can be observed in Section 4 in the case of $m = 1$.

This note is organized as follows: In Section 2, we use the explicit description of the eta correspondence we found in [4, 5, 6] to prove a lemma reducing the denominators of (5) and (6). In Section 3, we process the resulting expression in terms of oscillator representations, whose characters can be readily computed using the Schrödinger model. In Section 4, we perform an explicit computation of this step in the case of $O_3(\mathbb{F}_q)$.

2. The top term of the denominator

The purpose of this section is to re-express the denominators appearing in (5) and (6) by explicitly computing the effect the eta correspondence has on the dimension of an irreducible representation of an orthogonal group.

Lemma 3. *Consider a symplectic group $Sp(V)$ and an orthogonal group $O(W,B)$ in the symplectic stable range, meaning $dim(W) \leq dim(V)/2$. Then, for an irreducible representation $\pi \in \widehat{O(W,B)}$, the dimension of its image under the eta correspondence $\eta^V_{W,B}(\pi)$ may be considered as a polynomial in q.*

(1) *Suppose W is of odd dimension $2m+1$ and say V is of dimension $2N$ (so that $N \geq 2m+1$). For every $\pi \in \widehat{O(W,B)}$ the top term of this polynomial is the top term of $dim(\pi)$, multiplied by $q^{(2m+1)(N-m)}/2$. More specifically,*

$$dim(\eta^V_{W,B}(\pi)) = dim(\pi) \cdot (\frac{q^{(2m+1)(N-m)}}{2} + LOE_q). \tag{7}$$

(2) *Suppose W is of even dimension $2m$ and say V is of dimension $2N$ (so that $N \geq 2m$). For every $\pi \in \widehat{O(W,B)}$ the top term of this polynomial is the top term of $dim(\pi)$, multiplied by $q^{2m(N-m)+m}/2^{\alpha(\pi)+\beta(\pi)+1}$, where $\alpha(\pi)$ is 1 if the semisimple conjugacy class (s) associated to π in Lusztig's classification has 1 eigenvalues and 0 else, and $\beta(\pi)$ is 1 if (s) has -1 eigenvalues and 0 else. More specifically,*

$$dim(\eta^V_{W,B}(\pi)) = dim(\pi) \cdot (\frac{q^{2m(N-m)+m}}{2^{\alpha(\pi)+\beta(\pi)+1}} + LOE_q). \tag{8}$$

Given (7), the partial character ratio sum over the image of an odd eta correspondence in (5) has top term

$$\lim_{q\to\infty} \frac{2}{q^{(2m+1)(N-m)}} \cdot \left(\sum_{\pi \in \widehat{O_{2m+1}(\mathbb{F}_q,\sigma)}} \frac{\chi_{\eta^{2N}_{2m+1,\sigma}(\pi)}(g)}{dim(\pi)} \right). \tag{9}$$

Similarly, the partial character ratio sums over the images of the even eta correspondences appearing in (6) have top terms

$$\lim_{q\to\infty} \frac{1}{2 \cdot q^{2m(N-m)+m}} \cdot \left(\sum_{\pi \in \widehat{O^+_{2m}(\mathbb{F}_q)}} 2^{\alpha(\pi)+\beta(\pi)} \frac{\chi_{\eta^{2N}_{2m,+}(\pi)}(g)}{dim(\pi)} \right) \tag{10}$$

and

$$\lim_{q\to\infty} \frac{1}{2 \cdot q^{2m(N-m)+m}} \cdot \left(\sum_{\pi \in \widehat{O^+_{2m}(\mathbb{F}_q)}} 2^{\alpha(\pi)+\beta(\pi)} \frac{\chi_{\eta^{2N}_{2m,-}(\pi)}(g)}{dim(\pi)} \right) \tag{11}$$

by (8).

We note that for the final claim, we will also need to discuss the error "lower" terms appearing in the partial character ratio sums in (5) and (6). We postpone this discussion until we establish more terminology and notation for the irreducible representations of the orthogonal groups (see the end of Subsection 2.3 below).

We prove Lemma 3 by recalling the description of the eta correspondence done in [5]. To state it, we briefly recall G. Lusztig's classification of irreducible representations of a finite group of Lie type [8], in the case of odd orthogonal groups and symplectic groups.

2.1. **Lusztig's classification of irreducible representations.** In general, for a finite group G of Lie type, we consider a conjugacy class (s) of a semisimple element of its dual group G^D and an irreducible unipotent representation u of the dual of its centralizer $(Z_{G^D}(s))^D$. This data is associated to G-representation we denote by $\rho_{(s),u}$ which has dimension

$$dim(\rho_{(s),u}) = dim(u) \cdot \frac{|G|_{q'}}{|(Z_{G^D}(s))^D|_{q'}}$$

where $|?|_{q'}$ denotes the prime to q part of a group's order (noting that the order of a group is equal to the order of its dual). Considering a maximal torus $T \subseteq G$ containing s, the restriction of $\rho_{(s),u}$ to T is a sum of copies of the character corresponding to s along the identification of T with its Pontrjagin dual. These represetnations $\rho_{(s),u}$ are precisely the irreducible representations of G if its center is trivial. If the center is non-trivial (for example, in the case of symplectic groups), there may be cases where $\rho_{(s),u}$ splits further, according to the action of $Z(G)$. In all the cases we consider here, this can be described according to "central sign data," the precise form of which depends on the type of G, and $\rho_{(s),u}$ always splits into non-isomorphic equi-dimensional pieces.

Definition 4. *Consider an irreducible representation ρ of G. Suppose that ρ is a summand of the representation $\rho_{(s),u}$ corresponding to a semisimple conjugacy class (s) in G^D and an irreducible unipotent representation u of the dual of s's centralizer $(Z_{G^D}(s))^D$. Write $H = (Z_{G^D}(s))^D$. We say ρ has* centralizer-unipotent type

$$(H, u).$$

We write $\widehat{G}[H, u]$ for the set of all irreducible G-representations with centralizer-unipotent type (H, u).

We note that, in particular, in the cases we consider in this section, for irreducible representations ρ, ρ' of the same centralizer-unipotent type, we in fact have

$$dim(\rho) = dim(\rho').$$

Now let us discuss Luszitg's classification in more specific detail in the cases of symplectic and orthogonal groups:

One the one hand, in the case of odd orthogonal groups, since the center splits off $O_{2m+1}(\mathbb{F}_q) = \mathbb{Z}/2 \times SO_{2m+1}(\mathbb{F}_q)$, it suffices to describe the irreducible representations of $SO_{2m+1}(\mathbb{F}_q)$. The dual group of

$SO_{2m+1}(\mathbb{F}_q)$ is the symplectic group $Sp_{2m}(\mathbb{F}_q)$. The conjugacy classes of semisimple elements s in $Sp_{2m}(\mathbb{F}_q)$ are classified by orbits of the eigenvalues of s under the action of the Weyl group of $Sp_{2m}(\mathbb{F}_q)$. The centralizer of such an s is of the form

$$\prod_{i=1}^{k} U^{\pm}_{n_k}(\mathbb{F}_{q^{a_k}}) \times Sp_{2p}(\mathbb{F}_q) \times Sp_{2\ell}(\mathbb{F}_q) \tag{12}$$

(where we use the notation that $U^+ = GL$) such that

$$\sum_{i=1}^{k} n_k \cdot a_k + \ell + p = m. \tag{13}$$

The unitary factors of (12) correspond to eigenvalues of s not equal to ± 1, and the two symplectic factors correspond to eigenvalues equal to 1 and -1, respectively. An irreducible unipotent representation u of the dual of (12) can be expressed as a tensor product of irreducible unipotent representations of each individual factor. To describe the eta correspondence, we also recall that Lusztig classified the irreducible unipotent representations of the symplectic group $Sp_{2n}(\mathbb{F}_q)$ as corresponding to the combinatorial data of *symbols* $\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b}$ consisting of non-negative integers λ_i, μ_j such that $a - b$ is odd and

$$\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b} \mu_j = n^2 - \frac{(a+b-1)^2}{4}$$

(see [8]).

On the other hand, in the case of a symplectic group $Sp_{2N}(\mathbb{F}_q)$, its dual group is the odd special orthogonal group $SO_{2N+1}(\mathbb{F}_q)$, in which the conjugacy classes of semisimple elements s are again classified by orbits of the eigenvalues of s under the action of the Weyl group. Similarly to (12), the centralizer of a semisimple element in $SO_{2N+1}(\mathbb{F}_q)$ is of the form

$$\prod_{i=1}^{k} U^{\pm}_{n_k}(\mathbb{F}_{q^{a_k}}) \times SO_{2p+1}(\mathbb{F}_q) \times SO^{\pm}_{2\ell}(\mathbb{F}_q) \tag{14}$$

such that

$$\sum_{i=1}^{k} n_k \cdot a_k + \ell + p = N. \tag{15}$$

The unitary factors of (14) again correspond to eigenvalues of s not equal to ± 1, while the odd special orthogonal group factor corresponds to 1 eigenvalues and the even special orthogonal group factor corresponds to even eigenvalues. Consider u an irreducible unipotent representation of (14). Since the center of $Sp_{2N}(\mathbb{F}_q)$ is $\mathbb{Z}/2$, the representation $\rho_{(s),u}$ may split further. Specifically, it splits into two non-isomorphic summands $\rho_{(s),u,\pm 1}$ of dimension $dim(\rho_{(s),u})/2$ when s has -1 eigenvalues. According to Lusztig's classification (see the Appendix of [8]) irreducible unipotent representations of an odd special orthogonal group $SO_{2n+1}(\mathbb{F}_q)$ are classified by the same symbols as for $Sp_{2n}(\mathbb{F}_q)$, while the irreducible unipotent representations of $SO^{\pm}_{2n}(\mathbb{F}_q)$ correpsond to symbols $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ consisting of non-negative integers λ_i, μ_j such that $a-b$ is 0 mod 4 for $SO^{+}_{2n}(\mathbb{F}_q)$ and 2 mod 4 for $SO^{-}_{2n}(\mathbb{F}_q)$, and

$$\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b} \mu_j = n^2 - \frac{(a+b)(a+b-2)}{4}.$$

