INTERPOLATED EQUIVARIANT SCHEMES

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(Preliminary version)

ABSTRACT. We propose an approach to a theory of GL_c-equivariant schemes over $Spec(\mathbb{C})$ where c is a complex number. We investigate vector bundles and stacks of coherent modules in this context, and interpretations of these concepts in terms of moduli spaces of tensor categories. We give several examples, including flag varieties.

1. Introduction

Let X be a GL_n-equivariant affine scheme over $Spec(\mathbb{C})$ where n is a natural number. By definition, X is the spectrum of a GL_n-equivariant (commutative) $\mathbb{C}$-algebra A. We may think of the category of GL_n-equivariant associative commutative unital (ACU) $\mathbb{C}$-algebras as the category of ACU algebra objects in the symmetric tensor cateogry $\underline{Rep}(GL_n)$ of Ind-algebraic $GL_n(\mathbb{C})$-representations. The work of P. Deligne [2, 3] allows us to consider an analogous category $\underline{Rep}(GL_c)$, replacing the rank of the general linear group by a general value $c \in \mathbb{C} \smallsetminus \mathbb{Z}$ (the case of a negative integer rank general linear groups can be understood classically using the super- formalism). The case of $c \in \mathbb{C} \smallsetminus \mathbb{Z}$ is the context we will consider in the present paper.

When c is not an integer, the objects of $\underline{Rep}(GL_c)$ are Ind-objects in a semisimple tensor category $Rep(GL_c)$, but they cannot be understood as representations of a classical algebraic group (this is actually a theorem of Deligne [4] caused by their "super-exponential growth"). Because of this, the "points" of X we see are orbits, i.e. $Spec(A^{GL_c})$ where A^{GL_c} is the maximal trivial subrepresentation of an ACU algebra A in the tensor category $\underline{Rep}(GL_c)$. In the case of $c = n$ a positive integer, this corresponds to the subalgebra of actual GL_n-fixed points of A, i.e. regular functions on the orbit scheme $X/GL_n = Spec(A^{GL_n})$.

On $Spec(A^{GL_n})$, however, we can also consider A as a sheaf of ACU algebras in $\underline{Rep}(GL_n)$. More precisely, this is a stack, where isomorphisms of sections are isomorphisms of algebras in $\underline{Rep}(GL_n)$. (For example, when X/GL_n is a point, X is a GL_n-orbit by an algebraic subgroup H, and the automorphisms are given by the Weyl group $N(H)/H$.) In the case of a general GL_n-scheme X, we can recover a lot of information about X from gluing the stacks on U/GL_n where U are GL_n-equivarint affine open subsets.

All of this information has an interpolated version; this is the subject of the present paper.

Looking at these structures is motivated by the now extremely developed theory of $\mathbb{C}$-linear categories with ACU tensor product and strong duality which we refer to as *quasi-pre-Tannakian categories* (see e.g. [5, 6, 7, 9, 10, 11, 12, 13, 14, 15]). In particular, with a GL_c-equivariant scheme in our sense, one can associate two stacks of categories: The stack of vector bundles and the stack of quasicoherent modules (and coherent modules, in case of appropriate "local finiteness"). The focus in the existing literature has been mostly when X/GL_c is a point (i.e. when X is an orbit), but the general context describes parametric spaces of such structures. (The case of infinitesimal deformations was treated in [13].) In the case of an orbit, it turns out that, using a universal algebra formalism called T-*algebras*, GL_c-projective modules are precisely equivalent to the structure of $\mathbb{C}$-linear categories with associative commutative unital tensor product and strong duality generated by a "basic object" of dimension c [13].

In the case of local finiteness, it further turns out that the category of locally finite projective modules is semisimple if and only if it is abelian, which is equivalent to it being equivalent to the category of coherent modules [13]. For a general GL_c-equivariant scheme X, strong duality on coherent modules can be interpreted as a *smoothness condition* on X.

We begin this paper by giving the relevant definitions in Section 2. We then discuss some basic examples of GL_c-equivariant schemes, such as affine spaces and the affine and projective line of orbits (Sections 4 and 5). We also treat example of flag varieties (Sections 6, 7). Next, in Section 3, we briefly review the T-algebra formalism of [12] from the present point of view, and define the stacks of locally projective and quasicoherent modules. Finally, in Section 6, we describe the notion of affine GL_c-equivariant group schemes.

2. Interpolated Algebraic Geometry

The purpose of this section is to define GL_c-equivariant schemes. We begin by recalling in Subsection 2.1 the definition of P. Deligne's category $Rep(GL_c)$, and why it can be considered an interpolation of the representation of a general linear group with non-integer rank. We describe its algebra (Ind-)objects, which are our interpretation of "GL_c-equivariant algebras" from the classical point of view. In Subsection 2.2, we define GL_c-equivariant schemes. In Subsection 2.3, we briefly consider analogous notions for other interpolated algerbaic groups.

2.1. **The category** $Rep(GL_c)$**.**

Fix a complex number $c \in \mathbb{C}$. The purpose of this first subsection is to recall the definition from [3, 2] of the $\mathbb{C}$-linear semisimple tensor category $Rep(GL_c)$ which is meant to take the place of the classical representation category of $GL_n(\mathbb{C})$.

2.1.1. *Schur functors.* The basis of this "interpolation" of the general linear group is that the essence of a category tensor generated by a basic object X is completely captured in the data of the morphisms between the tensor powers of X and its dual.

In the case of $n \in \mathbb{N}$, the standard representation $X_n = \mathbb{C}^n$, on which $GL_n(\mathbb{C})$ acts by matrix multiplication, tensor generates all of the group's representations. We additionally recall that for any $m \leq n$, we may decompose the mth tensor power of X_n as

$$X_n^{\otimes m} = \bigoplus_{|\lambda|=m} dim(S_\lambda) \cdot \mathbb{S}_\lambda(X_n), \tag{1}$$

where in the direct sum, λ runs through all Young diagrams with m boxes, S_λ denotes the Specht module (in the classification of irreducible representations of the symmetric group on m elements) associated to λ, and $\mathbb{S}_\lambda$ denotes its corresponding Schur functor (giving an irreducible $GL_n(\mathbb{C})$-representation). Therefore, in particular, the endomorphism algebra of $X_n^{\otimes m}$ for $m \leq n$ is isomorphic to the group algebra on the symmetric group with m elements, which we shall denote by Σ_m:

$$End_{Rep(GL_n(\mathbb{C}))}(X_n^{\otimes m}) \cong \mathbb{C}\Sigma_m. \tag{2}$$

For a general Young diagram λ, the dimension of its corresponding Schur functor applied to a representation V is a polynomial in the dimension of the representation

$$dim(\mathbb{S}_\lambda(V)) = p_\lambda(dim(V)),$$

depending on the row and column lengths of λ (and not depending on the underlying group $GL_n(\mathbb{C})$). For a Young diagram λ with a column length greater than n (i.e. a Young diagram with more n rows), we find that

$$p_\lambda(n) = 0.$$

Putting Schur functors $\mathbb{S}_\lambda(X_n)$ to be equal to 0, we find that (1) may be extended to all values of m. However, because of the missing terms arising from Schur functors giving 0, (2) fails for $m > n$. Note that we also have

$$Hom(X_n^{\otimes m_1}, X_n^{\otimes m_2}) = 0,$$

for distinct m_1, m_2.

2.1.2. *Interpolation.* Now we will consider a category modelling this morphism structure, generated by a basic "standard representation" X_c, with categorical dimension

$$dim(X_c) = c.$$

Recall that the *categorical dimension* of an object Z of a $\mathbb{C}$-linear category $\mathscr{C}$ with associative, commutative, unital tensor product and strong duality is the composition of the unit and co-unit of duality

$$1 \to Z \otimes Z^\vee \to 1$$

which, in the most general context we consider in this paper, may be an element of an arbitrary $\mathbb{C}$-algebra $End_{\mathscr{C}}(1)$. The condition that the dimensions of objects be integers corresponds to a relationship between $\mathscr{C}$ and the category of (super) vector spaces.

We begin by constructing a "diagrammatic" category $Rep_0(GL_c)$ whose objects consist of pairs $(m, n) \in \mathbb{N}_0^2$, which we write as tensor products

$$X_c^{\otimes m} \otimes (X_c^\vee)^{\otimes n}.$$

Define the spaces of morphisms

$$Hom_{Rep_0(GL_c)}(X_c^{\otimes m_1} \otimes (X_c^\vee)^{\otimes n_1}, X_c^{\otimes m_2} \otimes (X_c^\vee)^{\otimes n_2})$$

to be the $\mathbb{C}$-vector space freely generated by biijections

$$[m_1] \amalg [n_2] \to [m_2] \amalg [n_1]. \tag{3}$$

To define composition, however, we prefer to represent a generator (3) graphically, as a certain kind of diagram on two columns of dots: Draw a line of m_1 black dots and a line of n_1 white dots in the left column, and draw a line of m_2 black dots and a line of n_2 white dots in the right column. Then draw connecting paths between the points corresponding to the matched elements of (3). Any diagram of paths is possible as long as each path connects two points which are either on opposite sides of the diagram, or are opposite colors. Then, to compose two such diagrams, we simply concatenate them side by side, and compose paths when possible. When we encounter a closed circle of paths, we remove it from the diagram and multiply the generator by a factor of the coefficient c (similarly as computing the operation in a Brauer algebra).

In particular, we have, for every m,

$$End_{Rep_0(GL_c)}(X_c^{\otimes m}) \cong \mathbb{C}\Sigma_m$$

and, for every $m_1 \neq m_2$,

$$Hom_{Rep_0(GL_c)}(X_c^{\otimes m_1}, X_c^{\otimes m_2}) = 0.$$

The diagrammatic category does not capture well the representation theory of the general linear groups on the level of objects. To correct the objects, we formally add (finite) direct sums, and then apply a pseudo-abelian envelope. The resulting category, denoted here by $Rep(GL_c)$, is the *interpolated category of representations of the general linear group*, at rank c. If we allowed infinite direct sums, we would get $\underline{Rep}(GL_c)$, the category of Ind-objects of $Rep(GL_c)$.

2.1.3. *Relation with the classical context.* For non-integer values of c, not only is $Rep(GL_c)$ an additive $\mathbb{C}$-linear category, but it is an abelian one. In fact, it actually turns out to be *semisimple.*

The simple objects of $Rep(GL_c)$ are indexed by pairs of Young diagrams (λ, μ). It first appears as a summand of $X^{\otimes m} \otimes (X^{\vee})^{\otimes n}$. In particular, $Y_{\lambda,\emptyset}$ can be conisdered as the application $\mathbb{S}_\lambda(X_c)$ of the Schur functor associated to λ to X_c, and has non-zero dimension

$$dim(Y_{\lambda,\emptyset}) = p_\lambda(c) \neq 0$$

(since the roots of p_λ are all integers).

Plugging in $c = n$ for an integer $n \in \mathbb{N}$, we can also recover the classical representation category $Rep(GL_n(\mathbb{C}))$ by applying *semisimplification* to $Rep(GL_c)$.

