THE ORTHOGONAL STABLE RANGE OF TYPE I HOWE DUALITY OVER A FINITE FIELD

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ABSTRACT. In [12], we considered Howe duality over finite fields for a certain range of type I reductive dual pairs with "small" orthogonal groups. In this note, we prove a version of Howe duality for an opposite range of type I reductive dual pairs consisting of a symplectic group and an orthogonal group on a space with a maximal isotropic subspace of dimension greater than or equal to that of the symplectic space.

1. Introduction

For a symplectic vector space $\mathbf{V}$ over a field k, given a non-trivial additive character

$$\mathbb{F}_q \to \mathbb{C},$$

one may consider the associated *oscillator* or *Weil-Shale* representation ω of the symplectic group $Sp(\mathbf{V})$ (when, for example, $k = \mathbb{C}$ or $\mathbb{F}_q$), or the metaplectic group $Mp(\mathbf{V})$ (when $k = \mathbb{R}$ or $\mathbb{Q}_p$). The question of Howe duality asks about the decomposition of the restriction of ω to the product of a reductive dual pair of subgroups of $Sp(\mathbf{V})$ or $Mp(\mathbf{V})$. Denote this reductive dual pair by G, H. Consider the decomposition

$$Res_{G\times H}(\omega) = \bigoplus_{\rho\in\widehat{G},\pi\in\widehat{H}} \mu(\rho,\pi)\cdot(\rho\otimes\pi) \tag{1}$$

for some multiplicities

$$\mu : \widehat{G}\times\widehat{H} \to \mathbb{N}_0.$$

(In this note, for a group G, $\widehat{G}$ denotes its Pontrjagin dual.)

For fields $k = \mathbb{C}, \mathbb{R}$, or $\mathbb{Q}_p$, there in fact exist subcollections $\mathcal{S}_G$ and $\mathcal{S}_H$ of the irreducible representations of G and H, and a bijective correspondence (called the *theta correspondence*)

$$\theta : \mathcal{S}_G \to \mathcal{S}_H,$$

such that

$$Res_{G\times H}(\omega) = \bigoplus_{\rho\in\mathcal{S}_G} \rho\otimes\theta(\rho) = \bigoplus_{\pi\in\mathcal{S}_H} \theta^{-1}(\pi)\otimes\pi. \tag{2}$$

This result, referred to as the *Howe duality conjecture*, is the culmination of a long history of work, (see, for example, [11] for the case of $k = \mathbb{R}$ and [4] for $k = \mathbb{Q}_p$), and has many deep applications in arithmetic geometry, number theory, and representation theory. It is an interesting and difficult question to find its appropriate analogue in the case when k is a finite field. This case, where $k = \mathbb{F}_q$ (for q a power of an odd prime), is the main topic of the present note.

It can be calculated that the exact statement (2) does not hold for a finite field k, due to the large isotropic subspace contained in any orthogonal space over a finite field $\mathbb{F}_q$. However, many meaningful patterns regarding which pairs $\rho\otimes\pi$ appear in the decomposition (1) have been observed and studied, see for example [1, 2, 3, 6, 7, 13, 14, 16].

In [12], we prove a version of Howe duality, fully decomposing the restricted oscillator representation, for a certain range of type I reductive dual pairs where the dimension of the involved symplectic space is at least double the dimension of the orthogonal space. We shall refer to this range of reductive dual pairs as the "symplectic stable range." We found that the restriction of the oscillator representation to a product of such a reductive dual pair of subgroups of a symplectic group decomposes into "levels" corresponding to the the sub-orthogonal groups obtainable from removing hyperbolic summands from the original symmetric bilinear form. At the top level (corresponding to removing no hyperbolic summands), the appearing summand is a direct sum of tensor product of each irreducible representation of the orthogonal group with its *eta correspondence*, which was observed by S. Gurevich and R. Howe to be "highest rank," in a certain sense (see [6, 7] for more details). The summand at level k consists of tensor products of the eta correspondence applied to each irreducible representation of the corresponding lower orthogonal group, tensored with its appropriate parabolic induction to a representation of the original orthogonal group (twisted by a certain character arising from the restriction of an oscillator representation to the general linear group).

In this follow-up note, we prove a symmetrical result for the range of reductive dual pairs where the dimension of the maximal isotropic subspace of the orthogonal space is greater than or equal to the dimension of the symplectic space, which we call the "orthogonal stable range." Specifically, we find a correspondence identifying irreducible representations of the symplectic groups with levels of irreducible representations of an orthogonal group with which they fall into the orthogonal stable range.