Finally, we also consider the case of even orthogonal groups $O^{\sigma}_{2m}(\mathbb{F}_q)$ for a sign $\sigma = \pm$. We cannot directly split off the $\mathbb{Z}/2$ center of $O^{\sigma}_{2m}(\mathbb{F}_q)$, and it is simpler in this case to work with the full orthogonal group directly. $O^{\sigma}_{2m}(\mathbb{F}_q)$ is its own dual group and the conjugacy classes of its semisimple elements are yet again classified by orbits of the eigenvalue of s under the action of the Weyl group. As in (12) and (14), the centralizer of a semisimple element in $O^{\sigma}_{2m}(\mathbb{F}_q)$ is of the form

$$\prod_{i=1}^{k} U^{\pm}_{n_k}(\mathbb{F}_{q^{a_k}}) \times O^{\pm}_{2p}(\mathbb{F}_q) \times O^{\pm}_{2\ell}(\mathbb{F}_q) \tag{16}$$

such that

$$\sum_{i=1}^{k} n_k \cdot a_k + \ell + p = N, \tag{17}$$

and the product of the signs in the superscripts of the factors is σ. As in the previous two cases, the unitary factors of (16) correspond to eigenvalues of s not equal to ± 1, while the even orthgonal group factors correpsond to 1 and -1 eigenvalues, respectively. Consider u an irreducible unipotent representation of the dual of (16). As in the case of the symplectic group, $O^{\sigma}_{2m}(\mathbb{F}_q)$ has center $\mathbb{Z}/2$, so the representation $\rho_{(s),u}$ may split further. Specifically, let us take α to be 1 if s has 1 eigenvalues (if $p > 0$) and 0 else and take β to be 1 if s has -1 eigenvalues (if $\ell > 0$) and 0 else. Then $\rho_{(s),u}$ splits into $2^{\alpha+\beta}$ equidimensional non-isomorphic irreducible summands. The irreducible unipotent representations of a group are the same as those of its quotient by its center, so the irreducible unipotent representations of $O^{\pm}_{2n}(\mathbb{F}_q)$ are those of $SO^{\pm}_{2n}(\mathbb{F}_q)$. This can be visualized as selecting one sign $\pm$ we call the *α-sign* if s has 1 eigenvalues, and independently selecting another sign we call the *β-sign* if s has -1 eigenvalues.

For more details in these specific cases, see also [5].

In particular, we also note that in all the above cases, the descriptions of which groups may appear as centralizers of semisimple conjugacy classes in the dual group is independent of q: in the cases of $SO_{2m+1}(\mathbb{F}_q), Sp_{2N}(\mathbb{F}_q)$, and $O^{\pm}_{2m}(\mathbb{F}_q)$, the number of possible centralizers is determined by choices of ranks satisfying (13), (15), and (17), respectively (and an appropriate choice of signs in the case of $O^{\pm}_{2m}(\mathbb{F}_q)$). Therefore, both the set of possible centralizers of semisimple conjugacy classes and their unipotent irreducible representations (corresponding to symbols, which are independent of q) in each of these cases is constant as q goes to infinity. Using the terminology introduced in Definition 4, the number of centralizer-unipotent types (H, u) is independent of q. In particular, the number of possible dimensions of irreducible representations of, say an orthogonal group $O(W, B) = O_{2m+1}(\mathbb{F}_q)$ or $O^{\pm}_{2m}(\mathbb{F}_q)$,

$$|\{dim(\pi) \mid \pi \in \widehat{O(W, B)}\}| \tag{18}$$

is constant as q increases to infinity.

2.2. **The description of the eta correspondence.** Now we recall the explicit description of the eta correspondence we gave in [5]. First we treat the case from $O(W, B)$ for W of odd dimension $2m + 1$. Consider an irreducible representation π of $O(W, B)$. Splitting off the center $O(W, B) = \mathbb{Z}/2 \otimes SO_{2m+1}(\mathbb{F}_q)$, we express it as $(\pm 1) \otimes \overline{\pi}$ where $\overline{\pi}$ denotes π's restriction to an irreducible $SO_{2m+1}(\mathbb{F}_q)$-representation. Suppose $\overline{\pi}$ corresponds to a semisimple conjugacy class (s) in $Sp_{2m}(\mathbb{F}_q)$ and an irreducible unipotent representation u of the centralizer $Z_{Sp_{2m}(\mathbb{F}_q)}(s)$. In the odd case of the eta correspondence, we specifically must consider the -1 eigenvalues of s. Say s has -1 as an eigenvalue of multiplicity 2ℓ and write its centralizer (12) as

$$Z_{Sp_{2m}(\mathbb{F}_q)}(s) = H \times Sp_{2\ell}(\mathbb{F}_q)$$

(denoting by H the product of factors from the eigenvalues not equal to -1). Factor u as a tensor product $u = u_H \otimes \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots \mu_b}$, where u_H denotes an irreducible unipotent representation of the dual of H and the second factor is a symbol of $SO_{2\ell+1}(\mathbb{F}_q)$, which is the dual of $Sp_{2\ell}(\mathbb{F}_q)$.

The eta correspondence $\eta^V_{W,B}(\pi)$ to be described is an irreducible representation of $Sp(V) = Sp_{2N}(\mathbb{F}_q)$, so we need to specify a semisimple conjugacy class $\phi^{\pm}(s)$ in $SO_{2N+1}(\mathbb{F}_q)$ and an irreducible unipotent representation $\phi^{\pm}(u)$ of its centralizer (the sign in the superscripts corresponds to the sign of the $\mathbb{Z}/2$-representation factor in the input representation $\pi = (\pm 1) \otimes \overline{\pi}$). There are exactly two conjugacy classes obtained by adding a single 1 eigenvalue and eigevnalues -1 of multiplicity $2(N-m)$ to s (corresponding to whether the symmetric bilinear form is split or non-split on the even-dimensional eigenspace of -1 for the resulting semisimple element). We denote these conjugacy classes by $(\phi^{\pm}(s))$, so that

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi^{\pm}(s)) = H^D \times SO^{\pm}_{2(N-m+\ell)}(\mathbb{F}_q). \tag{19}$$

(For more details see [5].) We then consider the irreducible unipotent representations

$$u_H \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\pi \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \text{ and } u_H \otimes \begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b < N'_\pi \end{pmatrix} \tag{20}$$

of (19) where we put

$$N'_\pi = N - m + \frac{a+b-1}{2}$$

so that both symbols have rank $N - m + \ell$. We take these to be $\phi^{\pm}(u)$ according to $a+1-b \bmod 4$ and $a-1-b \bmod 4$ (so that $\phi^{\pm}(u)$ is an irreducible unipotent representation of $Z_{SO_{2N+1}(\mathbb{F}_q)}(\phi^{\pm}(s))$). For the first tensor factor in (20), we again note that the unipotent representations of a group are identified with those of its dual group. Now, all $\phi^{\pm}(s)$ have -1 eigenvalues, so central sign data must be given to specify which of the two irreducible summand of the $Sp_{2N}(\mathbb{F}_q)$-representation determined by $(\phi^{\pm}(s))$, $\phi^{\pm}(u)$ the eta correspondence $\eta^V_{W,B}(\pi)$ gives. This sign data is determined by the discriminant of B (see [5] for details).

Now we consider the eta correspondence from even orthogonal groups $O(W,B) = O^{\sigma}_{2m}(\mathbb{F}_q)$. Consider an irreducible representation π of the group $O(W,B)$. Suppose that π is a summand of the representation associated to a semisimple conjugacy class (s) in $O^{\sigma}_{2m}(\mathbb{F}_q)$ and an irreducible unipotent representation u of the centralizer $Z_{O^{\sigma}_{2m}(\mathbb{F}_q)}(s)$. Now, in the even case of the eta correspondence, we specifically must consider the 1 eigenvalues of s. Say s has 1 as an eigenvalue of multiplicity $2p$ and write its centralizer (16) as

$$Z_{O^{\sigma}_{2m}(\mathbb{F}_q)}(s) = H \times O^{\pm}_{2p}(\mathbb{F}_q)$$

(denoting by H the product of factors from the eigenvalues not equal to 1). As before, we factor u as a tensor product $u = u_H \otimes \binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$, where u_H denotes an irreducible unipotent representation of the dual of H and the second factor is a symbol of $SO^{\pm}_{2p}(\mathbb{F}_q)$, which is self-dual (again, recall that the irreducible unipotent representations of $O^{\pm}_{2p}(\mathbb{F}_q)$ are those of $SO^{\pm}_{2p}(\mathbb{F}_q)$).