The procedure of semisimplification of a quasi-pre-Tannakian category is a more general construction, designed to accomodate subtelties which do not occur in this particular example. In the case of $Rep(GL_c)$ for c an integer, we do not conisder the category to be a semisimple pre-Tannakian category because there are non-zero simple objects of dimension 0. Applying semisimplification in this case simply removes these objects. However, the general procedure will play an important role later, so we discuss it in its full generality now:

Suppose $\mathscr{C}$ is a quasi-pre-Tannakian category. For some objects $Y, Z \in Obj(\mathscr{C})$, define a morphism

$$f : Y \to Z$$

to be *negligible* if for every morphism $g : Z \to Y$, we have

$$tr(f \circ g) = 0.$$

Negligible morphisms form a tensor ideal, and the quotient of $\mathscr{C}$ identifying them with the 0 morphism is called the *semisimplification* of $\mathscr{C}$, which we denote by $\overline{\mathscr{C}}$. However, it can happen that the semisimplification of a general quasi-pre-Tannakian category may not be semisimple, or even abelian (see Section 5.8 of [2]).

In $\mathscr{C} = Rep(GL_c)$ at $c = n$ for an integer n, the negligible morphisms consist of linear combinations of the identity morphisms on the simple objects $Y_{\lambda,\mu}$ of dimension 0. Therefore, the semisimplification $\overline{Rep(GL_c)}$ precisely identifies these simple objects with the 0 object, recovering the classical representation category

$$\overline{Rep(GL_c)} \cong Rep(GL_n(\mathbb{C})).$$

2.2. GL_c-equivariant schemes.

2.2.1. *The topology of a GL_c-equivariant spectrum.* We have discussed the underlying category $Rep(GL_c)$ meant to replace $Rep(GL_n(\mathbb{C}))$. The next step should be to propose a GL_c-version of the spectrum functor $Spec_{GL_c}$. Similarly as in the classical framework, $Spec_{GL_c}(\mathcal{A})$, for a GL_c-equivariant algebra $\mathcal{A}$, consists of an underlying topological space, and a "structure stack" on it. The purpose of this subsection is to discuss the underlying topological space.

Recall that in the classical story of $GL_n(\mathbb{C})$-equivariant sheaves, we would apply *Spec* to an associative commutative unital algebra A which is equivariant with respect to a group action of $GL_n(\mathbb{C})$. In the interpolated setting, we do not have an actual group GL_c. We instead take $Spec_{GL_c}$ to have input data in the form of algebra objects of the category $\underline{Rep}(GL_c)$. By this, we mean objects $\mathcal{A} \in Obj(\underline{Rep}(GL_c))$ with multiplication and unit morphisms

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

$$\eta : 1 \to \mathcal{A}$$

(1 denoting the unit object of $\underline{Rep}(GL_c)$), which (strictly) satsify the typical diagrams encoding associativity, commutativity, and unitality.

We may refer to these objects as GL_c*-equivariant algebras*. Denote the subcategory of GL_c-equivariant algebras (and morphisms in $\underline{Rep}(GL_c)$ preserving algebra structures) by

$$\underline{Rep}(GL_c)\text{-Algebra}.$$

Fix an ACU algebra object $\mathcal{A}$ in $\underline{Rep}(GL_c)$. We begin by considering the decomposition of $\mathcal{A}$ into a direct sum of simple objects

$$\bigoplus_{\lambda,\mu} \bigoplus_{i \in I_{\lambda,\mu}} Y_{\lambda,\mu}$$

over pairs of Young diagrams λ, μ, for some indexing sets $I_{\lambda,\mu}$ enumerating the copies of $Y_{\lambda,\mu}$ appearing. For each choice of λ, μ, we write

$$\mathcal{A}_{\lambda,\mu} := \bigoplus_{i \in I_{\lambda,\mu}} Y_{\lambda,\mu}, \tag{4}$$

and call this the *degree* (λ, μ)*-piece* of $\mathcal{A}$. (We may sometimes also use this terminology for the space of coefficients $\mathbb{C}I_{\lambda,\mu}$ of (4).)

In particular, $\mathcal{A}_{\emptyset,\emptyset}$ can be considered as a classical commutative, unital $\mathbb{C}$-algebra. We may consider the classical affine $\mathbb{C}$-scheme $Spec(\mathcal{A}_{\emptyset,\emptyset})$. This (with the Zariski topology) is taken to be the underlying topological space of $Spec_{GL_c}(\mathcal{A})$.

2.2.2. *Interpolated schemes and structure stacks.* We conclude our definition of $Spec_{GL_c}(\mathcal{A})$ by defining the GL_c*-equivariant structure stack* $\mathscr{O}_{\mathcal{A}}$ on the underlying topological space $Spec(\mathcal{A}_{\emptyset,\emptyset})$. In this subsection, we also use this to define general GL_c-equivariant schemes.

Recall that $Spec(\mathcal{A}_{\emptyset,\emptyset})$ (with the Zariski topology) has a neighborhood basis consisting of localizations

$$Spec(f^{-1}\mathcal{A}_{\emptyset,\emptyset}) \subseteq Spec(\mathcal{A}_{\emptyset,\emptyset})$$

for some element f. We assign the $\underline{Rep}(GL_c)$-algebra of sections of $\mathscr{O}_{\mathcal{A}}$ on such an open set by putting

$$\mathscr{O}_{\mathcal{A}}(Spec(f^{-1}\mathcal{A}_{\emptyset,\emptyset})) = f^{-1}\mathcal{A}, \tag{5}$$

where the right hand side is considered as a groupoid by taking its morphisms to consist of automorphisms

$$\phi : f^{-1}\mathcal{A} \to f^{-1}\mathcal{A}$$

such that the $(\emptyset, \emptyset)$-part acts by identity on $(f^{-1}\mathcal{A})_{\emptyset,\emptyset} = f^{-1}\mathcal{A}_{\emptyset,\emptyset}$

$$\phi_{\emptyset,\emptyset} = Id_{f^{-1}\mathcal{A}_{\emptyset,\emptyset}}. \tag{6}$$

To define $\mathscr{O}_{\mathcal{A}}(U)$ for a general open subset $U \subseteq Spec(\mathcal{A}_{\emptyset,\emptyset})$ of commutative algebras in the category $\underline{Rep}(GL_c)$: For any point $p \in Spec(\mathcal{A}_{\emptyset,\emptyset})$, we may consider it also as a prime ideal in the whole $Rep(GL_c)$-algebra $\mathcal{A}$, and may therefore define the localization $\mathcal{A}_p$. For an open set U of the underlying topological space, we then take $\mathscr{O}_{\mathcal{A}}(U)$ to consist functions from U into the disjoint union of localizations $\mathcal{A}_p$ for $p \in U$ such that each point is sent into its corresponding localization and the analogue of "locally being a quotient" is satisfied, exactly analogously to the classical treatment of the structure sheaf on an affine scheme. (U is covered by neighborhoods $V \subseteq U$ such that there exist $x, y \in \mathcal{A}$ such that for any $p \in V$, $y \notin p$, and the function applied to p is equal to $x/y \in \mathcal{A}_p$.

2.2.3. *Affinoid and general GL_c-schemes.* We define $Spec_{GL_c}(\mathcal{A})$ to be the pair of this underlying topological space and structure stack

$$Spec_{GL_c}(\mathcal{A}) = (Spec(\mathcal{A}_{\emptyset,\emptyset}), \mathscr{O}_{\mathcal{A}}). \tag{7}$$

We warn, however, that non-affineness can exist intrinsically in a $Rep(GL_c)$-algebra (embodied, for example, by the case of the flag varieties described in Sections 6, 7 below), so we will *not* count all $Spec_{GL_c}(\mathcal{A})$ as affine GL_c-equivariant schemes. Instead, we will refer to them as the *affinoid GL_c-equivariant schemes*:

Definition 1. *For a topological space X and a stack of GL_c-equivariant algebras $\mathscr{S}$ on it, we say that $(X, \mathscr{S})$ forms an* affinoid GL_c-equivariant scheme *if there exists a GL_c-equivariant algebra $\mathcal{A}$ such that*

$$(X, \mathscr{S}) \cong (Spec(\mathcal{A}_{\emptyset,\emptyset}), \mathscr{O}_{\mathcal{A}}) = Spec_{GL_c}(\mathcal{A}).$$

Then, similarly as for classical definitions, we require a general GL_c-equivariant scheme to be locally isomorphic to an affinoid one:

Definition 2. *A GL_c-equivariant scheme is a pair $(X, \mathscr{S})$ of a topological space X and a stack $\mathscr{S}$ on X of GL_c-equivariant algebras such that for every point $x \in X$, there exists an open neighborhood $U \ni x$ such that U (with the subspace topology), together with the restriction of $\mathscr{S}$ to U gives an affinoid GL_c-equivariant scheme.*

For a GL_c-equivariant scheme $\mathcal{X} = (X, \mathscr{S})$, we write $\mathscr{S} = \mathscr{O}_{\mathcal{X}}$ and call it the *structure stack* of $\mathcal{X}$.

For a GL_c-equivariant scheme $\mathcal{X} = (X, \mathscr{O}_{\mathcal{X}})$, define its *scheme of orbits* $\mathcal{X}/GL_c$ to consist of the topological space X and the sheaf $(\mathscr{O}_{\mathcal{X}})_{\emptyset,\emptyset}$

of $\mathbb{C}$-algebras defined by taking, for a open subset $U \subseteq X$, taking

$$(\mathscr{O}_{\mathcal{X}})_{\emptyset,\emptyset}(U) = (\mathscr{O}_{\mathcal{X}}(U))_{\emptyset,\emptyset}$$

with restriction and gluing structure from $\mathscr{O}_{\mathcal{X}}$. In particular, for a GL_c-equivariant algebra $\mathcal{A}$

$$(Spec_{GL_c}(\mathcal{A}))_{\emptyset,\emptyset} = Spec(\mathcal{A}_{\emptyset,\emptyset}).$$

Then for GL_c-equivariant schemes $\mathcal{X} = (X, \mathscr{O}_{\mathcal{X}})$, $\mathcal{Y} = (Y, \mathscr{O}_{\mathcal{Y}})$, define a *morphism of GL_c-equivariant schemes* $\mathcal{X} \to \mathcal{Y}$ as a continuous map

$$f : X \to Y$$

and a morphism of stacks of ACU algebras in $\underline{Rep}(GL_c)$

$$\phi : f^* \mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}},$$

such that the restriction to the scheme of points (consisting of f and the restriction of ϕ to a morphism of sheaves $\phi_{\emptyset,\emptyset} : f^*(\mathscr{O}_{\mathcal{Y}})_{\emptyset,\emptyset} \to (\mathscr{O}_{\mathcal{X}})_{\emptyset,\emptyset}$) is a classical morphism of schemes

$$\mathcal{X}/GL_c \to \mathcal{Y}/GL_c.$$

This defines the category of GL_c-equivariant schemes, which we denote by

$$Sch_{GL_c}.$$

We denote the full subcategory of affinoid schemes by $Sch_{GL_c}^{\text{Affinoid}}$.