In the remaining range of reductive dual pairs where neither symplectic nor orthogonal space has dimension less than or equal to the maximal isotropic subspace of the other (the "unstable range"), the restriction of the oscillator representation remains unknown.

The present paper is organized as follows: In Section 2, we introduce notation and precisely state our version of Howe duality. In Section 3, we recall the vector space structure of the endomorphism algebra of an oscillator representation. In Section 4, we consider its algebra operation, and find generators giving a subalgebra isomorphic to the group algebra of the symplectic group part of a reductive dual pair in the orthogonal stable range. In Section 5, we conclude the proofs of our results using an inductive argument and several combinatorial observations.

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2. Notation and Statements

Let us denote by U an n-dimensional $\mathbb{F}_q$-vector space with a non-degenerate bilinear form B, and let us denote by V a $2N$-dimensional $\mathbb{F}_q$-vector space with a non-degenerate symplectic form S. Without loss of generality, we may assume that B, as a matrix, is of the form

$$B = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix} \tag{3}$$

for some $a_1, \dots, a_n \in \mathbb{F}_q^\times$.

The maximal dimension of an isotropic subspace of V with respect to S is N. We denote by $m(U,B)$ the maximum dimension of an isotropic subspace of U with respect to B. Over a finite field, in the case when $n = 2m$ is even, we either have

$$m(U,B) = m-1 \text{ or } m$$

(giving rise to the two orthogonal groups $O^+_{2m}(\mathbb{F}_q)$ and $O^-_{2m}(\mathbb{F}_q)$, respectively). In the case when $n = 2m+1$ is odd, then we must have

$$m(U, B) = m.$$

For i the dimension of an isotropic space Z in V (resp. U), we denote by $(V[2N-2i], S[2N-2i])$ (resp. $(U[n-2i], B[n-2i])$) the subspace of dimension $2N-2i$ (resp. $n-2i$) and its accompanying non-degenerate symplectic (resp. symmetric) form, which are obtained by projecting away from Z and its dual in V with respect to S (resp. B).

Note then that $U \otimes V$ forms a $2nN$ dimensional vector space with symplectic form $B \otimes S$. The type I reductive dual pairs in a symplectic group $Sp(\mathbf{V})$ consist precisely of subgroups $Sp(V)$, $O(U)$ where $\mathbf{V} \cong U \otimes V$ (with consistent forms). Write

$$f : O(U) \times Sp(V) \to Sp(U \otimes V). \tag{4}$$

Further, let us denote by $P_{V,i}$ (resp. $P_{U,i}$) the parabolic subgroups of $Sp(V)$ (resp. $O(U)$) corresponding to an i-dimensional isotropic subspace of V (resp. U) with Levi subgroup

$$GL_i(\mathbb{F}_q) \times Sp_{2N-2i}(\mathbb{F}_q)$$

(resp.

$$GL_i(\mathbb{F}_q) \times O_{n-2i}(\mathbb{F}_q)).$$

For $a \in \mathbb{F}_q^\times$, we may associate to a a non-trivial additive character of $\mathbb{F}_q$

$$\psi_a : \mathbb{F}_q \to \mathbb{C}^\times$$
$$x \mapsto e^{\frac{2\pi i}{p} \cdot \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a \cdot x)}.$$

For a symplectic space V, let us denote by $\omega[V]$ the oscillator representation arising from the non-trivial additive character corresponding to $1 \in \mathbb{F}_q$. We write

$$\psi := \psi_1. \tag{5}$$

More generally, we write $\omega[V]_a$ for the oscillator representation arising from the additive character corresponding to $a \in \mathbb{F}_q^\times$. (Recall that $\omega[V]_a \cong \omega[V]_b$ when a and b are equal in the quotient $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$, so there are only two non-isomorphic oscillator representations over a finite field.)

The problem of type I Howe duality then concerns the restriction of the oscillator representation $\omega[U \otimes V]$ to an $O(U) \times Sp(V)$-representation, and its resulting decomposition into tensor products of irreducible representations of $O(U)$ and $Sp(V)$. We may consider this decomposition to be indexed either by the irreducible representations of $O(U)$

$$f^*(\omega[U\otimes V]) = \bigoplus_{\rho\in\widehat{O(U)}} \rho\otimes\Phi(\rho), \tag{6}$$

or by the irreducible representations of $Sp(V)$

$$f^*(\omega[U\otimes V]) = \bigoplus_{\pi\in\widehat{Sp(V)}} \Psi(\pi)\otimes\pi \tag{7}$$

for some (not necessarily irreducible or non-zero) $Sp(V)$- and $O(U)$-representations $\Phi(\rho)$ and $\Psi(\pi)$ In the classical cases of Howe duality over $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{Q}_p$ (with the symplectic group replaced by the metaplectic group in the latter two cases), the functions Φ and Ψ, both restricted away from the elements of $\widehat{O(U)}, \widehat{Sp(V)}$ where they give 0, give irreducible representations and are inverse to each other.