Again, the eta correspondence $\eta^V_{W,B}(\pi)$ to be described is an irreducible representation of $Sp(V) = Sp_{2N}(\mathbb{F}_q)$, specified by a choice of semisimple conjugacy class $\psi(s)$ in $SO_{2N+1}(\mathbb{F}_q)$ and an irreducible unipotent representation of its centralizer (and possibly central sign data). We take $\psi(s)$ to be the semisimple element in $SO_{2N+1}(\mathbb{F}_q)$ obtained by adding the eigenvalue 1 with multiplicity $2(N-m)+1$ to s. We could write $\psi(s) = (s) \oplus I_{2(N-m)+1}$, considering s in one of the standard maximal tori. The centralizer of $\psi(s)$ in $SO_{2N+1}(\mathbb{F}_q)$ can be written as

$$Z_{SO_{2N+1}(\mathbb{F}_q)}(\psi(s)) = H^D/Z(H^D) \times SO_{2(N-m+p)+1}(\mathbb{F}_q), \tag{21}$$

(noting that H^D has a non-trivial centralizer when s has eigenvalues -1, in which case H^D has a corresponding factor $O^{\pm}_{2\ell}(\mathbb{F}_q)$, while the factor corresponding to -1 eigenvalues in the centralizer of $\psi(s)$ is $SO^{\pm}_{2\ell}(\mathbb{F}_q)$). Again, the irreducible unipotent representations of $H^D/Z(H^D)$ are the same as those of H^D (which in this case is equal to H). We may consider the irreducible unipotent representations of (21) which are still of the form (20) where now we put

$$N'_\pi = N - m + \frac{a+b}{2},$$

so that both symbols have rank $N-m+p$. Both form symbols of $SO_{2(N-m+1)}(\mathbb{F}_q)$, since $a+b+1$ is odd. Let us denote them by $\psi^{\pm}(u)$. They are distinct precisely when $p > 0$ (if $p = 0$, both symbols correspond to the trivial representation of $SO_{2(N-m)+1}(\mathbb{F}_q)$). In the case when $\lambda_i = \mu_i$, there are still two choices of representations, since then the symbol $\binom{\lambda_1 < \dots < \lambda_a}{\mu_1 < \dots < \mu_b}$ splits in half, see [8]. Therefore, when we need to, we may choose which $\psi^{\pm}(u)$ to use in order to make $\eta^V_{W,B}(\pi)$ according to the α-sign of π. Also, we must select the central sign of $\eta^V_{W,B}(\pi)$ when $\psi(s)$ has -1 eigenvalues which is equivalent to s itself having -1 eigenvalues. We may therefore choose this central sign according to the β-sign from the original representation π. Again, for more detail, see [5].

In particular, we also note that the description of the eta correspondence is stable as q increases to infinity since the sets of symbols corresponding to irreducible unipotent representations do not change. More specifically, consider two irreducible representations π and π' of an orthogonal group $O(W,B)$ with the same dimension, coming from Lusztig classification data arising from different semisimple conjugacy classes with the same centralizers, paired with the same unipotent irreducible representations (and possible central sign data). We then have

$$dim(\eta^V_{W,B}(\pi)) = dim(\eta^V_{W,B}(\pi')).$$

2.3. **The proof of Lemma 3.** Given the above description of the eta correspondence, the claim follows directly from Lusztig's dimension formula for symbols (see [8], Appendix):

Proof of Lemma 3. First we treat the case of $O(W,B)$ for W of odd dimension $2m+1$. Fix an irreducible representation of $O(W,B)$. Assume the above notation and without loss of generality that N'_π is added to the top row $(\lambda_1 < \cdots < \lambda_a)$ of the symbol when consturction $\eta^V_{W,B}(\pi)$. The dimensions of π and $\eta^V_{W,B}(\pi)$ are

$$dim(\pi) = \frac{|SO_{2m+1}(\mathbb{F}_q)|_{q'}}{|H \times Sp_{2\ell}(\mathbb{F}_q)|_{q'}} \cdot dim(u_H) \cdot dim\begin{pmatrix} \lambda_1 < \cdots < \lambda_a \\ \mu_1 < \cdots < \mu_b \end{pmatrix} \tag{22}$$

(23)

$$dim(\eta^V_{W,B}(\pi)) = \frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{2 \cdot |H^D \times SO^{\pm}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}} \cdot dim(u_H) \cdot dim\begin{pmatrix} \lambda_1 < \cdots < \lambda_a < N'_\pi \\ \mu_1 < \cdots < \mu_b \end{pmatrix}.$$

(Recall that $|H| = |H^D|$.)

Now we apply the dimension formula given in Appendix of [8] which states that for a symbol $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ specifying a representation of a finite group G (of Lie type B, C, D, or ${}^2D$), its dimension is $|G|_{q'}$, multiplied by

$$\frac{\prod\limits_{1\le i<j\le a}(q^{\lambda_j} - q^{\lambda_i}) \prod\limits_{1\le i<j\le b}(q^{\mu_j} - q^{\mu_i}) \prod\limits_{1\le i\le a, 1\le j\le b}(q^{\lambda_i} + q^{\mu_j})}{2^{c(a,b)} \prod\limits_{1\le i\le a}\prod\limits_{k=1}^{\lambda_i}(q^{2k}-1) \prod\limits_{1\le j\le a}\prod\limits_{k=1}^{\mu_j}(q^{2k}-1) q^{d(a,b)}} \tag{24}$$

where

$$c(a,b) = \lfloor \frac{a+b-1}{2} \rfloor$$

and

$$d(a,b) = \sum_{i=1}^{\lfloor (a+b)/2 \rfloor} \binom{a+b-2i}{2}. \tag{25}$$

We now use this to compare the factors $dim\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}/|Sp_{2\ell}(\mathbb{F}_q)|_{q'}$ and $dim\binom{\lambda_1<\cdots<\lambda_a<N'_\pi}{\mu_1<\cdots<\mu_b}/|SO^{\pm}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}$ in (22) and (23). We note that $a+b$ is fixed to be odd in this case since $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ is a symbol of $Sp_{2\ell}(\mathbb{F}_q)$. To compare the powers of q in the denominators of these factors, we note that

$$\begin{gathered} d(a+1,b)-d(a,b) = \\ \sum_{i=1}^{(a+b+1)/2} \binom{a+b+1-2i}{2} - \sum_{i=1}^{(a+b-1)/2} \binom{a+b-2i}{2} = \\ \sum_{i=1}^{(a+b-1)/2} (a+b-2i) = \frac{(a+b-1)^2}{4}. \end{gathered} \tag{26}$$

The powers of 2 in the denominator of the factors are the same, since $c(a,b) = c(a+1,b)$ if $a+b$ is odd. We find

$$\begin{gathered} \frac{dim\binom{\lambda_1<\cdots<\lambda_a<N'_\pi}{\mu_1<\cdots<\mu_b}}{|SO^{\pm}_{2(N-m+\ell)}(\mathbb{F}_q)|_{q'}} = \\ \frac{\prod_{i=1}^{a}(q^{N'_\pi}-q^{\lambda_i})\prod_{j=1}^{b}(q^{N'_\pi}+q^{\mu_j})}{\prod_{k=1}^{N'_\pi}(q^{2k}-1)q^{(a+b-1)^2/4}} \cdot \frac{dim\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}}{|Sp_{2\ell}(\mathbb{F}_q)|_{q'}}. \end{gathered} \tag{27}$$

Therefore, since

$$|Sp_{2n}(\mathbb{F}_q)|_{q'} = |SO_{2n+1}(\mathbb{F}_q)|_{q'} = \prod_{k=1}^{n}(q^{2k}-1), \tag{28}$$

comparing (22) and (23) gives that $dim(\eta^V_{W,B}(\pi))$ is exactly $dim(\pi)$ multiplied by

$$\frac{\prod_{k=N'_\pi+1}^{N}(q^{2k}-1)\prod_{i=1}^{a}(q^{N'_\pi}-q^{\lambda_i})\prod_{j=1}^{b}(q^{N'_\pi}+q^{\mu_j})}{2\prod_{k=1}^{m}(q^{2m}-1)q^{(a+b-1)^2/4}}. \tag{29}$$

It suffices to find the top term of (29) as a polynomial in q. The coefficient is $1/2$ and the top q degree of (29) is

$$2(\sum_{k=N'_\pi+1}^{N} k) + (a+b)N'_\pi - 2(\sum_{k=1}^{m} k) - \frac{(a+b-1)^2}{4} =$$
$$(2N - m + \frac{a+b-1}{2} + 1)(m - \frac{a+b-1}{2}) +$$
$$(a+b)(N - m + \frac{a+b-1}{2}) - \frac{(a+b-1)^2}{4} - m(m+1) =$$
$$(2m+1)(N-m),$$

as claimed.

Now we treat the case of $O(W,B)$ for W of even dimension $2m$. Assume the notation of the above paragraphs. Fix an irreducible representation of $O(W,B)$.

The dimensions of π and $\eta^V_{W,B}(\pi)$ are

$$dim(\pi) =$$
$$\frac{|O^\sigma_{2m}(\mathbb{F}_q)|_{q'}}{2^{\alpha(\pi)+\beta(\pi)} \cdot |H \times O^{\pm}_{2p}(\mathbb{F}_q)|_{q'}} \cdot dim(u_H) \cdot dim\binom{\lambda_1 < \cdots < \lambda_a}{\mu_1 < \cdots < \mu_b} \tag{30}$$

$$dim(\eta^V_{W,B}(\pi)) =$$
$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{2^{\beta(\pi)} \cdot |H^D/Z(H^D) \times SO_{2(N-m+p)+1}(\mathbb{F}_q)|_{q'}}$$
$$dim(u_H) \cdot dim\binom{\lambda_1 < \cdots < \lambda_a < N'_\pi}{\mu_1 < \cdots < \mu_b}.$$

Now the center of $Z(H^D)$ is only non-trivial when $\beta(\pi) = 1$, in which case $Z(H^D) = \mathbb{Z}/2$. Therefore the dimension of $\eta^V_{W,B}(\pi)$ can be processed to

$$dim(\eta^V_{W,B}(\pi)) =$$
$$\frac{|Sp_{2N}(\mathbb{F}_q)|_{q'}}{|H^D \times SO_{2(N-m+p)+1}(\mathbb{F}_q)|_{q'}} \cdot dim(u_H) \cdot dim\binom{\lambda_1 < \cdots < \lambda_a < N'_\pi}{\mu_1 < \cdots < \mu_b}. \tag{31}$$