As usual, we have a

Proposition 3. *The GL_c-equivariant spectrum defines an equivalence of categories*

$$Spec_{GL_c} : (\underline{Rep}(GL_c)\text{-}Algebras)^{Op} \to Sch_{GL_c}^{Affinoid}$$

One sees that if $\mathcal{X}$ is a GL_c-equivariant scheme and $U \subseteq \mathcal{X}/GL_c$ is an open subscheme, then there is a GL_c-equivariant scheme $\widetilde{U} = (U, \mathscr{O}_{\widetilde{U}})$ where $\mathscr{O}_{\widetilde{U}}$ is the restriction of the stack $\mathscr{O}_{\mathcal{X}}$. This determines a morphism of GL_c-equivariant schemes

$$\widetilde{U} \to \mathcal{X}.$$

We call $\widetilde{U}$ an *open subscheme* of $\mathcal{X}$.

On the other hand, for a GL_c-equivariant scheme $\mathcal{X}$, there is an obvious notion of a *stack of ideals* $\mathscr{I}$ of $\mathscr{O}_{\mathcal{X}}$ and we have a GL_c-equivariant scheme $\mathcal{Z}(\mathscr{I})$ with underlying scheme having the zero locus $Z(\mathscr{I}_{\emptyset,\emptyset})$ of $\mathscr{I}_{\emptyset,\emptyset}$ in $\mathcal{X}/GL_c$, and the structure stack is the restriction of $\mathscr{O}_{\mathcal{X}}/\mathscr{I}$. We call $Z(\mathscr{I})$ a GL_c-equivariant *closed subscheme* of $\mathcal{X}$. A GL_c-equivariant scheme $\mathcal{X}$ where $\mathcal{X}/GL_c$ is a single point will be called a *GL_c-orbit*.

On a GL_c-equivariant scheme $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$, we will also define its categories of *vector bundles* and *quasi-coherent stacks of modules* (*coherent* in the case of local finiteness), denoted by

$$Bun_{GL_c}(\mathcal{X}), \quad QCoh_{GL_c}(\mathcal{X}), \quad Coh_{GL_c}(\mathcal{X})$$

and consisting of stacks of projective modules and general modules (resp. locally finite modules) in $\underline{Rep}(GL_c)$, over $\mathscr{S}$, see Subsection 3.3 below.

2.2.4. *Affine GL_c-equivariant schemes and relative semisimplification.* To define truly affine GL_c-equivariant schemes, we need to first describe a process of *relative semisimplification* for a GL_c-equivariant algebra $\mathcal{A}$, which eliminates its possible "intrinsic non-affineness." (This procedure is related to the process of semisimplification of categories. We will explain this in Subsection 3.4 below.)

Consider the class of ideals $\mathcal{J} \subseteq \mathcal{A}$ in $\underline{Rep}(GL_c)$ such that the projection $\mathcal{A}_{\emptyset,\emptyset} \to (\mathcal{A}/\mathcal{J})_{\emptyset,\emptyset}$ is an isomorphism. We claim the following

Proposition 4. *There exists a unique maximal ideal $\mathcal{J}_{\mathcal{A}}$ that*

$$(\mathcal{A}/\mathcal{J}_{\mathcal{A}})_{\emptyset,\emptyset} = \mathcal{A}_{\emptyset,\emptyset}.$$

Proof. In $\underline{Rep}(GL_c)$, we have unique (up to $\mathbb{C}^\times$-multiple) direct summand inclusions

$$Y_{\emptyset,\emptyset} \xrightarrow{\eta} Y_{\lambda,\mu} \otimes Y_{\mu,\lambda} \tag{8}$$

Let $\mathcal{A}$ be an ACU algebra in $\underline{Rep}(GL_c)$. A morphism in $\underline{Rep}(GL_c)$ $\alpha : Y_{\lambda,\mu} \to \mathcal{A}$ is called *negligible* if for every morphism $\beta : Y_{\mu,\lambda} \to \mathcal{A}$, the composition

$$Y_{\emptyset,\emptyset} \xrightarrow{\eta} Y_{\lambda,\mu} \otimes Y_{\mu,\lambda} \xrightarrow{\alpha\otimes\beta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mu} \mathcal{A}$$

is 0. It follows immediately that the image of the sum of all negligible morphisms forms an ideal in $\mathcal{A}$, which is therefore equal to $\mathcal{J}_{\mathcal{A}}$. $\square$

We call $\mathcal{J}_{\mathcal{A}}$ the *relative maximal ideal* in $\mathcal{A}$. We call the quotient algebra

$$\widetilde{\mathcal{A}} := \mathcal{A}/\mathcal{J}_{\mathcal{A}} \tag{9}$$

its *relative semisimplification*. (This can also be conisdered as the terminal quotient algebra of $\mathcal{A}$, factoring over $\mathcal{A}_{\emptyset,\emptyset}$.)

Lemma 5. *For a point $f \in \mathcal{A}_{\emptyset,\emptyset}$, localization by f preserves the relative maximal ideal of $\mathcal{A}$:*
$$\mathcal{J}_{f^{-1}\mathcal{A}} = f^{-1}\mathcal{J}_{\mathcal{A}}.$$

2.3. **Other groups.** There are a number of versions of interpolated reductive algebraic groups known at this point, notably orthogonal and symplectic groups [2]. Also algebraic groups over finite fields can be interpolated [9, 10]. A generalized context of interpolating finite groups using *oligomorphic groups* has be developed by N. Harman and A. Snowden [7]. Interesting new examples arise [8, 11, 12, 13, 14]. Each of these contexts has, in principle, a theory analogous to the one presented here. Furthermore, we have the usual group change functors. We do not develop these theories here systematically, in part because we will see that the GL_c-equivariant context is in some sense universal in that the other contexts can be discussed within it.

Roughly speaking, for $H \subseteq GL_c$, an H-scheme $\mathcal{X}$ has an avatar $GL_c \times_H \mathcal{X}$ in the category of GL_c-schemes. One case which will get some attention, however, is the inclusion
$$GL_{a_1} \times \cdots \times GL_{a_n} \subseteq GL_c,$$
for $a_1 + \cdots + a_n = c$ (see Section 7 below).

3. Vector bundles and (quasi-)coherent modules

In this section, we define for a GL_c-equivariant algebra $\mathcal{A}$ the categories of projective $\mathcal{A}$-modules and (quasi-)coherent $\mathcal{A}$-modules. In other words, the categories of vector bundles or (quasi-)coherent stacks of modules on $Spec_{GL_c}(\mathcal{A})$. The case of coherent modules applies when it is locally finite. (See Subsection 3.3 below.) These notions readily pass to general GL_c-equivariant schemes.

As usual, while the category of vector bundles is always a $\mathbb{C}$-linear additive category with strong duality, it may not be an abelian category. On the other hand, the category of (quasi-)coherent stacks of modules is always a $\mathbb{C}$-linear abelian tensor category, but it may not have duality. We recall results of [13] and interpret them from this point of view. We also discuss the significance of affine GL_c-equivariant schemes and semisimplification from this point of view, and define smooth GL_c-equivariant schemes (see Definition 9 below).

3.1. **A universal algebra approach to quasi-pre-Tannakian categories.** In this subsection, we recall the universal algebra concept of T-*algebras*. T-algebras are a universal algebra construction that precisely characterize the data of an additive $\mathbb{C}$-linear category with associative, commutative, unital tensor product, strong duality, and a basic object X tensor-generating the category:

Definition 6. *A* T-algebra $\mathcal{T}$ *consists of the data of*

- *a collection of functorial vector spaces* $\mathcal{T}[S,T]$ *indexed by pairs of finite sets* S,T
- *partial trace maps*
$$\tau_\sigma : \mathcal{T}[S,T] \to \mathcal{T}[S \smallsetminus S', T \smallsetminus T']$$
for a bijection $\sigma : S' \to T'$ *for subsets* $S' \subseteq S$, $T' \subseteq T$,
- *tensor product operations*
$$\pi_{(S,T),(S',T')} : \mathcal{T}[S,T] \otimes \mathcal{T}[S',T'] \to \mathcal{T}[S \amalg S', T \amalg T']$$
for disjoint S, S', T, T'
- *and unit data*
$$1 \in \mathcal{T}[\emptyset,\emptyset], \ \ \iota \in \mathcal{T}[[1],[1]]$$

The axioms we require are that the tensor product operations are associative, commutative, and unital with respect to 1*, that the composition of two partial trace maps is the same as the single partial trace map on the disjoint union of their two matchings, that tensor product commutes with partial trace, and that the "composition operations" defined from a combination of product and partial trace are associative and unital with respect to* ι *(and it's tensor powers).*

Clearly, given a quasi-pre-Tannakian category $\mathscr{C}$ with basic object X, we can define a T-algebra by putting, for finite sets S and T,
$$\mathcal{T}_{\mathscr{C}}[S,T] = Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T}).$$
For a bijection $\sigma : S' \to T'$, $S' \subseteq S$, $T' \subseteq T$, the partial trace operation τ_σ is defined by considering the identification
$$Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T}) \cong Hom_{\mathscr{C}}(X^{\otimes (S \smallsetminus S')}, (X^\vee)^{\otimes S'} \otimes X^{\otimes T})$$
and composing any element with the co-unit of duality
$$\bigotimes_{s \in S'} (X^\vee)^{\otimes \{s\}} \otimes X^{\otimes \{\sigma(s)\}} \to 1,$$
tensored with $Id_{X^{\otimes (T \smallsetminus T')}}$. Product operations are the tensor product, and we take unit data $1 = Id_1$ and $\iota = Id_X$.

On the other hand, given a T-algebra, a corresponding quasi-pre-Tannakian category $\mathscr{C}(\mathcal{T})$ can be produced by a similar construction as the definition of $Rep(GL_c)$, by first constructing a diagrammatic category $\mathscr{C}_0(\mathcal{T})$ with formal objects $X^{\otimes S} \otimes (X^\vee)^{\otimes T}$, and spaces of morphisms

$$Hom_{\mathscr{C}(\mathcal{T})}(\mathcal{T})(X^{\otimes S_1} \otimes (X^\vee)^{\otimes T_1}, X^{\otimes S_2} \otimes (X^\vee)^{\otimes T_2}) = \mathcal{T}[S_1 \amalg T_2, S_2 \amalg T_1]$$

(as in the case of $Rep(GL_c)$, composition operations can be constructed from the partial trace data).

3.2. **A correspondence with GL_c-equivariant algebras.** In this subsection, we recall a correspondence proved in [13] between the data of a quasi-pre-Tannakian categories with basic object of dimension c (encoded using T-algebras) and GL_c-equivariant algebras. We interpret this as a version of Tannakian duality for vector bundles over GL_c-equivariant schemes.

Definition 7. *For a fixed $c \in \mathbb{C}^\times$, a* T_c*-algebra is a* T*-algebra $\mathcal{T}$ where in*

$$\mathcal{T}[\emptyset, \emptyset]$$

(which is required to be a commutative, unital $\mathbb{C}$-algebra by the general T-algebra axioms), the partial trace of the composition unit $\iota \in \mathcal{T}[[1],[1]]$ along the matching $Id_{[1]} : [1] \to [1]$ is

$$\tau_{Id_{[1]}}(\iota) = c \cdot 1.$$

We find then that a T_c-algebra is equivalent to the data of a quasi-pre-Tannakian category $\mathscr{C}$ which is generated by tensor powers of a basic object $X \in Obj(\mathscr{C})$ such that the composition of the unit and co-unit of duality

$$1 \to X \otimes X^\vee \to 1$$

is $c \cdot 1 \in End_{\mathscr{C}}(1)$ (where 1 on the left hand side denotes the algebra unit of $End_{\mathscr{C}}(1)$). Note that we allow $End_{\mathscr{C}}(1)$ to be a general $\mathbb{C}$-algebra. T_c-algebras are still a universal algebra structure.