We again compare the factor $dim\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}/|O^{\pm}_{2p}(\mathbb{F}_q)|_{q'}$ in (30) with the factor $dim\binom{\lambda_1<\cdots<\lambda_a<N'_\pi}{\mu_1<\cdots<\mu_b}/|SO_{2(N-m+p)+1}(\mathbb{F}_q)|_{q'}$ in (31) using the dimension formula (24). Now $a+b$ is fixed to be even, since $\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}$ is a symbol of $SO^{\pm}_{2p}(\mathbb{F}_q)$. We note that then, instead of (26), we have

$$d(a+1,b)-d(a,b)=\sum_{i=1}^{(a+b)/2}(a+b-2i)=\frac{(a+b)(a+b-2)}{4}.$$

The power of 2 in the denominator in $dim\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}/|SO^{\pm}_{2p}(\mathbb{F}_q)|_{q'}$ is $c(a,b)=(a+b-2)/2$, so the of 2 in the denominator of the factor $dim\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}/|O^{\pm}_{2p}(\mathbb{F}_q)|_{q'}$ is $(a+b)/2$, matching the power $c(a+1,b)$ in the denominator of $dim\binom{\lambda_1<\cdots<\lambda_a<N'_\pi}{\mu_1<\cdots<\mu_b}/|SO_{2(N-m+p)+1}(\mathbb{F}_q)|_{q'}$. We find

$$\frac{dim\binom{\lambda_1<\cdots<\lambda_a<N'_\pi}{\mu_1<\cdots<\mu_b}}{|SO_{2(N-m+p)+1}(\mathbb{F}_q)|_{q'}}=$$

$$\frac{\prod_{i=1}^{a}(q^{N'_\pi}-q^{\lambda_i})\prod_{j=1}^{b}(q^{N'_\pi}+q^{\mu_j})}{\prod_{k=1}^{N'_\pi}(q^{2k}-1)q^{(a+b)(a+b-2)/4}}\cdot\frac{dim\binom{\lambda_1<\cdots<\lambda_a}{\mu_1<\cdots<\mu_b}}{|O^{\pm}_{2p}(\mathbb{F}_q)|_{q'}}\tag{32}$$

Therefore, using (28) and

$$|O^{\pm}_{2n}(\mathbb{F}_q)|_{q'}=2(q^n\mp 1)\cdot\prod_{k=1}^{n-1}(q^{2k}-1),$$

we see that $dim(\eta^V_{W,B}(\pi))$ is exactly $dim(\pi)$ multiplied by

$$2^{\alpha(\pi)+\beta(\pi)}\frac{\prod_{k=N'_\pi+1}^{N}(q^{2k}-1)\prod_{i=1}^{a}(q^{N'_\pi}-q^{\lambda_i})\prod_{j=1}^{b}(q^{N'_\pi}+q^{\mu_j})}{2(q^m\mp 1)\cdot\prod_{k=1}^{m-1}(q^{2k}-1)q^{(a+b)(a+b-2)/4}}.\tag{33}$$

The top coefficient is $2^{\alpha(\pi)+\beta(\pi)-1}$, and the top q degree of (33) is

$$2(\sum_{k=N'_\pi+1}^{N}k)+(a+b)N'_\pi-m-2(\sum_{k=1}^{m-1}k)-\frac{(a+b)(a+b-2)}{4}=$$

$$(2N-m+\frac{a+b}{2}+1)(m-\frac{a+b}{2})+$$

$$(a+b)(N-m+\frac{a+b}{2})-\frac{(a+b)(a+b-2)}{4}-m^2=$$

$$2m(N-m)+m,$$

as claimed. $\square$

Now we get that the top term of the partial character ratio sum appearing in (5) is (9), and similarly, the top terms of the partial character ratio sums appearing in (6) corresponding to the two even orthogonal groups are (10) and (11). Further, though, we now note that in the description of the eta correspondence for two irreducible representations π, π' with the same centralizer-unipotent type over an orthogonal group $O(W,B)$, the eta correspondences $\eta^V_{W,B}(\pi)$, $\eta^V_{W,B}(\pi')$ also have the same centralizer-unipotent type over $Sp(V)$. In particular, we have $dim(\pi) = dim(\pi')$ and

$$dim(\eta^V_{W,B}(\pi)) = dim(\eta^V_{W,B}(\pi')).$$

Therefore, the error difference term of (5) and (9) can be expressed as a sum over all possible centralizer-unipotent types (H,u) over $O(W,B) = O_{2m+1}(\mathbb{F}_q,\sigma)$ of terms

$$K_{H,u} \cdot \left(\sum_{\pi \in \widehat{O_{2m+1}(\mathbb{F}_q,\sigma)}[H,u]} \frac{\chi_{\eta^{2N}_{2m+1,\sigma}(\pi)}(g)}{dim(\pi)} \right) \tag{34}$$

for coefficients $K_{H,u}$ all of q-degree strictly less than $-(2m+1)(N-m)$. Similarly, the error difference of the summands of (6) and their corresponding top terms (10) or (11) can be expressed as a sum over all possible character-unipotent types (H,u) over $O(W^\pm,B^\pm) = O^\pm_{2m}(\mathbb{F}_q)$ of terms

$$K^\pm_{H,u} \cdot \left(\sum_{\pi \in \widehat{O^\pm_{2m}(\mathbb{F}_q)}[H,u]} 2^{\alpha(\pi)+\beta(\pi)} \frac{\chi_{\eta^{2N}_{2m,\pm}(\pi)}(g)}{dim(\pi)} \right) \tag{35}$$

for coefficients $K^\pm_{H,u}$ all of q-degree strictly less than $-2m(N-m)+m$.

Again, we recall that the number of centralizer-unipotent type does not vary as q increases, so the number of error terms of the form (34) and (35) also do not vary as q increases.

3. The oscillator representation and the proof of Theorem 2

In the previous section we described how the partial character ratio sums appearing in Theorem 2 can be expressed as the top term (9), summed with error terms of the form (34) for the case of odd orthogonal groups $O_{2m+1}(\mathbb{F}_q,\sigma)$, and in the case of even orthogonal groups

$O^{\pm}_{2m}(\mathbb{F}_q)$, they can be expressed as the top terms (10) and (11) summed with error terms of the form (35). We prove that the limits of these terms vanish for every non-trivial element g of the symplectic group $Sp_{2N}(\mathbb{F}_q)$ as q goes to infinity. However, the characters of the eta correspondence are difficult to compute explicilty. Instead, for the present purpose, we prefer to work with whole oscillator representations. To simplify notation, let us write (W, B) for the fixed $(2m+1)$-dimensional orthogonal space. Let us write $(W^{\pm}, B^{\pm})$ for the $2m$-dimensional orthogonal spaces such that $O(W^{\pm}, B^{\pm}) = O^{\pm}_{2m}(\mathbb{F}_q)$. Write V_N for the $2N$-dimensional symplectic space.

3.1. **Processing into linear combinations of oscillator representation characters.** Our next step is, in the odd case of $O(W, B) = O_{2m+1}(\mathbb{F}_q)$, to process the sum

$$\sum_{\pi\in \widehat{O(W,B)}} \frac{\chi_{\eta^{V_N}_{(W,B)}(\pi)}(g)}{dim(\pi)} \tag{36}$$

appearing in (9), and in the even cases of $O(W^{\pm}, B^{\pm}) = O^{\pm}_{2m}(\mathbb{F}_q)$, to process the sum

$$\begin{gathered}\sum_{\pi\in \widehat{O(W^+,B^+)}} 2^{\alpha(\pi)+\beta(\pi)}\frac{\chi_{\eta^{V_N}_{W^+,B^+}(\pi)}(g)}{dim(\pi)}+\\ \sum_{\pi\in \widehat{O(W^-,B^-)}} 2^{\alpha(\pi)+\beta(\pi)}\frac{\chi_{\eta^{V_N}_{W^-,B^-}(\pi)}(g)}{dim(\pi)}.\end{gathered} \tag{37}$$

We will process these sums into linear combinations of the form

$$\sum_{(h)\in O(W,B)} c_{(h)}\cdot \chi_{\omega^{top}[V_N\otimes W]}(g\otimes h), \tag{38}$$

and

$$\begin{gathered}\sum_{(h)\in O(W^+,B^+)} c^+_{(h)}\cdot \chi_{\omega^{top}[V_N\otimes W^+]}(g\otimes h)+\\ \sum_{(h)\in O(W^-,B^-)} c^-_{(h)}\cdot \chi_{\omega^{top}[V_N\otimes W^-]}(g\otimes h),\end{gathered} \tag{39}$$

with terms consisting of some constant $c_{(h)}$ or $c^{\pm}_{(h)}$ multiples of characters of the *top part* of an oscillator representation of $V_N \otimes W$, which is given as

$$\omega^{top}[V_N\otimes W] = \bigoplus_{\pi\in \widehat{O(W,B)}} \eta^V_{W,B}(\pi)\otimes \pi. \tag{40}$$

This top part $\omega^{top}[V_N\otimes W]$ is the summand of the oscillator representation $\omega[V_N\otimes W]$ such that all other summands involve $\eta_{W,B}^{V_{N'}}$ for $N'<N$. We argue then that, for $N>>m$, the lower parts of $\omega[V_N\otimes W]$ can be ignored, reducing the claim to a statement about character of the full, genuine oscillator representation, which can be readily computed using the Schrödinger model.

By (40), re-expressions of the form (38) and (39) are possible by the character theory of the orthogonal groups, specifically by choosing $c_{(h)}$ and $c_{(h)}^{\pm}$ so that, in the case of $O(W,B)=O_{2m+1}(\mathbb{F}_q)$, for every irreducible representation π

$$\frac{1}{dim(\pi)}=\sum_{(h)\in O(W,B)}c_{(h)}\cdot\chi_\pi(h),\tag{41}$$

and in the case of $O(W^{\pm},B^{\pm})=O_{2m}^{\pm}(\mathbb{F}_q)$,

$$\frac{2^{\alpha(\pi)+\beta(\pi)}}{dim(\pi)}=\sum_{(h)\in O(W^{\pm},B^{\pm})}c_{(h)}^{\pm}\cdot\chi_\pi(h).\tag{42}$$

We must determine information about the coefficients $c_{(h)}$ and $c_(h)^{\pm}$ to reduce the top terms (9), (10), and (11) appearing in the partial character ratio sums (5) and (6), respectively.

For a complete argument, we must also process the error terms (34) and (35). We will process these terms similarly into

$$K_{H,u}\cdot\left(\sum_{(h)\in O(W,B)}a_{(h)}^{H,u}\cdot\chi_{\omega^{top}[V_N\otimes W]}(g\otimes h)\right)\tag{43}$$

in the odd case $O(W,B)=O_{2m+1}(\mathbb{F}_q)$ and

$$K_{H,u}\cdot\left(\sum_{(h)\in O(W^{\pm},B^{\pm})}a_{(h)}^{\pm,(H,u)}\cdot\chi_{\omega^{top}[V_N\otimes W]}(g\otimes h)\right)\tag{44}$$

in the even case $O(W^{\pm},B^{\pm})=O_{2m}^{\pm}(\mathbb{F}_q)$ for coefficients $a_{(h)}^{H,u}$ and $a_{(h)}^{\pm,(H,u)}$. We do this by taking the coefficients $a_{(h)}^{H,u}$ for the odd case defined so that

$$\begin{aligned}\frac{1}{dim(\pi)}&=\sum_{(h)\in O(W,B)}a_{(h)}^{H,u}\cdot\chi_\pi(h)\text{ for }\pi\in\widehat{O(W,B)}[H,u]\\0&=\sum_{(h)\in O(W,B)}a_{(h)}^{H,u}\cdot\chi_\pi(h)\text{ else,}\end{aligned}$$

and $a^{\pm,(H,u)}_{(h)}$ for the even case defined so that

$$\begin{aligned} \frac{2^{\alpha(\pi)+\beta(\pi)}}{dim(\pi)} &= \sum_{(h)\in O(W^\pm,B^\pm)} a^{\pm,(H,u)}_{(h)} \cdot \chi_\pi(h) \text{ for } \pi \in O(\widehat{W^\pm,B^\pm})[H,u] \\ 0 &= \sum_{(h)\in O(W^\pm,B^\pm)} a^{\pm,(H,u)}_{(h)} \cdot \chi_\pi(h) \text{ else.} \end{aligned}$$

We must also determine information about the coefficients $a^{H,u}_{(h)}$, $a^{\pm,(H,u)}_{(h)}$.

Specifically, we prove the following

Lemma 5. *In the odd case where $O(W,B) = O_{2m+1}(\mathbb{F}_q)$, the coefficients $c_{(h)}$, $a^{H,u}_{(h)}$ satisfy the following properties:*

1. *Only conjugacy classes $(h) \in SO(W,B)$ are used, i.e. $c_{(h)} = 0$ for every $h \in O(W,B)$ with determinant -1. Similarly, for every possible centralizer-unipotent type (H,u) for $O(W,B)$, the coefficients $a^{H,u}_{(h)} = 0$ for every $h \in O(W,B)$ with determinant -1.*
2. *The coefficient at the identity matrix $h = I \in O(W,B)$, as a rational function of q, has degree*
$$deg_q(|c_{(1)}|) = -2m^2.$$
For any centralizer-unipotent type (H,u) for $O(W,B)$, we have $deg_q(a^{H,U}_{(I)}) \le -2m^2$.

In the even case where $O(W^\pm,B^\pm) = O^\pm_{2m}(\mathbb{F}_q)$, the coefficients $c^\pm_{(\pm I)}$ at the central elements $h = I, -I \in O(W^\pm,B^\pm)$, as a rational functions of q, have degrees

$$deg_q(|c^\pm_{(-I)}|) \le deg_q(|c^\pm_{(I)}|) = -2m(m-1).$$

For any centralizer-unipotent type (H,u) for $O(W^\pm,B^\pm)$, we have

$$deg_q(a^{\pm,(H,U)}_{(I)}), deg_q(a^{\pm,(H,U)}_{(-I)}) \le -2m(m-1).$$

Proof. We first again note that by the splitting $O(W,B) = \mathbb{Z}/2 \times SO(W,B)$, every irreducible representation π of $O(W,B)$ can be expressed as $(\pm1)\otimes\overline{\pi}$ for (±1) denoting an irreducible representation of $\mathbb{Z}/2$ and $\overline{\pi}$ denoting the restriction of π to $SO(W,B)$. We know the $\overline{\pi}$ is still irreducible as an $SO(W,B)$-representation, and it has the same dimension and character values at $SO(W,B)$ elements as π does. The character theory of $SO(W,B)$ then ensures the existence of constants

$$\frac{1}{dim(\overline{\pi})} = \sum_{(h)\in SO(W,B)} c_{(h)} \cdot \chi_{\overline{\pi}}(h),$$

which is equivalent to (41). Therefore, we get (1).

Now we approach the claim that $deg_q(c_{(I)}) = -2m^2$. First, let us consider for each individual irreducible representation π, the coefficients $b^{\pi}_{(h)}$ such that

$$(45)\qquad \begin{aligned} &\sum_{(h)\in SO(W,B)} b^{\pi}_{(h)} \cdot \chi_{\pi}(h) = 1 \\ &\sum_{(h)\in SO(W,B)} b^{\pi}_{(h)} \cdot \chi_{\pi'}(h) = 0 \text{ for } \pi' \neq \pi \in \widehat{SO(W,B)} \end{aligned}$$

(we use $SO(W,B)$ instead of $O(W,B)$ by the above argument). We can see that $c_{(h)}$ can then be expressed as

$$c_{(h)} = \sum_{\pi\in\widehat{SO(W,B)}} \frac{b^{\pi}_{(h)}}{dim(\pi)}.$$

Now, let us write $CT_{SO(W,B)}$ for the character table of $SO(W,B)$, considered as a matrix with entries $\chi_\pi(h)$, with columns indexed by conjugacy classes (h) in $SO(W,B)$ and rows indexed by irreducible representations of $SO(W,B)$. Then, by their definition according to (45), the inverse matrix $(CT_{SO(W,B)})^{-1}$ consists of the entries $b^{\pi}_{(h)}$ with rows indexed by conjugacy classes (h) and columns indexed by irreducible representations π. We also recall that by character orthogonality, $(CT_{SO(W,B)})^{-1}$ can be expressed as the transpose matrix $(CT_{SO(W,B)})^{T}$ with each row multiplied by the size of the corresponding conjugacy class, divided by the group order $|SO(W,B)|$. In particular, the top row of $(CT_{SO(W,B)})^{-1}$, consisting of $b^{\pi}_{(I)}$, is described by

$$b^{\pi}_{(I)} = \frac{1}{|SO(W,B)|} \cdot \chi_\pi(I) = \frac{dim(\pi)}{|SO(W,B)|}.$$

Therefore,

$$(46)\qquad c_{(I)} = \sum_{\pi\in\widehat{SO(W,B)}} \frac{1}{|SO(W,B)|} = \frac{|\widehat{SO(W,B)}|}{|SO(W,B)|}.$$

Again, we recall

$$|SO(W,B)| = |SO_{2m+1}(\mathbb{F}_q)| = q^{m^2} \cdot \prod_{i=1}^{m}(q^{2i}-1),$$

which has q-degree $2m^2+m$. The number of irreducible representations of $SO(W,B)$ (i.e. the number of conjugacy classes) can be computed according to the results of [1], Specifically, in the case of odd orthogonal groups, Theorem 7.5.1 gives that the number of conjugacy classes in $SO_{2m+1}(\mathbb{F}_q)$ (i.e. half of the number of conjugacy classes in $O_{2m+1}(\mathbb{F}_q)$) is $1/4$ times the coefficient of t^{2m+1} in the taylor expansion of

$$\prod_{i=1}^{\infty} \frac{(1+t^{2i-1})^4}{1-qt^{2i}}. \tag{47}$$

Expanding this gives

$$(1+t)^4 \cdot (1+qt^2+q^2t^4+\dots) \cdot (1+t^3)^4 \cdot (1+qt^4+q^2t^8+\dots) \dots . \tag{48}$$

We can see that to find the highest q-degree term of the coefficient of t^{2m+1} in (47), we must consider the summand 1 in each factor of (47) corresponding to $i \geq 2$ and in the factors corresponding to $i = 1$, consider the summand $4t$ of $(1+t)^4$ and the summand $q^m t^{2m}$ of $(1+qt^2+q^2t^4+\dots)$. Therefore, the top q-term of the coefficient of t^{2m+1} in (47) is $4q^m$. Hence, q-degree of the number of conjugacy classes in $SO(W,B)$ is

$$deg_q(|\widehat{SO(W,B)}|) = -m.$$

Therefore, (46) gives the claim $deg_q(c_{(I)}) = -2m^2$. A similar argument applies to $a^{H,u}_{(I)}$ for each centralizer-unipotent type (H,u) for $O(W,B)$. Analogous to (46), $a^{H,u}_{(I)}$ is equal to the number of irreducible representations π of centralizer-unipotent type (H,u), divided by $|SO(W,B)|$. In particular, $|a^{H,u}_{(I)}| \leq |c_{(I)}|$, giving

$$deg_q(a^{H,u}_{(I)}) \leq deg_q(c_{(I)}) = -2m^2, \tag{49}$$

as claimed.