Let us denote the cateogry of ind-objects in $Rep(GL_c)$ by $\underline{Rep}(GL_c)$. The main result of [13] is the following

Theorem 8. *There is an equivalence of categories*

$$\begin{aligned} \underline{Rep}(GL_c)\text{-}Algebra &\to \mathrm{T}_c\text{-}Algebra \\ \mathcal{A} &\mapsto \mathcal{T}_{\mathcal{A}} \end{aligned} \tag{10}$$

This equivalence sends $\underline{Rep}(GL_c)$-algebra $\mathcal{A}$ to the T_c algebra $\mathcal{T}_\mathcal{A}$ defined by

$$\mathcal{T}_\mathcal{A}[S,T] = Hom_{\underline{Rep}(GL_c)}(\mathcal{A}, X^{\otimes S} \otimes (X^\vee)^{\otimes T})$$

(with the data of partial trace and product defined by compositions along the system of morphisms in $\underline{Rep}(GL_c)$ on $X^{\otimes S} \otimes (X^\vee)^{\otimes T}$ arising from composition with co-units and tensor product).

We consider Theorem 8 as a version of Tannakian duality, with the category $\mathscr{C}(\mathcal{T}_\mathcal{A})$ being equivalent to the category of projective modules over $\mathcal{A}$, or vector bundles over $Spec_{GL_c}(\mathcal{A})$.

3.3. **The stacks of vector bundles and (quasi-)coherent modules.** Let $\mathcal{X}$ be a GL_c-equivariant scheme. Over an affinoid open subset $\mathcal{U} \subseteq \mathcal{X}$, we define the category of *vector bundles* over U as the quasi-pre-Tannakian category associated with the ACU algebra $\mathscr{O}_\mathcal{U}$ in $\underline{Rep}(GL_c)$. These categories glue into a stack which we call the *stack of vector bundles over $\mathcal{X}$*. We denote it by $Bun_{GL_c}(\mathcal{X})$.

On the other hand, we can also assign to $\mathcal{U}$ as above the $\mathbb{C}$-linear tensor category $\mathscr{O}_\mathcal{U}$-Mod of $\mathscr{O}_\mathcal{U}$-modules in $\underline{Rep}(GL_c)$. These categories glue into a stack which we call the *stack of quasi-coherent modules on X*. We denote it by $QCoh_{GL_c}(\mathcal{X})$.

We call an object of $\underline{Rep}(GL_c)$ *locally finite* if it contains only finitely many copies of $Y_{\lambda,\mu}$ for each λ, μ. We call a GL_c-equivariant scheme $\mathcal{X}$ *locally finite* if for $\mathcal{U} \subseteq \mathcal{X}$ open affinoid, $\mathscr{O}_\mathcal{U}$, considered as an object of $\underline{Rep}(GL_c)$, is locally finite.

If $\mathcal{X}$ is locally finite, then a *coherent module* over $\mathcal{U} \subseteq \mathcal{X}$ open affinoid is a locally finite $\mathscr{O}_\mathcal{U}$-module in $\underline{Rep}(GL_c)$. Coherent sheaves then also form a stack over $\mathcal{X}$, which we call the *stack of coherent modules* and denote by $Coh_{GL_c}(\mathcal{X})$.

Definition 9. *For a locally finite GL_c-equivariant scheme $\mathcal{X}$ we call $\mathcal{X}$* smooth *if for every affinoid open $\mathcal{U} \subseteq \mathcal{X}$, the category of global sections of $Coh_{GL_c}(\mathcal{U})$ has strong duality.*

3.4. **Vector bundles and coherent sheaves over an affine orbit.** We begin by considering the procedure of semisimplification on quasi-pre-Tannakian categories and relate it to the definition of relative semisimplification on a GL_c-equivariant algebra and affine GL_c-equivariant schemes.

In a quasi-pre-Tannakian category $\mathscr{C}$, recall that we form its semisimplification $\overline{\mathscr{C}}$ by taking a quotient which identifies all negligible morphisms with 0.

Suppose $\mathscr{C}$ is tensor-generated by a basic object X of dimension c, and denote its defining T_c-algbera by $\mathcal{T}_{\mathscr{C}}$ and the corresponding GL_c-equivariant algebra by $\mathcal{A}$. Then let us consider for every S, T, the vector subspace of $\mathcal{T}[S,T]$ consisting of negligible morphisms

$$f : X^{\otimes S} \to X^{\otimes T} \tag{11}$$

In order to consider semisimplification, let us unpack briefly the statement that negligible morphisms form a tensor ideal in this notation, and see how they are preserved by T-algebra structure: The partial trace of a negligible morphism (11) is also negligible, since for any bijection $\sigma : S' \to T'$ and any morphism $g : X^{\otimes(S\smallsetminus S')} \to X^{\otimes(T\smallsetminus T')}$, we have

$$tr(\tau_\sigma(f) \circ g) = tr(f \circ (\sigma \otimes g)) = 0,$$

where we consider σ as its action permuting tensor factors

$$X^{\otimes S'} \to X^{\otimes T'}.$$

Similarly the tensor product of (11) with any other morphism g is also negligible, since for any morphism h, the trace $tr((f \otimes g) \circ h)$ will always be expressible as the trace of a composition of f with a new morphism h' (possibly multiplied by some constants arising from pieces of g that remain "disjoint" from f).

Therefore, we have

$$\overline{\mathscr{C}} = \mathscr{C}(\mathcal{T}_{\mathcal{A}/\mathcal{J}}) \tag{12}$$

An important consequence of (4) is that for any maximal ideal $m \subset \mathcal{A}_{\emptyset,\emptyset}$, $\mathcal{A}$ contains a unique maximal ideal containing m (generated by the relative maximal ideal $J_{\mathcal{A}}$, together with m). Quotients of $\mathcal{A}$ by these ideals (giving fields) correspond to the true categorical semisimplifications in the quasi-pre-Tannakian categories of their categories of modules in $\underline{Rep}(GL_c)$.

In summary, while the process of relative semisimplification on a GL_c-equivariant algebra $\mathcal{A}$ quotients out the unique maximal ideal not affecting the $(\emptyset, \emptyset)$-degree, if one quotients out a true maximal ideal of $\mathcal{A}$, it corresponds to a semisimplification of the category

$$Bun_{GL_c}(Spec_{GL_c}(\mathcal{A})).$$

Further, e call a GL_c-equivariant algebra $\mathcal{A}$ *field-like*, then, if it has no non-zero ideals. In this case (the underlying topological space) of $Spec_{GL_c}(\mathcal{A})$ is a point. Recalling Theorem 5.10 of [13], we in fact have

Theorem 10. *For a field-like GL_c-equivariant algebra $\mathcal{A}$, the following are equivalent:*

- *There is an equivalence of categories*
$$Bun_{GL_c}(Spec_{GL_c}(\mathcal{A})) \cong Coh_{GL_c}(Spec_{GL_c}(\mathcal{A}))$$
- *The category $Coh_{GL_c}(Spec_{GL_c}(\mathcal{A}))$ has strong duality.*
- *The category $Bun_{GL_c}(Spec_{GL_c}(\mathcal{A}))$ is semisimple.*

4. First Examples: Affine spaces, and an affine and projective line orbits

The purpose of this section is to give two different kinds of examples of GL_c-equivariant schemes. Our first examples can be interpreted as affine spaces. They arise naturally from a certain "free" construction of categories, which we are sometimes referred to as "skein categories." We will describe the categories of vector bundles on affine spaces. We will also look at a particular GL_c-equivariant scheme whose scheme of orbits is an affine line $Spec(\mathbb{C}[x])$, parametrizing the orbits $GL_c/(GL_x \times GL_{c-x})$.

We will also show how to add a point at ∞, thus exhibiting a GL_c-equivariant scheme with scheme of orbits $\mathbb{P}^1_{\mathbb{C}}$.

We note, however, that we do not have a general model of the projective space of a GL_c-representation. This is related to the fact that even for $c = n \in \mathbb{N}$, the space of orbits of the projective space of GL_n-representation is usually not a scheme.

4.1. Affine spaces.

Take an ACU algebra in $\underline{Rep}(GL_c)$ of the form

$$\mathcal{A}[\underline{(m_i, n_i)} \mid i \in I] := Sym(\bigoplus_{i \in I} X^{\otimes m_i} \otimes (X^\vee)^{\otimes n_i}), \tag{13}$$

for some chosen pairs (m_i, n_i) in $\mathbb{N}_0^2$. (The symmetric tensor algebra corresponds to the commutativity of the tensor product.)

Definition 11. *Define the* interpolated affine spaces on generators of degree $\underline{(m_i, n_i)}$, for $i \in I$ *by*

$$\mathbb{A}^{\{\underline{(m_i,n_i)}|i\in I\}} := Spec_{GL_c}(\mathcal{A}[\underline{(m_i, n_i)}|i \in I]). \tag{14}$$

(Note that the spaces $\mathbb{A}^{\{\underline{(m_i,n_i)}|i\in I\}}$ are indeed affine GL_c-equivariant schemes, since any relation on the free variables corresponding in (13) to the coefficients of the summand $Sym^1(X^{\otimes m_i} \otimes (X^\vee)^{\otimes n_i})$ can be traced down to the $(\emptyset, \emptyset)$-degree.)

We can describe the category of vector bundles on $\mathbb{A}^{\{\underline{(m_i,n_i)}|i\in I\}}$. Let $\varphi : I \to \mathbb{N}_0^2$ denote the function $\varphi(i) = (m_i, n_i)$.

By an (M, N)-(I, φ)-*skein*, we shall mean an oriented graph whose vertices are labelled by I, where each vertex with label i has m_i ordered incoming edges and n_i ordered outgoing edges, with a total of M ordered open incoming edges and N ordered outgoing edges. A $(0, 0)$-(I, φ)-skein will also be called a *closed* (I, φ)-*skein.*

Proposition 12. *Let $\mathcal{T}$ be the T-algebra defining the category of vector bundles on $\mathbb{A}^{\{\underline{(m_i,n_i)}|i\in I\}}$. Then $\mathcal{T}[[M], [N]]$ is naturally identified with the free $\mathbb{C}$-vector space on the set of (M, N)-(I, φ)-skeins.*

□

Corollary 13. *The $\mathbb{C}$-algebra $(\mathcal{A}[\underline{(m_i, n_i)} \mid i \in I])_{\emptyset,\emptyset}$ is the polynomial algebra on closed connected (I, φ)-skein.*

□

The most basic example is

$$\mathcal{A}[\underline{(1, 1)}] = Sym(X \otimes X^\vee),$$

in which case, we find that

$$(\mathcal{A}[\underline{(1, 1)}])_{\emptyset,\emptyset} \cong \mathbb{C}[x_1, x_2, x_3, \dots], \tag{15}$$

(since the $(\emptyset, \emptyset)$-part of $\mathcal{A}[\underline{(1, 1)}]$ sees generators corresponding to skeins, which are of the form of the trace of some n successive compositions of the non-trivial degree $(1, 1)$-morphism in $\mathcal{A}$-modules, which we represent by x_n in (15)). Therefore, the underlying topological space of $Spec_{GL_c}(\mathcal{A}[\underline{(1, 1)}])$ is an infinite-dimensional affine space.