Now we consider the case of the even orthogonal groups $O(W^\pm, B^\pm) = O^\pm_{2m}(\mathbb{F}_q)$. We may use a similar argument as above with $SO(W,B)$ replaced by $O(W^\pm, B^\pm)$ to see that

$$\frac{|\widehat{O(W^\pm, B^\pm)}|}{4 \cdot |O(W^\pm, B^\pm)|} \leq c^\pm_{(I)} \leq \frac{|\widehat{O(W^\pm, B^\pm)}|}{|O(W^\pm, B^\pm)|} \tag{50}$$

similarly as in (46). To be more specific, as before, the top row of the inverse of the character matrix of $O(W^{\pm}, B^{\pm})$ consists of terms $dim(\pi)/|O(W^{\pm}, B^{\pm})|$, and $c^{\pm}_{(I)}$ is the sum of these positive entries, each with coefficient $2^{\alpha(\pi)+\beta(\pi)}dim(\pi)$. Then considering $1 \leq 2^{\alpha(\pi)+\beta(\pi)} \leq 4$ gives (50). Now

$$|O(W^{\pm}, B^{\pm})| = |O^{\pm}_{2m}(\mathbb{F}_q)| = 2q^{m(m-1)}(q^m \mp 1)\prod_{i=1}^{m-1}(q^{2i}-1),$$

which has q-degree $2m^2 - m$. On the other hand, we can again apply Theorem 7.5.1 of [1], which in this case gives that the number of conjugacy classes in $O^{\pm}_{2m}(\mathbb{F}_q)$ is half of the coefficient of t^{2m} in the expansion of (47) plus or minus the coefficient of t^{2m} in the expansion of

$$\prod_{i=1}^{\infty}\frac{1-t^{4i-2}}{1-qt^{4i}}. \tag{51}$$

In the expansion (48) of (47), to find the top q-term of the coefficient of t^{2m}, we must consider the summand 1 in each factor of (47) corresponding to $i \geq 2$ and in the factors corresponding to $i = 1$, consider the summand 1 of $(1+t)^4$ and the summand $q^m t^{2m}$ of $(1+qt^2+q^2t^4+\dots)$. Similarly, we can expand (51) to see the q-degree of the coefficient of t^{2m} is less than or equal to $m/2$.

Therefore, the top q-degree of $|\widehat{O(W^{\pm}, B^{\pm})}|$ is q^m. Therefore, by (50), we see

$$deg_q(c^{\pm}_{(I)}) = -2m(m-1),$$

as claimed. As in (49), we can apply a similar argument to the coefficients $a^{\pm,(H,u)}_{(h)}$, obtaining

$$deg_q(a^{\pm,(H,u)}_{(I)} \leq deg_q(c_{(I)}) = -2m(m-1).$$

Now, we know $deg_q(c^{\pm}_{(-I)}) \leq deg_q(c^{\pm}_{(I)})$ and $deg_q(a^{\pm,(H,u)}_{(-I)}) \leq deg_q(a^{\pm,(H,u)}_{(I)})$, since for every $\pi \in \widehat{O(W^{\pm}, B^{\pm})}$,

$$|\frac{\chi_\pi(-I)}{dim(\pi)}| \leq 1.$$

$\square$

Finally, since we assume $N >> m$, the coefficient $1/q^{(2m+1)(N-m)}$ in (9) or the coefficient $1/q^{2m(N-m)+m}$ in (10) and (11) will allow us to replace all characters of $\omega^{top}[V_N \otimes W]$ or $\omega^{top}[V_N \otimes W^{\pm}]$ by characters of the genuine oscillator representation $\omega[V_N \otimes W]$ or $\omega[V_N \otimes W^{\pm}]$, which we use the Schrödinger model to compute: The full decomposition of the oscillator representation as a $Sp(V) \times O(W, B)$-representation (in the symplectic stable range we consider here) is

$$\omega[V_N \otimes W] =$$
$$\bigoplus_{k=0}^{m} \bigoplus_{\pi \in \widehat{O(W[-k], B[-k])}} Ind^{P_k}(\pi \otimes \epsilon(det)) \otimes \eta^{V_N}_{(W[-k],B[-k])}(\pi)$$

where $(W[-k], B[-k])$ denotes the orthogonal subspace of W dimension $(2m-2k)+1$ with symmetric bilinear form satsifying $disc(B) = disc(B[-k])$, and Ind^{P_k} denotes parabolic induction from the standard parabolic with Levi factor $O(W[-k], B[-k]) \times GL_k(\mathbb{F}_q) \subseteq O(W, B)$, considering $\epsilon(det)$ as a representation of $GL_k(\mathbb{F}_q)$ (see [4]). For terms $k > 0$ corresponding to "lower" pieces of the oscillator representation, assuming $N >> m$ the denominator $q^{(2m+1)(N-m)}$ or $q^{2m(N-m)+m}$ kills the contributed terms since the coefficient $(2m+1)$ or $2m$ of N is larger than the coefficient $(2(m-k)+1)$ or $2(m-k)$ of N corresponding to the lower eta correspondences.

Therefore, we reduce the vanishing of (9) to the claim that

$$\lim_{q\to\infty} \frac{1}{q^{(2m+1)(N-m)}} \sum_{(h)\in SO(W,B)} c_{(h)} \cdot \chi_{\omega[V_N\otimes W]}(g \otimes h) = 0 \tag{52}$$

(dividng (9) by 2). Similarly, we reduce the vanishing of the sum of (10) and (11) to the claim

$$\lim_{q\to\infty} \frac{\sum_{(h)\in O(W^+,B^+)} c^+_{(h)} \cdot \chi_{\omega[V_N\otimes W^+]}(g\otimes h)}{q^{2m(N-m)+m}} + \frac{\sum_{(h)\in O(W^-,B^-)} c^-_{(h)} \cdot \chi_{\omega[V_N\otimes W^-]}(g\otimes h)}{q^{2m(N-m)+m}} = 0 . \tag{53}$$

3.2. **The Schrödinger model and the proof of Theorem 2.** Now we conclude Theorem 2 by explicitly computing that the characters of the oscillator representation $\chi_{\omega[V_N\otimes W]}(g \otimes h)$ cannot attain higher q powers than the denominators we are given in (52).

First, we recall (a mild variant of) the Schrödinger model of an oscillator representation. Consider our symplectic space $V_N = \mathbb{F}_q^{2N}$. Take the symplectic form to be standard, i.e. to correspond to the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The Schrödinger model typically is described as functions on a Lagrangian $\mathbb{F}_q^N$. Here, since we are working over finite fields, we find it simpler for the present purpose to identify the indicator function of a vector with the vector itself and work directly on $\mathbb{C}\mathbb{F}_q^N$. Then, for a basis element $(v) \in \mathbb{F}_q^N$, the action of an oscillator representation ω_a is given by

$$\omega_a \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} (v) = \psi_a(\frac{vAv^T}{2}) \cdot (v)$$

$$\omega_a \begin{pmatrix} 0 & B \\ -B^{-1} & 0 \end{pmatrix} (v) = \sum_{w \in \mathbb{F}_q^N} \frac{\psi_a(wBv^T)}{\sum_{u \in \mathbb{F}_q^N} \psi_a(uBu^T)} \cdot (w),$$

$$\omega_a \begin{pmatrix} (C^T)^{-1} & 0 \\ 0 & C \end{pmatrix} (v) = \epsilon(det(C)) \cdot (Cv)$$

for A, B symmetric $N \times N$ matrices and C an invertible $N \times N$ matrix, where ψ_a denotes the additive character associated to a under the identification of $\mathbb{F}_q$ with its Pontrjagin dual.

This can be used to obtain the following very rough statement about the q-degree of character values of an oscillator representation:

Say, for a general matrix g, an eigenvalue λ has *true multiplicity* r if in the Jordan normal form of g (over $\overline{\mathbb{F}_q}$), there are r trivial one-by-one Jordan blocks with eigenvalue λ.

Lemma 6. *Consider a conjugacy class $h \in O(W, B)$. Consider the distinct eigenvalues $\lambda_1, \dots, \lambda_a$ not equal to 1 of h. Write $r_1, \dots, r_a$ for the true multiplicity of each of these eigenvalues. Write ℓ for the true multiplicity of 1 as an eigenvalue of g and p for its total multiplicity. Then for $1 \neq g \in Sp(V_N)$, the maximal degree of the character value of $\omega[V_N \otimes W]$ at $g \otimes h$ is*

$$deg_q(|\chi_{\omega[V_N \otimes W]}(g \otimes h)|) =$$
$$max(r_1 \cdot N, \dots, r_a \cdot N, \frac{p \cdot N + \ell \cdot (N-1)}{2})$$

In particular, we can use this to conclude Theorem 2:

Proof of Theorem 2. First, we treat the odd case, when $dim(W) = 2m + 1$. For an element h of $SO(W, B) = SO_{2m+1}(\mathbb{F}_q)$, let us consider its Jordan normal form over $\overline{\mathbb{F}_q}$. First, h must have 1 as an eigenvalue of multiplicity at least one.

In particular, for $N >> m$, the terms contributed by $h \neq I$ can all be seen to vanish in (52), since the multiple of N in the q degree of $\chi_{\omega[V_N\otimes W]}(g\otimes h)$ will be strictly less than the $(2m+1)$ coefficient appearing in the denominator.

It remains to show

$$\lim_{q\to\infty}\frac{c_{(1)}}{q^{(2m+1)(N-m)}}\cdot\chi_{\omega[V_N\otimes W]}(g\otimes I)=0.$$

As we computed in Lemma 5, $deg_q(|c_{(1)}|)\leq -2m^2$, so replacing $c_{(1)}$ by $1/q^{2m^2}$, it in fact suffices to show the vanishing

$$\lim_{q\to\infty}\frac{1}{q^{(2m+1)N-m}}\cdot\chi_{\omega[V_N\otimes W]}(g\otimes I)=0. \tag{54}$$

Now, we note that

$$\chi_{\omega[V_N\otimes W]}(g\otimes I)=\prod_{i=1}^{2m+1}\chi_{\omega_{a_i}[V_N]}(g),$$

where $a_1,\ldots,a_{2m+1}$ denote the eigenvalues of the symmetric bilinear form B fixed on W, and in particular,

$$deg_q(|\chi_{\omega[V_N\otimes W]}(g\otimes I)|)=(2m+1)\cdot deg_q(|\chi_{\omega[V_N]}(g)|). \tag{55}$$

Now, applying Lemma 6 again, we find that the maximal q-degree of $|\chi_{\omega[V_N]}(g)|$ for non-trivial g occurs when all of g's eigenvalues are 1, and the true multiplicity of 1 is $2(N-1)$, where $|\chi_{\omega[V_N]}(g)|=q^{N-\frac{1}{2}}$. For a general $g\neq 1\in Sp(V_N)$, we have

$$deg_q(|\chi_{\omega[V_N]}(g)|)\leq N-\frac{1}{2}. \tag{56}$$

Combining this with (55), we reduce (54) to

$$\lim_{q\to\infty}\frac{q^{(2m+1)(N-\frac{1}{2})}}{q^{(2m+1)N-m}}=\lim_{q\to\infty}\frac{1}{\sqrt{q}}=0. \tag{57}$$

Hence, the top term (9) of the claimed sum (5) therefore vanishes. Now we consider the error terms (43). Similarly as above, for each centralizer-unipotent type (H,u), it suffices to consider the term occuring at $h=I$

$$\lim_{q\to\infty}K_{H,u}\cdot a_{(I)}^{H,u}\cdot\chi_{\omega[V_N\otimes W]}(g\otimes I).$$

By (56) and Lemma 5, to prove this vanishes, it suffices to prove

$$\lim_{q\to\infty}K_{H,u}\cdot\frac{q^{(2m+1)(N-\frac{1}{2})}}{q^{2m^2}}=0,$$

which follows since $deg_q(K_{H,u})<-(2m+1)(N-m)$, and therefore this term has strictly smaller q-degree than $-1/2$ (see (57)).

Now we consider the even case, at dimension $2m$. By a similar argument as in the odd case, the factor $1/q^{2m(N-m)}$ in (53) ensures that the terms contributed by $h \neq I, -I$ all vanish, since, again, the multiple of N in the q degree of $\chi_{\omega[V_N \otimes W]}(g \otimes h)$ will be strictly less than the $2m$ coefficient appearing in the numerator. It now suffices to show

$$\lim_{q\to\infty} \frac{c^+_{(1)} \cdot \chi_{\omega[V_N\otimes W^+]}(g\otimes 1) + c^-_{(1)} \cdot \chi_{\omega[V_N\otimes W^-]}(g\otimes 1)}{q^{2m(N-m)+m}} \tag{58}$$

and

$$\lim_{q\to\infty} \frac{c^+_{(-I)} \cdot \chi_{\omega[V_N\otimes W^+]}(g\otimes -I) + c^-_{(-I)} \cdot \chi_{\omega[V_N\otimes W^-]}(g\otimes -I)}{q^{2m(N-m)+m}} \tag{59}$$

vanish. As we computed in Lemma 5, $deg_q(|c^+_{(I)}|)$, $deg_q(|c^-_{(I)}) \leq 2m(m-1)$, and $c^+_{(I)}$ and $c^-_{(I)}$ have the same top coefficient (and similarly for $c^{\pm}_{(-I)}$). Consider replacing $c^{\pm}_{(I)}$ by $1/q^{2m(m-1)}$:

$$\lim_{q\to\infty} \frac{1}{q^{2mN-m}} (\chi_{\omega[V_N\otimes W^+]}(g\otimes I) + \chi_{\omega[V_N\otimes W^-]}(g\otimes I)) = 0. \tag{60}$$

Similarly as before, writing $a^{\pm}_1, \ldots, a^{\pm}_{2m}$ for the eigenvalues of $B^{\pm}$, we have

$$\chi_{\omega[V_N\otimes W^{\pm}]}(g\otimes I) = \prod_{i=1}^{2m} \chi_{\omega_{a_i^{\pm}}[V_N]}(g).$$

Now we recall (56). For the character of $\omega[V_N]$ at an element $1 \neq g \in Sp_{2N}(\mathbb{F}_q)$ to attain q degree $N - \frac{1}{2}$, g must be a transvection, in which case

$$\chi_{\omega_a[V_N]}(g) = (-1)^{(N-1)(s+1)} \cdot \epsilon(-a/2) \cdot \sqrt{\epsilon(-1)} \cdot q^{N-\frac{1}{2}}, \tag{61}$$

where q is the sth power of a prime (see [4] for more details). Therefore, in the case of g a transvection, reduces (60) to a constant multiple of

$$\lim_{q\to\infty} \frac{+q^{2m(N-\frac{1}{2})} - q^{2m(N-\frac{1}{2})}}{q^{2mN-m}} = 1 - 1 = 0.$$

All other g follow immediately, since the power of q in the numerator is strictly less. With $c^{\pm}_{(I)}$ replaced by $1/q^{2m(m-1)}$, the top q-degree terms which can be obtained only in the case of transvection elements g have q-degree 0. The case of the $h = -I$ term proceeds similarly. Therefore, (60) implies the vanishing of the sum of the top terms (10) and (11).

Finally, the treatment of the error terms occuring from the lower order expressions in (8) proceeds similarly as in the odd case: Consider the error terms (44). Similarly as above, for each centralizer-unipotent type (H, u), it suffices to consider the terms occuring at $h = I$ and transvection g

$$\lim_{q\to\infty} K^{\pm}_{H,u} \cdot a^{\pm,(H,u)}_{(I)} \cdot \chi_{\omega[V_N \otimes W^{\pm}]}(g \otimes I).$$

By (61) and Lemma 5, to prove this vanishes, it suffices to prove

$$\lim_{q\to\infty} K_{H,u} \cdot \frac{q^{2m(N-\frac{1}{2})}}{q^{2m(m-1)}} = 0, \tag{62}$$

which follows since $deg_q(K_{H,u}) < -2m(N-m)+m$, and therefore, the q-degree of the term (62) is strictly less than 0.

□

4. Explicit Computation for $O_3(\mathbb{F}_q)$

In this section, we work out some examples, from rank 1 orthogonal groups. We calculate the coefficients $c_{(h)}$ in (52) explicitly, since we fully know the character tables of small orthogonal groups.

We note that the full decomposition of restricted of the oscillator representations we wrote down in [4, 5, 6] can theoretically be used to explicitly inductively compute the character table of finite orthogonal and symplectic groups in general. With that information, the method of the $O_3(\mathbb{F}_q)$ example could be used to explicitly compute all the coefficients $c_{(h)}$ used in (38). However, this has not been done yet.

We now consider the case of $O_3(\mathbb{F}_q)$in explicit terms. Following the proof given above, according to (52), we want to prove the vanishing

$$\lim_{1\to\infty} \frac{1}{q^{3N-3}} \sum_{(h)\in SO_3(\mathbb{F}_q)} c_{(h)} \cdot \chi_{\omega[V_N\otimes \mathbb{F}_q^3]}(g \otimes h) = 0$$

for every $1 \neq g \in Sp_{2N}(\mathbb{F}_q)$.

Here, since the character table of $SO_3(\mathbb{F}_q)$ can be explicitly written down, we can fully calculate the coefficients $c_{(h)}$. The conjugacy classes (h) in $SO_3(\mathbb{F}_q)$ are

- the trivial conjugacy class (I)
- one conjugacy class we denote by (f) of an element with all 1 eigenvalues conjugate over $\overline{\mathbb{F}_q}$ to a three-by-three Jordan block
- one semisimple conjugacy class we denote by (h_λ) with eigenvalues $\{\lambda, \lambda^{-1}\}$ and a single eigenvalue 1, per choice of $\lambda \in (\mathbb{F}_q^\times \smallsetminus \{\pm 1\})/\lambda \sim \lambda^{-1}$
- one semisimple conjugacy class we denote by (h_μ) with eigenvalues $\{\mu, \mu^{-1}\}$ and and a single eigenvalue 1, per choice of $\mu \in (\mu_{q+1} \smallsetminus \{\pm 1\})/\mu \sim \mu^{-1}$, identifying μ_{q+1} with $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$.

- two semisimple conjugacy classes we denote by $(h^{\pm}_{-1})$ with eigenvalue -1 of multiplicity two and eigenvalue 1 of multiplicity one, differentiated by whether the symmetric bilinear form is split or non-split on the eigenspace of -1

$$h^{+}_{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h^{-}_{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We want to calculate the coefficients $c_{(h)}$ for each of these conjugacy classes. We recall that for $\lambda \in \mathbb{F}_q^\times$ not a square (resp. $\mu \in \mu_{q+1}$ not a square), we take $c_{(h_\lambda)}$ (resp. $c_{(h_\mu)}$) to be 0. All the $c_{(h_\lambda)}$ (resp. $c_{(h_\mu)}$) for $\lambda \in \mathbb{F}_q^\times$ a square (resp. $\mu \in \mu_{q+1}$ a square) are equal.

There are four possible dimensions of irreducible representations of $SO_3(\mathbb{F}_q)$, of dimension $1, q-1, q$, and $q+1$. The value of the characters of representations with these dimension on f are 1, -1, 0, and 1 respectively.