Of course, the category defined by a free generator in such a way is not locally finite. One must add some relations on the variables, i.e. consider an affine subvariety. In the general case of $\mathbb{A}^{\{\underline{(m_i,n_i)}|i\in I\}}$, this can be difficult to approach, but in the case of a single "degree 1" generator as in $\mathcal{A}[\underline{(1,1)}]$, it can be fully discussed, and will be in the following subsections.

4.2. **An example of an affine line of orbits.** In this subsection, we consider two subvarieties of $\mathbb{A}^{\underline{(1,1)}}$ which can be considered to model a (non-trivial) "affine point" and an "affine line" made up of these points. Specifically, we will find ideals of relations

$$\mathcal{I}^{\text{line}} \subseteq \mathcal{I}_a^{\text{point}} \subseteq \mathcal{A}[\underline{(1,1)}]$$

(where a is a fixed constant) such that the underlying vector spaces of

$$Spec_{GL_c}(\mathcal{A}[\underline{(1,1)}]/\mathcal{I}_a^{\text{point}}) \subseteq Spec_{GL_c}(\mathcal{A}[\underline{(1,1)}]/\mathcal{I}^{\text{line}})$$

are isomorphic to $Spec(\mathbb{C})$ and $\mathbb{A}^1_{\mathbb{C}}$, respectively. We will design these relations so that the whole category of vector bundles on the "affine point" is equivalent to

$$Bun_{GL_c}(Spec_{GL_c}(\mathcal{A}[\underline{(1,1)}]/\mathcal{I}_a^{\text{point}})) \cong Rep(GL_a \times GL_{c-a}) \tag{16}$$

(see Subsection 3.4 for a more detailed discussion of why such a condition on vector bundles corresponds to being a point). In the "affine line," we will have that at a closed point $(x-a) \in \mathbb{A}^1_{\mathbb{C}}$, the category of stalks of vector bundles there is equivalent to (16).

We begin with defining the "affine point" relations $\mathcal{I}_a^{\text{point}}$. Fix a constant $a \in \mathbb{C}$. We will give an example of an ideal $\mathcal{I} \subseteq \mathcal{A}[\underline{(1,1)}]$ whose category of vector bundles is equivalent to

$$Rep(GL_a \times GL_{c-a}).$$

In $\mathcal{A}[\underline{(1,1)}] = \bigoplus Sym^n(X \otimes X^\vee)$, denote by y the (coefficient of the) degree $n = 1$ term $X \otimes X^\vee$ (representing in the category of vector bundles on $\mathbb{A}^{\underline{(1,1)}}$, a non-trivial morphism on the basic object). Denoting the algebra operation of $\mathcal{A}[\underline{(1,1)}]$ by π, we have that

$$y \;\pi\; y \in Sym^2(X \otimes X^\vee).$$

The simplest way to get an ideal $\mathcal{I}$ such that

$$Bun_{GL_c}(Spec_{GL_c}(\mathcal{A}[\underline{(1,1)}]/\mathcal{I})) \cong Rep(GL_a \times GL_{c-a})$$

is to identify the endomorphism determined by y of the basic vector bundle to be an idempotent of dimension a. Both of these relations

$$y \circ y = y, \qquad \text{and} \qquad tr(y) = a,$$

are encoded by relations on the coefficients of decompositions of $Sym^1(X\otimes X^\vee)$ in $Sym^2(X \otimes X^\vee)$.

Recall that

$$Sym^1(X \otimes X^\vee) = X \otimes X^\vee = Y_{1,1} \oplus 1. \tag{17}$$

Since there are no terms of higher multiplicities in (17), there is no ambiguity to writing $(y)_{1,1}$ and $(y)_{\emptyset,\emptyset}$ for the coefficients of $Y_{1,1}$ and 1. (Note that the coefficient of the whole summand (17), is by definition y, so we automatically have that $(y)_{1,1} = y-(y)_{\emptyset,\emptyset}$.) To encode $tr(y) = a$, we put the element

$$(y)_{\emptyset,\emptyset} - a \ \in \mathcal{I}_a^{\text{point}} \tag{18}$$

in the relation ideal.

To encode $y \circ y = y$, we need to involve the degree 2 product $y \ \pi \ y$. Recall that

$$Sym^2(X \otimes X^\vee) = (Y_{(2),(2)} \oplus Y_{(1^2),(1^2)}) \oplus (Y_{1,1} \oplus Y_{1,1}) \oplus (1 \oplus 1). \tag{19}$$

From the point of view of the category of vector bundles on $\mathbb{A}^{\underline{(1,1)}}$, the element $y \ \pi \ y$ and its coefficients in the summands (19) describe all possible "partial traces" and permutation actions on $y \otimes y$. One can choose to perform an empty partial trace, recovering There are two different options on how to do a partial trace on $y \otimes y$ to get a on an endomorphism of a single tensor factor of the basic object in the category of vector bundles of $\mathbb{A}^{\underline{(1,1)}}$: matching the source coordinate of a copy of y with its own target coordinate, or matching the source coordinate of a copy y with the other copy's target coordinate. These two options give

$$tr(y) \cdot y, \qquad \text{or} \qquad y \circ y, \tag{20}$$

respectively. Finally, there are two options to take a "full trace" of $y \otimes y$, giving

$$tr(y)^2, \qquad \text{or} \qquad tr(y \circ y). \tag{21}$$

From this point of view, we denote the coefficients of the terms

$$(y \ \pi \ y)_{(2),(2)},\ (y \ \pi \ y)_{(1^2),(1^2)},\ (y \ \pi \ y)_{1,1}^{Id},\ (y \ \pi \ y)_{1,1}^{\sigma}, (y \ \pi \ y)_{\emptyset,\emptyset}^{Id},\ (y \ \pi \ y)_{\emptyset,\emptyset}^{\sigma}$$

with the Id and σ labels in the summands with multiplicity 2, corresponding to (20) and (21), respectively (corresponding from the vector bundle perspective to applying a trace to $y \otimes y$ or $\sigma \circ (y \otimes y)$). For example, $y \circ y$ is represented by

$$(y \ \pi \ y)_{1,1}^{\sigma} + (y \ \pi \ y)_{\emptyset,\emptyset}^{\sigma}.$$

Therefore, the condition that y is an idempotent can be expressed by the relation

$$(y \;\pi\; y)^{\sigma}_{1,1} + (y \;\pi\; y)^{\sigma}_{\emptyset,\emptyset} - y \tag{22}$$

(note that once we have, also that once we have a relation, it generates all partial traces, e.g. in this case, (22) generates the relation that $tr(y \circ y) = tr(y)$, i.e.

$$(y \;\pi\; y)^{\sigma}_{\emptyset,\emptyset} - (y)_{\emptyset,\emptyset}). \tag{23}$$

In summary, to define an "affine point"

$$Spec(\mathcal{A}[\underline{(1,1)}]/\mathcal{I}^{\text{point}}_a),$$

we put

$$\mathcal{I}^{\text{point}}_a =$$
$$((y)_{\emptyset,\emptyset} - a,\;\; (y \;\pi\; y)^{\sigma}_{1,1} + (y \;\pi\; y)^{\sigma}_{\emptyset,\emptyset} - y).$$

However, we could also consider the GL_c-equivariant algebra given by only imposing (22) in $\mathcal{A}[\underline{(1,1)}]$, and not (18). Take

$$\mathcal{I}^{\text{ line}} = ((y \;\pi\; y)^{\sigma}_{1,1} + (y \;\pi\; y)^{\sigma}_{\emptyset,\emptyset} - y).$$

Definition 14. *We call the affine subvariety*

$$Spec_{GL_c}(\mathcal{A}[\underline{(1,1)}]/\mathcal{I}^{line}) \subseteq \mathbb{A}^{\underline{(1,1)}}$$

the true interpolated affine line*.*

We will see in the next section other kinds of "more degenerate" GL_c-equivariant affine subvarieties of $\mathbb{A}^{\underline{(1,1)}}$ which can be interpreted as affine lines.

Lemma 15. *The scheme of orbits of the true interpolated affine line*

$$(Spec(\mathcal{A}[\underline{(1,1)}]/\mathcal{I}^{line}))/GL_c$$

is isomorphic to to $\mathbb{A}^1_{\mathbb{C}}$.

Proof. The relation (22) generates other elements in $\mathcal{I}^{\text{line}}$ obtained taking a product with other elements of $\mathcal{A}[\underline{(1,1)}]$ and applying any partial trace. For example, the relation that (in the vector bundle context) $tr(y \circ y) = tr(y)$ comes from applying a trace to (22) to get (23). Similarly, by taking a product and (switched) trace with copies of y, we find that for every n, (22) generates the relation forcing

$$tr(y^{\circ n}) = tr(y)$$

(represented in $\mathcal{A}[\underline{(1,1)}]$ by $(y)_{\emptyset,\emptyset}$).

In (15), recalling that x_n represents $tr(y^{\circ n})$, we therefore find that

$$(\mathcal{A}[\underline{(1,1)}]/\mathcal{I}^{\text{line}})_{\emptyset,\emptyset} \cong \mathbb{C}[(y)_{\emptyset,\emptyset}].$$

□

4.3. **A projective line of orbits.** One may observe now that we can make many example of affine subvarieties of the interpolated affine spaces, for example modelling affine curves, surfaces, etc. The purpose of this subsection is to show how we may. For example, here we will construct the "true interpolated projective line" by two (true) interpolated affine lines to obtain a GL_c-equivariant scheme such that the underlying topological space is isomorphic to the projective line $\mathbb{P}^1_{\mathbb{C}}$.

First, let us consider two copies of the interpolated affine space $\mathbb{A}^{\underline{(1,1)}}$ generated by y_1, y_2. The true interpolated projective line is defined as the pushforward union of the two copies of the true interpolated affine line along the relation that

$$(y_1)_{\emptyset,\emptyset} \cdot (y_2)_{\emptyset,\emptyset} = 1 \in \mathbb{C}.$$

5. A closer look at $\mathbb{A}^{\underline{(1,1)}}$

In this section, we give a classification of the categories of vector bundles over the closed orbits in $\mathbb{A}^{\underline{(1,1)}}$. We also give a description of its corresponding T_c-algebra in the case when it is semisimple.

5.1. **Closed orbits of $\mathbb{A}^{\underline{(1,1)}}$.** The purpose of this section is to prove the following

Proposition 16. *Suppose $\mathcal{J} \subseteq \mathcal{A}[\underline{(1,1)}]$ is a maximal ideal. Then one of the following occurs:*

(1) The closed orbit $Spec(\mathcal{A}[\underline{(1,1)}]/\mathcal{J})$ is not locally finite.