Suppose q is 1 mod 4. In this case, -1 is not a square in μ_{q+1}, so the values of

$$\sum_{\mu \in (\mu_{q+1})^2} \chi_\pi(h_\mu)$$

at π of dimension $1, q-1, q$, and $q+1$ are $(q-1)/4$, 1, $-(q-1)/4$, and 0, respectively. Since -1 is a square in $\mathbb{F}_q^\times$, we consider the sum

$$2 \cdot \sum_{\lambda \in (\mathbb{F}_q^\times)^2} \chi_\pi(h_\lambda) + h^{+}_{-1}$$

at π of dimension $1, q-1, q$, and $q+1$, which evaluate to $(q-3)/2$, 0, $(q-3)/2$, and -2, respectively. Therefore, we take $c_{(h^{-}_{-1})} = 0$, $c_{(h^{+}_{-1})} = c_{(h_\lambda)}/2$. Then the coefficients $c_{(I)}, c_{(f)}$, $c_{(h_\lambda)}$ for $\lambda \in (\mathbb{F}_q^\times)^2$, and $c_{(h_\mu)}$ for $\mu \in (\mu_{q+1})^2$ must solve the equations

$$\begin{aligned} 1 &= c_{(I)} + c_{(f)} + c_{(h_\lambda)} \frac{q-3}{2} + c_{(h_\mu)} \frac{q-1}{4} \\ \frac{1}{q-1} &= c_{(I)}(q-1) - c_{(f)} + c_{(h_\mu)} \\ \frac{1}{q} &= c_{(I)} q + c_{(h_\lambda)} \frac{q-3}{2} - c_{(h_\mu)} \frac{q-1}{4} \\ \frac{1}{q+1} &= c_{(I)}(q+1) + c_{(f)} - 2c_{(h_\lambda)} \end{aligned}$$

Solving these equations, we find the top terms of the coefficients are

$$c_{(I)} = \frac{q+2}{q(q^2-1)}, \quad c_{(f)} = \frac{2}{q+1}$$

$$c_{(h_\mu)} = \frac{2(q^2-q+1)}{q(q^2-1)}, \quad c_{(h_\lambda)} = \frac{q^2+q+1}{q(q^2-1)}$$

In particular, note that $c_{(I)} = |\widehat{SO_3(\mathbb{F}_q)}|/|SO_3(\mathbb{F}_q)|$.

Similarly, we can write down equations for $a_{(h)}^{SO_3(\mathbb{F}_q),1}$, $a_{(h)}^{SO_3(\mathbb{F}_q),\binom{0<1}{1}}$, $a_{(h)}^{\mu_{q+1},1}$, and $a_{(h)}^{\mathbb{F}_q^\times,1}$ and solve to find

$$a_{(I)}^{SO_3(\mathbb{F}_q),1} = \frac{2}{q(q^2-1)}, \quad a_{(f)}^{SO_3(\mathbb{F}_q),1} = \frac{2}{q}$$

$$a_{(h_\mu)}^{SO_3(\mathbb{F}_q),1} = \frac{2}{(q+1)}, \quad a_{(h_\lambda)}^{SO_3(\mathbb{F}_q),1} = \frac{1}{q-1}$$

$$a_{(I)}^{SO_3(\mathbb{F}_q),\binom{0<1}{1}} = \frac{2}{q(q^2-1)}, \quad a_{(f)}^{SO_3(\mathbb{F}_q),\binom{0<1}{1}} = 0$$

$$a_{(h_\mu)}^{SO_3(\mathbb{F}_q),1} = -\frac{2}{q(q+1)}, \quad a_{(h_\lambda)}^{SO_3(\mathbb{F}_q),1} = \frac{1}{q(q-1)}$$

$$a_{(I)}^{\mu_{q+1},1} = \frac{1}{2q(q+1)}, \quad a_{(f)}^{\mu_{q+1},1} = -\frac{1}{2q}$$

$$a_{(h_\mu)}^{\mu_{q+1},1} = \frac{2}{(q^2-1)}, \quad a_{(h_\lambda)}^{\mu_{q+1},1} = 0$$

$$a_{(I)}^{\mathbb{F}_q^\times,1} = \frac{q-3}{2q(q^2-1)}, \quad a_{(f)}^{\mathbb{F}_q^\times,1} = \frac{q-3}{2q(q+1)}$$

$$a_{(h_\mu)}^{\mathbb{F}_q^\times,1} = 0, \quad a_{(h_\lambda)}^{\mathbb{F}_q^\times,1} = -\frac{1}{q^2-1}$$

Now suppose q is 3 mod 4. In this case, -1 is not a square in $\mathbb{F}_q^\times$, so the values of

$$\sum_{\lambda \in (\mathbb{F}_q^\times)^2} \chi_\pi(h_\lambda)$$

at π of dimension $1, q-1, q$, and $q+1$ are $(q-3)/4$, 1, $(q-3)/4$, and 0. Since -1 is a square in μ_{q+1}, we consider the sum

$$2 \cdot \sum_{\mu \in (\mu_{q+1})^2} \chi_\pi(h_\mu) + h_{-1}^-$$

at π of dimension $1, q-1, q$, and $q+1$, which evaluate to $(q-1)/2$, 2, $-(q-1)/2$, and 0. Therefore, we take $c_{(h_{-1}^+)} = 0$, $c_{(h_{-1}^-)} = c_{(h_\mu)}/2$. Again, the coefficients $c_{(I)}, c_{(f)}$, $c_{(h_\lambda)}$ for $\lambda \in (\mathbb{F}_q^\times)^2$, and $c_{(h_\mu)}$ for $\mu \in (\mu_{q+1})^2$ must solve the equations

$$\begin{aligned}
1 &= c_{(I)} + c_{(f)} + c_{(h_\lambda)}\frac{q-3}{4} + c_{(h_\mu)}\frac{q-1}{2} \\
\frac{1}{q-1} &= c_{(I)}(q-1) - c_{(f)} + 2c_{(h_\mu)} \\
\frac{1}{q} &= c_{(I)}q + c_{(h_\lambda)}\frac{q-3}{4} - c_{(h_\mu)}\frac{q-1}{2} \\
\frac{1}{q+1} &= c_{(I)}(q+1) + c_{(f)} - c_{(h_\lambda)}
\end{aligned}$$

Solving these equations as in the previous case gives

$$c_{(I)} = \frac{q+2}{q(q^2-1)}, \quad c_{(f)} = \frac{2}{q+1}$$

$$c_{(h_\mu)} = \frac{2(q^2+q+1)}{q(q^2-1)}, \quad c_{(h_\lambda)} = \frac{q^2-q+1}{q(q^2-1)}$$

Similarly, we can solve for $a_{(h)}^{SO_3(\mathbb{F}_q),1}$, $a_{(h)}^{SO_3(\mathbb{F}_q),\binom{0<1}{1}}$, $a_{(h)}^{\mu_{q+1},1}$, and $a_{(h)}^{\mathbb{F}_q^\times,1}$, getting

$$a_{(I)}^{SO_3(\mathbb{F}_q),1} = \frac{2}{q(q^2-1)}, \quad a_{(f)}^{SO_3(\mathbb{F}_q),1} = \frac{2}{q}$$

$$a_{(h_\mu)}^{SO_3(\mathbb{F}_q),1} = \frac{1}{(q+1)}, \quad a_{(h_\lambda)}^{SO_3(\mathbb{F}_q),1} = \frac{2}{q-1}$$

$$a_{(I)}^{SO_3(\mathbb{F}_q),\binom{0<1}{1}} = \frac{2}{q(q^2-1)}, \quad a_{(f)}^{SO_3(\mathbb{F}_q),\binom{0<1}{1}} = 0$$

$$a_{(h_\mu)}^{SO_3(\mathbb{F}_q),1} = -\frac{1}{q(q+1)}, \quad a_{(h_\lambda)}^{SO_3(\mathbb{F}_q),1} = \frac{2}{q(q-1)}$$

$$a_{(I)}^{\mu_{q+1},1} = \frac{1}{2q(q+1)}, \quad a_{(f)}^{\mu_{q+1},1} = -\frac{1}{2q}$$

$$a_{(h_\mu)}^{\mu_{q+1},1} = \frac{1}{(q^2-1)}, \quad a_{(h_\lambda)}^{\mu_{q+1},1} = 0$$

$$a_{(I)}^{\mathbb{F}_q^\times,1} = \frac{q-3}{2q(q^2-1)}, \quad a_{(f)}^{\mathbb{F}_q^\times,1} = \frac{q-3}{2q(q+1)}$$

$$a_{(h_\mu)}^{\mathbb{F}_q^\times,1} = 0, \quad a_{(h_\lambda)}^{\mathbb{F}_q^\times,1} = -\frac{2}{q^2-1}$$

References

[1] G. De Franceschi, M. W. Liebeck, E. A. O'Brien. Conjugacy in Finite Classical Groups *Springer Monogr. Math. Springer*, Cham, 2025, xi+176 pp.

[2] S. Gurevich, R. Howe. Small representations of finite classical groups, *Progr. Math.*, 323 *Birkhäuser/Springer*, Cham, 2017, 209-234.

[3] S. Gurevich, R. Howe. Rank and duality in representation theory, *Jpn. J. Math.* 15 (2020), 223-309.

[4] S. Kriz. Howe duality over finite fields I: the two stable ranges. arXiv:2412.15346.

[5] S. Kriz. Howe duality over finite fields II: explicit stable computation. arXiv:2506.22983.

[6] S. Kriz. Howe duality over finite fields III: full computation and the Gurevich-Howe conjectures. arXiv:2506.22986.

[7] M. W. Liebeck, E. A. O'Brien, A. Shalev, P. H. Tiep. The Ore conjecture. *JEMS* 12 (2010), 939-1008.

[8] G. Lusztig: *Characters of Reductive Groups over a Finite Field*. Ann. of Math. Stud., 107 *Princeton University Press*, Princeton, NJ, 1984, xxi+384 pp.

[9] G. Malle. The proof of Ore's conjecture [after Ellers-Gordeev and Liebeck-O'Brien-Shalev- Tiep]. *Sém. Bourbaki, Astérisque* 361 (2014), 325-348.

[10] O. Ore. Some remarks on commutators. *Proc. Amer. Math. Soc.* 2 (1951), 307-314.

[11] A. Shalev. Commutators, Words, Conjugacy Classes and Character Methods. *Turk J Math* 31 (2007), Suppl, 131-148.

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