(2) There exist some $a_1, \ldots a_r \in \mathbb{C}^\times$ such that $a_1 + \cdots + a_r = c$ and

$$Bun_{GL_c}(Spec(\mathcal{A}[\underline{(1,1)}]/\mathcal{J})) \cong Rep(GL_{a_1} \times \cdots \times GL_{a_r}).$$

(3) The category $Bun_{GL_c}(Spec(\mathcal{A}[\underline{(1,1)}]/\mathcal{J}))$ is a locally finite quasi-pre-Tannakian category which is not semisimple.

Comment: Note that in case (2), the closed orbit $Spec(\mathcal{A}[\underline{(1,1)}]/\mathcal{J})$ is smooth, and in case (3) it is not. The existence of non-smooth affine orbits is a key feature of the theory of GL_c-equivariant schemes. In the language of categories, this phenomenon was first noticed by P. Deligne in [2], Subsection 5.8.

Proof. Recall (15), with x_n representing the unique connected closed skein with n vertices (in this case, we only have one kind of vertex, which has one input and one output edge). In this case, the connected skeins with one total incoming edge and one total outgoing edge are all also determined by their number of vertices. Denote the element of $(\mathcal{A}[\underline{(1,1)})_{1,1}$ corresponding to the connected skein with one open input edge and one open outgoing edge which has n vertices by y_n. We have that $\mathcal{A}[\underline{(1,1)}]_{1,1}$ is then the free module over $\mathcal{A}[\underline{(1,1)}]_{\emptyset,\emptyset}$ generated by $y_1, y_2, \dots$, i.e.

$$(\mathcal{A}[\underline{(1,1)}])_{1,1} = \mathbb{C}[x_1, x_2, \dots]\{y_1, y_2, \dots\}. \tag{24}$$

We have

$$tr(y_n) = x_n, \quad y_n \circ y_m = y_{n+m} \tag{25}$$

(where, again, from the point of view of $\mathcal{A}[\underline{(1,1)}]$, a trace is interpreted as a $(\emptyset, \emptyset)$-part coefficient of an element and composition corresponds to the coefficient of a certain summand $X \otimes X^\vee$ of $Sym^2(X \otimes X^\vee)$).

Now let us fix a maximal ideal $\mathcal{J} \subseteq \mathcal{A}[\underline{(1,1)}]$ (in $\underline{Rep}(GL_c)$). Then the $(\emptyset, \emptyset)$-part

$$\mathcal{J}_{\emptyset,\emptyset} \subseteq \mathcal{A}[\underline{(1,1)}]_{\emptyset,\emptyset} = \mathbb{C}[x_1, x_2, \dots]$$

forms a classical maximal ideal. Therefore, there exist some constants α_n for $n \in \mathbb{N}$ such that $\mathcal{J}_{\emptyset,\emptyset}$ is generated by $x_n - \alpha_n$

$$\mathcal{J}_{\emptyset,\emptyset} = (x_1 - \alpha_1, x_2 - \alpha_2, \dots),$$

giving

$$\mathcal{A}[\underline{(1,1)}]_{\emptyset,\emptyset}/\mathcal{J}_{\emptyset,\emptyset} = \mathbb{C}.$$

Now, for every $i \in \mathbb{N}$, consider a $\mathbb{N}_0$-tuple

$$A_i = (\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \dots),$$

whose nth coordinate for $n \in \mathbb{N}$ is

$$(A_i)_n = \alpha_{i+n} = tr(y_n \circ y_i)$$

(interpreting y_0 as the identity morphism on the basic object of the category of vector bundles). Linear relations between the $\mathbb{N}_0$-tuples A_i are equivalent to linear relations between the y_i themselves in $\mathcal{A}[\underline{(1,1)}]/\mathcal{J}$, since a morphism is determined by its trace with every other morphism after semisimplification.

If the $\mathbb{N}_0$-tuples A_i are all linearly independent, meaning that after semisimplification, all the y_i are independent, then

$$Spec_{GL_c}(\mathcal{A}[\underline{(1,1)}]/\mathcal{J})$$

is not locally finite (since, in particular, $\mathcal{A}[\underline{(1,1)}]/\mathcal{J}$ contains a new $Y_{1,1}$ summand for each y_i). This gives scenario (1).

Suppose that the A_i are linearly dependent, and let r be minimal such that A_{r+1} can be expressed as a linear combination

$$A_{r+1} = b_r A_r + b_{r-1} A_{r-1} + \cdots + b_1 A_1$$

for some $b_i \in \mathbb{C}$. In fact, this gives a linear recurrence relation

$$A_{r+k} = b_r A_{r+k-1} + b_{r-1} A_{r+k-2} + \cdots + b_1 A_k$$

for $k \in \mathbb{N}$ (by shifting). On y_i's, this gives

$$y_{r+1} = b_r y_r + b_{r-1} y_{r-1} + \cdots + b_1 y_1,$$

which, by composing with y_{k-1} for $k \in \mathbb{N}$ recalling (25), gives

$$y_{r+k} = b_r y_{r+k-1} + b_{r-1} y_{r+k-2} + \cdots + b_1 y_k. \tag{26}$$

Now we can consider

$$(\mathcal{A}[\underline{(1,1)}]/\mathcal{J})_{1,1} = \mathbb{C}\{y_1, \ldots y_r\}.$$

In fact, we can identify the graded algebra of all of the $((n),(n))$-pieces of $\mathcal{A}[\underline{(1,1)}]/\mathcal{J}$ with respect to tensor product with the polynomial $\mathbb{C}$-algebra on y_i's (graded by degree)

$$\bigoplus_{n\in\mathbb{N}} (\mathcal{A}[\underline{(1,1)}]/\mathcal{J})_{(n),(n)} = \mathbb{C}[y_1, \ldots, y_r]$$

where multiplication on the right hand side of y_i, y_j corresponds to placing y_i and y_j next to each other to obtain a skein with more (in this case two) open incoming and outgoing edges.

The linear recurrsion relation (26) has characteristic polynomial

$$\lambda^r - b_r \lambda^{r-1} - \cdots - b_1. \tag{27}$$

If we have distinct eigenvalues (i.e. roots of (27)) $\lambda_1, \ldots, \lambda_r$, then we can write

$$y_k = a_1 \lambda_1^k + \ldots a_n \lambda_n^k$$

for some $a_1, \dots, a_r \in \mathbb{C}^\times$ (they must be non-zero, or else r was not minimal). Plugging in $k = 0$ gives that $a_1 + \cdots + a_r = c$. We get

$$Bun_{GL_c}(Spec(\mathcal{A}[\underline{(1,1)}]/\mathcal{J})) \cong Rep(GL_{a_1} \times \cdots \times GL_{a_r}),$$

giving scenario (2).

Finally, in the case where we have degenerate, non-distinct eigenvalues, this gives that $Bun_{GL_c}(Spec(\mathcal{A}[\underline{(1,1)}]/\mathcal{J}))$ is a locally finite quasi-pre-Tannakian category which is not semisimple (see Subsection 5.8 of [2]).

□

5.2. **The T_c-algebras of semisimple closed orbits of $\mathbb{A}^{\underline{(1,1)}}$.** In this section, we describe a T_c-algebra $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}$ corresponding to the category $Rep(GL_{a_1} \times \cdots \times GL_{a_r})$:

Definition 17. *For an r-tuple $(a_1, \dots, a_r)$, define the T-algebra*

$$\mathcal{T}^{GL}_{(a_1,\dots,a_r)}$$

by the following collection of data:

- *For finite sets S, T, take $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}[S,T]$ to be 0 if $|S| \neq |T|$. When $|S| = |T|$, take $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}[S,T]$ to be the free vector space with basis elements corresponding to the data of decompositions of S and T into r disjoint (not necessarily nonempty) subsets*

$$S = \coprod_{i=1}^{r} S_i, \quad T = \coprod_{i=1}^{r} T_i$$

  and bijections

$$\phi_i : S_i \to T_i$$

- *Consider subsets $S' \subseteq S$, $T' \subseteq T$ with a bijection*

$$\sigma : S' \to T'.$$

  Then, we define the partial trace operation

$$\tau_\sigma : \mathcal{T}^{GL}_{(a_1,\dots,a_r)}[S,T] \to \mathcal{T}^{GL}_{(a_1,\dots,a_r)}[S \smallsetminus S', T \smallsetminus T']$$

  by sending a basis element as described above of the source to 0 unless for every $i \in [r]$,

$$\sigma(S_i \cap S') = T_i \cap T'.$$

In this case, the image is the basis element of the target of τ_σ *corresponding to taking*

$$(S \smallsetminus S')_i = S_i \smallsetminus S', (T \smallsetminus T')_i = T_i \smallsetminus T',$$

and composing repeatedly the ϕ_i*'s with* σ^{-1} *as is possible, and multiplying by factors (see the diagrammatic description and Figure 1 below for examples).*

- *For disjoint* S, T, S', T'*, tensor product operations*

$$\mathcal{T}^{GL}_{(a_1,\dots,a_r)}[S,T] \otimes \mathcal{T}^{GL}_{(a_1,\dots,a_r)}[S',T'] \to \mathcal{T}^{GL}_{(a_1,\dots,a_r)}[S \amalg S', T \amalg T']$$

arise by applying disjoint union to pairs of basis data for $[S,T]$ *and* $[S',T']$*.*

- *The tensor unit* 1 *is precisely the single basis element in the above description for* $S = \emptyset, T = \emptyset$*. The composition unit* ι *corresponding to the identity on the basic object is the sum over* $i \in [r]$ *of the basis elemenets of* $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}[[1],[1]]$ *corresponding to choosing*

$$S_i = [1], T_i = [1], \phi_i = Id_{[1]},$$

and all other sets to be empty.

The generators of the vector spaces $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}[S,T]$ for a pair of finite sets S, T may be graphically represented by drawing two columns of dots, with the left and right columns corresponding to S and T, respectively, and the bijections ϕ_i drawn as lines between the two columns, with color i indicating the index. Up to r colors may therefore be involved. To apply a partial trace τ_σ to this data, we draw uncolored lines along σ between the two columns. The trace is 0 unless uncolored lines connect on both ends to lines of the same color. We assign the uncolored lines the color they connect to on both sides, otherwise. Then we compose each color of lines as possible, and multipy by a factor of a_i when a loop of color i appears in the diagram.

Note in the case of $r = 1$, this is exactly the description given by P. Deligne in [2], Section of the category $Rep(GL_c)$.

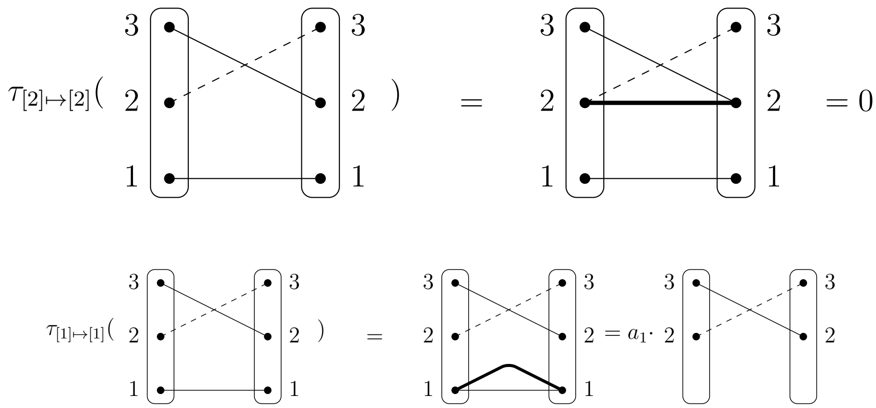

FIGURE 1. Two examples of applying partial trace to a generator of $\mathcal{T}^{GL}_{(a_1,a_2)}[[3],[3]]$. The two colors corresponding to $1,2$ are represented as solid and dashed lines, respectively. The points are labelled as their corresponing elements in [3] to avoid confusion. Colorless lines are represented by thick lines.

6. Interpolated flag varieties

We will now introduce a class of examples of non-affine smooth GL_c-equivariant orbits, which can be interpreted as interpolated flag varieties (i.e. Schubert varieties of standard parabolics of GL_c).

In this section, we will introduce these GL_c-obrits by means of their associated T_c-algebras, and will prove their smoothness. In the next section, we will justify their geometric interpretations.

6.1. **The category of representations of an interpolated parabolic subgroup of GL_c.**

Recalling Definition 17 which described the representation category $Rep(GL_{a_1} \times \cdots \times GL_{a_r})$ of the Levi subgroup, define the T-algebra corresponding to $Rep(P_{GL}(a_1, \dots, a_r))$ is defined as follows:

Definition 18. *For an r-tuple $(a_1, \dots, a_r)$, define the T-algebra*

$$\mathcal{T}^P_{(a_1,\dots,a_r)}$$

by the following collection of data:

- *For finite sets S, T, take $\mathcal{T}^P_{(a_1,\dots,a_r)}[S,T]$ to be 0 if $|S| \neq |T|$. When $|S| = |T|$, take $\mathcal{T}_{(a_1,\dots,a_r)}[S,T]$ to be the free vector space with basis elements corresponding to the data of decompositions* *of S and T into $r+\binom{r}{2}$ (possibly empty) disjoint subsets indexed by subsets of $[r]$ of cardinality ≤ 2*

(28)
$$S = \coprod_{i=1}^{r} S_i \amalg \coprod_{\{i<j\}\subseteq[r]} S_{\{i<j\}}$$
$$T = \coprod_{i=1}^{r} T_i \amalg \coprod_{\{i<j\}\subseteq[r]} T_{\{i<j\}}$$

and bijections

(29)
$$\phi_i : S_i \to T_i,\ \ \phi_{\{i<j\}} : S_{\{i<j\}} \to T_{\{i<j\}}.$$

- *Consider subsets $S' \subseteq S$, $T' \subseteq T$ with a bijection*
$$\sigma : S' \to T'.$$
Then, we define the partial trace operation
$$\tau_\sigma : \mathcal{T}^P_{(a_1,\dots,a_r)}[S,T] \to \mathcal{T}^P_{(a_1,\dots,a_r)}[S \smallsetminus S', T \smallsetminus T']$$
by sending a basis element of the source, corresponding to some choice of decomposition (28) of S, T and bijections (29) to 0 unless for every $i \in [r]$,
$$\sigma^{-1}(T' \cap T_i) \subseteq S_i \amalg \left(\coprod_{j=i+1}^{r} S_{\{i<j\}} \right).$$
In this case, the basis element is sent to the basis element of the target of τ_σ corresponding to as in the GL case, deleting the elements of S' and T' in each set S_i, T_i, and similarly in $S_{\{i<j\}}$, $T_{\{i<j\}}$, and defining new bijections by composing the original choices of ϕ_i, $\phi_{\{i<j\}}$ with σ^{-1} as long as possible, with final indexing given by taking the smallest (resp. largest) i_{min} (resp. j_{max}) appearing in the composition when $\phi_{\{i<j\}}$ are involved, multiplied by the coefficient
$$c_{a_1,\dots,a_r}(\sigma|_{\coprod_i S_i \cap \sigma^{-1} T_i}),$$
corresponding to the restriction of σ mapping a subset of each S_i to a subset of T_i, arising in the T-algebra for
$$Rep(GL_{a_1} \times \cdots \times GL_{a_r}).$$
(A diagrammatic description is described in the below remark.)

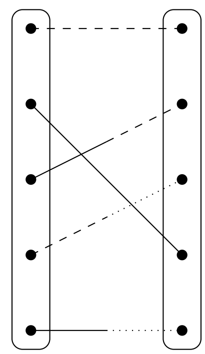

Figure 2. An example of a basis element of $\mathcal{T}^P_{(a_1,a_2,a_3)}[[5],[5]]$ where the colors $1,2,3$ are represented by solid, dashed, and dotted lines, respectively. In this case, the rules can be considered to force any line to remain the same or become "less solid" in its right half

- *As in the description of $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}$, for disjoint S,T,S',T', tensor product operations arise by applying disjoint union to pairs of basis data for $[S,T]$ and $[S',T']$.*

- *The unit data is the same as in $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}$ with data indexed by pairs taked to be empty.*

Remarks: 1. First, note that the trace of the composition unit $\iota \in \mathcal{T}^P_{(a_1,\dots,a_r)}[[1],[1]]$ along $Id_{[1]}:[1]\to[1]$ gives, as in $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}$,

$$(a_1+\dots+a_r)\cdot 1 \in \mathcal{T}^P_{(a_1,\dots,a_r)}[\emptyset,\emptyset]\cong\mathbb{C}$$

2. There is a natural diagrammatic description of the basis elements of $\mathcal{T}^P_{(a_1,\dots,a_r)}[S,T]$, adding to the diagrammatic description in the case of $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}$. Again, we draw two columns of points corresponding to S and T on the left and right, respectively. We draw lines between the two columns in i different colors corresponding to the bijections ϕ_i. To represent the bijections $\phi_{\{i<j\}}$, we draw lines which are half-colored i on the left and half-colored j on the right. For example, an example when $r=3$ is pictured in Figure 2 below.

In this diagrammatic description, trace can be described diagrammatically by drawing additional (a priori colorless) lines corresponding to σ between the two columns of dots, and then attempting to compose where possible. The answer is 0 if a colorless line connects the ends of two lines with different colors. Otherwise, a colorless line assumes the single color it immediately connects to on both sides. In a multicolored composition, colors in the middle (not connecting to a dot on the left or right that survives in the trace) are ignored, and the composition only remembers the data of the beginning and ending color in its index. Note that by these rules, no multicolored loops can occur. As in $Rep(GL_{a_1}\times\cdots\times GL_{a_r})$, when a loop of color i occurs, it is deleted and becomes a coefficient factor of a_i. An example of applying a partial trace to the generator pictured in Figure 2 is shown in Figure 3 below.

Theorem 19. *The category $Rep(P_{GL}(a_1,\ldots,a_r))$ is an abelian category with strong duality. There is a tensor functor*

$$Rep(GL_{a_1}\times\cdots\times GL_{a_r})\to Rep(P_{GL}(a_1,\ldots,a_r))$$

such that the images of the simple objects of $Rep(GL_{a_1}\times\cdots\times GL_{a_r})$ are precisely the simple objects of $Rep(P_{GL}(a_1,\ldots,a_r))$. In fact, the semisimplification of $Rep(P_{GL}(a_1,\ldots,a_r)$ again recovers

$$\overline{Rep(P_{GL}(a_1,\ldots,a_r))}\cong Rep(GL_{a_1}\times\cdots\times GL_{a_r}).$$

Comment: In particular, applying $Spec_{GL_c}$ to the GL_c-equivariant algebra corresponding to the T_c-algebra $\mathcal{T}^P_{(a_1,\ldots,a_r)}$ gives a smooth GL_c-orbit which is *not affine* (though it is, by definition, affinoid). This GL_c-orbit, which we denote by $GL_c/P_{GL}(a_1,\ldots,a_r)$, is the interpolated flag variety.

6.2. **The proof of Theorem 19, and restriction functors.** In this subsection, we prove Theorem 19. We will also describe the restriction functors

$$Rep(GL_{a_1+\cdots+a_r})\to Rep(P_{GL}(a_1,\ldots,a_r))\to Rep(GL_{a_1}\times\cdots\times GL_{a_r}).$$

Proof of Theorem 19. We begin with describing the simple objects of

$$Rep(P_{GL}(a_1,\ldots,a_r)).$$

First, we will define a decreasing filtration on morphisms capturing "tensor degree." By the T-algebra description given in the previous section, it suffices to define a filtration on the vector spaces

$$\mathcal{T}^P_{(a_1,\ldots,a_r)}[S,T] \tag{30}$$

for every choice of finite sets S,T.

For a basis elements of (30) corresponding to a choice of data

$$D:=(S_i,T_i,S_{\{i<j\}},T_{\{i<j\}},\phi_i,\phi_{\{i<j\}}\mid i,j\in[r]),$$

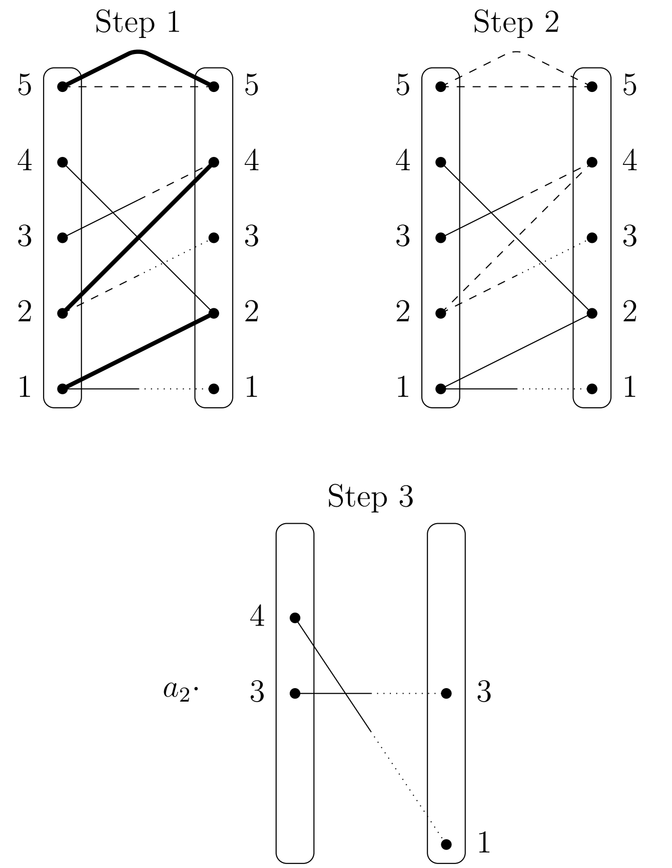

FIGURE 3. Applying the trace along the bijection pairing $\sigma : \{1, 2, 5\} \to \{2, 4, 5\}$ by $5 \mapsto 5$, $2 \mapsto 4$ to the generator in Figure 2. We now label the dots with the elements of [5] they correspond to, to avoid confusion. In Step 1, we draw "colorless lines" in the diagram, which due to graphic constraints are represented by thick lines. In Step 2, we assign the forced colors to these lines. In Step 3, we complete the composition and multiply by the arising coefficient.

consider the number, for $i \in [r]$

$$N(D)_i = |\coprod_{j=i+1}^{r} S_{\{i<j\}}| - |\coprod_{k=1}^{i-1} S_{\{k<i\}}|.$$

(This measures the difference of "inputs" of color i and "outputs" of color i in the diagrammatic representation of D.) We may then define a well-ordering on all basis elements D of vector spaces (30) from the lexicographic ordering on the r-tuples

$$(N(D)_1, N(D)_2, \dots, N(D)_r) \in \mathbb{Z}^r.$$

We then define a filtration

$$\mathcal{F}(\mathcal{T}^P_{(a_1,\dots,a_r)}[S,T])$$

graded by this ordering.

Since the ordering is additive with respect to the tensor product on the T-algebra, the filtration is as well. Let us consider the associated graded T-algebra

$$Gr_{\mathcal{F}(\mathcal{T}^P_{(a_1,\dots,a_r)})}.$$

The simple objects of $Rep(P_{GL}(a_1,\dots,a_r))$ will be precisely the simple objects of the category defined by this new T-algebra.

However, the associated graded T-algebra obtained from $\mathcal{F}_i$ precisely recovers the T-algebra

$$\mathcal{T}^{GL}_{(a_1,\dots,a_r)},$$

since the degree of data D only changes when bicolored lines are present (bijections $\phi_i : S_i \to T_i$ do not contribute). Hence, the simple objects of the two categories are the same.

Further, note that since $Rep(GL_{a_1} \times \cdots \times GL_{a_r})$ is a semisimple category with the same simple objects as, the quotient functor of T-algberas

$$\mathcal{T}^P_{(a_1,\dots,a_r)} \to Gr_{\mathcal{F}}(\mathcal{T}^P_{(a_1,\dots,a_r)})$$

which gives a quotient functor of tensor categories

$$Rep(P_{GL}(a_1,\dots,a_r)) \to Rep(GL_{a_1} \times \cdots \times GL_{a_r}),$$

in fact describes a semisimplification functor. Hence, $Rep(P_{GL}(a_1,\dots,a_r))$ must have originally been abelian (and therefore pre-Tannakian) category.

□

Now consider the restriction functor

$$Rep(GL_{a_1+\cdots+a_r}) \to Rep(GL_{a_1} \times \cdots \times GL_{a_r}). \tag{31}$$

We would like to understand how this functor (31) factors through the intermediate category $Rep(P_{GL}(a_1,\dots,a_r))$. Of course, these functors can be understood by the above Theorem, since as previously discussed,

$$Rep(P_{GL}(a_1,\dots,a_r)) \to Rep(GL_{a_1} \times \cdots \times GL_{a_r})$$

can be expressed as the semisimplification functor, sending simple objects to themselves, and the functor

$$Rep(GL_{a_1+\cdots+a_r}) \to Rep(P_{GL}(a_1, \dots, a_r)$$

arises, for example, from the universal property of interpolations of the general linear group, since the basic object of $Rep(P_{GL}(a_1, \dots, a_r))$ has dimension $a_1 + \cdots + a_r$.

On the level of T-algebras, the functor (31) occurs by considering a bijection $[n] \to [n]$ in $\mathcal{T}^{GL}_{(a_1+\cdots+a_r)}[[n],[n]]$ (labelled a single color), to the sum of all choices of basis elements in $\mathcal{T}^{GL}_{(a_1,\dots,a_r)}[[n],[n]]$ over all choices of possible colorings of the bijection in r colors. To get to $\mathcal{T}^{P}_{(a_1,\dots,a_r)}$ instead, we instead sum over all choices of replacing every line of a bijection $[n] \to [n]$ by a line colored a single (one of r) colors, or a pair of colors (beginning with the color corresponding to the lower index $i \in [r]$).

7. Geometric interpretation; interpolated equivariant group schemes

In this section, we will give some geometric motivation for the definitions made in the last section, which also motivate the definition of interpolated equivariant group schemes.

7.1. **Classical standard parabolics.** We begin with recalling the standard parabolics of $GL_n(\mathbb{C})$, $n \in \mathbb{N}$.

Let us begin with the parabolic subgroup $P_{GL}(a, b)$ of $GL_n(\mathbb{C})$ with Levi subgroup $GL_a(\mathbb{C}) \times GL_b(\mathbb{C})$, for $a, b \in \mathbb{N}$ with $a+b = n$. We use the convention defining $P_{GL}(a, b)$ to be the subgroup of GL_{a+b} consisting of matricies of the form

$$\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

where $A \in GL_a(\mathbb{C})$, $B \in GL_b(\mathbb{C})$, and M is a general $a \times b$ matrix. Specifically, we may consider the space of $a \times b$ matrices as the tensor product

$$X_a \otimes X_b^\vee$$

where $X_a = \mathbb{C}^a$, $X_b = \mathbb{C}^b$ are the standard representations of the general linear groups. In fact, then we may describe the parabolic by

$$P_{GL}(a, b) \cong (GL_a \times GL_b) \ltimes (X_a \otimes X_b^\vee)$$

with semidirect product along the natural action of GL_a, GL_b on X_a, X_b (transpose for $X_b^\vee$).

Now, for the case of a general number of blocks r, fix an r-tuple $(a_1, \dots, a_r)$ of constants $a_i \in \mathbb{N}$ such tht $a_1 + \cdots + a_r = n$.

Similarly as for $r = 2$, for a general r-tuple $(a_1, \dots, a_r)$ we define the parabolic $P_{GL}(a_1, \dots, a_r)$ to be the subgroup of $GL_n(\mathbb{C})$ consisting on matricies of the form

$$\begin{pmatrix} A_1 & M_{\{1<2\}} & M_{\{1<3\}} & \dots & M_{\{1<r\}} \\ 0 & A_2 & M_{\{2<3\}} & \dots & M_{\{2<r\}} \\ 0 & 0 & A_3 & \dots & \vdots \\ 0 & 0 & 0 & \dots & A_r \end{pmatrix} \tag{32}$$

where $A_i \in GL_{a_i}(\mathbb{C})$ and for each pair $\{i < j\} \subseteq [r]$, $M_{\{i<j\}}$ is a general $a_i \times a_j$ matrix. Hence, again writing $X_{a_i} = \mathbb{C}^{a_i}$, we have

$$P_{GL}(a_1, \dots, a_r) \cong (GL_{a_1} \times \cdots \times GL_{a_r}) \ltimes \Big(\bigoplus_{\{i<j\} \subseteq [r]} X_{a_i} \otimes X_{a_j}^\vee \Big). \tag{33}$$

In this semidirect product, the sum of $X_{a_i} \otimes X_{a_j}^\vee$, (i.e. the normal subgroup consisting of matrices (32) where each $A_i = I_{a_i}$ is the identity matrix), is the unipotent radical of $P_{GL}(a_1, \dots, a_r)$.

7.2. **The interpolated theory.** Now we will consider r-tuples

$$(a_1, \dots, a_r)$$

of $a_i \in \mathbb{C} \smallsetminus \mathbb{Z}$. In this subsection, we will freely operate in the category of $GL_{a_1} \times \cdots \times GL_{a_r}$-equivariant schemes, which is introduced following the same steps as for $r = 1$ (as we mentioned in Subsection 2.3).

As above, we want to think of the parabolic as a semidirect product of a Levi subgroup $GL_{a_1} \times \cdots \times GL_{a_r}$ with a unipotent radical N

$$P_{GL}(a_1, \dots, a_r) = (GL_{a_1} \times \cdots \times GL_{a_r}) \ltimes N. \tag{34}$$

We interpret this by considering N as a $GL_{a_1} \times \cdots \times GL_{a_r}$*-equivariant affine group scheme:*

Definition 20. *We define a* $GL_{a_1} \times \cdots \times GL_{a_r}$-equivariant affine group scheme *to be* $Spec_{GL_{a_1} \times \cdots \times GL_{a_r}}(\mathcal{A})$ *for an ACU algebra object*

$$\mathcal{A} \in Obj(\underline{Rep}(GL_{a_1} \times \cdots \times GL_{a_r}))$$

with the additional data of coproduct, co-unit, and conjugation

$$\psi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$
$$\epsilon : \mathcal{A} \to 1$$
$$\gamma : \mathcal{A} \to \mathcal{A}$$

satisfying co-associativity, co-unitality, and co-inversion.

For a $GL_{a_1} \times \cdots \times GL_{a_r}$-equivariant group scheme

$$G = Spec_{GL_{a_1} \times \cdots \times GL_{a_r}}(\mathcal{A}),$$

*we define the category of G-*representations

$$Rep_{GL_{a_1} \times \cdots \times GL_{a_r}}(G)$$

to be the category of $\mathcal{A}$-comodules in $\underline{Rep}(GL_{a_1} \times \cdots \times GL_{a_r})$, i.e. objects M with a co-multiplication morphism

$$\mu : M \to M \otimes A$$

(satisfying co-associativity and co-unitality). If G is locally finite, we can define locally finite G-modules to be $\mathcal{A}$-comodules of finite type (again defined by containing only finitely many copies of each simple object in $\underline{Rep}(GL_{a_1} \times \cdots \times GL_{a_r})$).

Example: The unipotent radical. In $\underline{Rep}(GL_{a_1} \times \cdots \times GL_{a_r})$, consider the object X_{a_i} which is the summand of the basic object of dimension a_i, corresponding to the basic object of $Rep(GL_{a_i})$. Consider the algebra

$$\mathcal{A}_{(a_1,\dots,a_r)} :=$$
$$Sym\left(\bigoplus_{\{i<j\}\subseteq[r]} X_{a_i} \otimes X_{a_j}^\vee\right) \cong \bigotimes_{\{i<j\}\subseteq[r]} Sym(X_{a_i} \otimes X_{a_j}^\vee)$$

in $\underline{Rep}(GL_{a_1} \times \cdots \times GL_{a_r})$. It has a coproduct operation defined by, for each $i < j$,

$$\psi : X_{a_i} \otimes X_{a_j}^\vee \to \bigoplus_{i\leq s\leq j} (X_{a_i} \otimes X_{a_s}^\vee) \otimes (X_{a_s} \otimes X_{a_j}^\vee),$$

coming from the units of duality for each $X_{a_s}^\vee$. (A unique conjugation and counit exist.)

Theorem 21. *The category $Rep(P_{GL}(a_1, \dots, a_r))$ is equivalent to the category of representations*

$$Rep_{GL_{a_1} \times \cdots \times GL_{a_r}}(Spec_{GL_{a_1} \times \cdots \times GL_{a_r}}(\mathcal{A}_{(a_1,\dots,a_r)}))$$

of the affine group scheme $Spec_{GL_{a_1}\times\cdots\times GL_{a_r}}(\mathcal{A}_{(a_1,\ldots,a_r)})$.

$\square$

Comment: The above example suggests that in the notion of a stack in the definition of GL_c-equivariant schemes, the automorphism groups could also be interpolated. As further evidence, note that one can, in fact, give an interpolated interpretation of the absolute Weyl group of GL_c as the interpolated symmetric group Σ_c, and the normalizer of the maximal torus as $\Sigma_c \wr \mathbb{G}_m$. This will be done in future work.

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