In the case when $dim(U)\leq dim(V)/2$, it can be calculated that all the irreducible representations of $O(U)$ appear when further restricting $f^*(\omega[U\otimes V])$ to $O(U)$ (see [6, 7, 12]). The eta correspondences then can be expressed as injections

$$\eta_{U,V}:\widehat{O(U)}\hookrightarrow\widehat{Sp(V)}$$

which assign to an irreducible representation ρ of $O(U)$, the "highest rank" irreducible summand of $\Phi(\rho)$ (see [6, 7] for more details). In [12], we proved that in fact,

$$f^*(\omega[U\otimes V]) =$$

$$\bigoplus_{k=0}^{m(U,B)} \bigoplus_{\rho\in\widehat{O(U[n-2k])}} Ind_{O(U)}^{P_{U,k}}(\rho\otimes\epsilon(det))\otimes\eta_{U[n-2k],V}(\rho) \tag{8}$$

where $\epsilon:\mathbb{F}_q^\times\rightarrow\{\pm1\}$ denotes the non-trivial multiplicative character of $\mathbb{F}_q$ of order two, and

$$\epsilon(det): GL_k(\mathbb{F}_q)\rightarrow\mathbb{F}_q^\times\rightarrow\{\pm1\}$$

is considered as a representation of the $GL_k(\mathbb{F}_q)$ factor of the Levi subgroup of $P_{U,k}$.

In the present note, we prove that in the case when $dim(V)\leq m(U,B)$, all the irreducible representations of $Sp(V)$ occur in the restriction of $f^*(\omega[U\otimes V])$ to $Sp(V)$, meaning that for every $\pi\in\widehat{Sp(V)}$, the $O(U)$-representation $\Psi(\pi)$ in (7) is non-zero. Further, we determine a "top" irreducible summand

$$\zeta_{U,V}(\pi)\subseteq\Psi(\pi)$$

which can be considered an analogue of the eta correspondence. Finally, we obtain a symmetric formula to (8):

Theorem 1. *When $dim(V) \leq m(U, B)$, we have injections*

$$\zeta_{V,U} : \widehat{Sp(V)} \hookrightarrow \widehat{O(U)}$$

sending irreducible $Sp(V)$-representations to irreducible $O(U)$-representations, with disjoint images for non-isomorphic choices of (V, S), (U, B), such that

$$f^*(\omega[U \otimes V]) \cong$$

(9) $$\bigoplus_{k=0}^{N} \bigoplus_{\pi \in \widehat{Sp(V[2N-2k])}} Ind_{Sp(V)}^{P_{V,k}}(\pi \otimes \epsilon(det)) \otimes \zeta_{V[2N-2k],U}(\pi).$$

3. Endomorphisms of the Oscillator Representation

As in [12], our main method to approaching Theorem 1 is by examining the endomorphism algebra of $\omega[U \otimes V]$. In this section, we recall a certain description of the endomorphism algebra of a oscillator representation as a linearization of its symplectic vector space, which we can use to restate Theorem 1.

First, consider a general symplectic space and form $(\mathbf{V}, \mathbf{S})$. Recall that for an oscillator representation $\omega[\mathbf{V}]_a$, its dual is the oscillator representation of opposite character $\omega[\mathbf{V}]_{-a}$ (see [8]). Their tensor product gives the standard representation $\mathbb{C}\mathbf{V}$, where $Sp(\mathbf{V})$ acts geometrically.

Hence, for a subgroup $G \subseteq Sp(\mathbf{V})$, the endomorphism algebra of the restriction of the oscillator representation to G is, as a vector space,

$$End_G(\omega[\mathbf{V}]_a) \cong Hom_G(1, \omega[\mathbf{V}]_a \otimes \omega[\mathbf{V}]_{-a}) \cong (\mathbb{C}\mathbf{V})^G,$$

which can also be considered as the $\mathbb{C}$-vector space with a basis indexed by G-orbits on $\mathbf{V}$.

In fact, we can introduce an operation $\star_{\mathbf{V}}$ on $\mathbb{C}\mathbf{V}$ corresponding to composition in the endomorphism algebra such that, as $\mathbb{C}$-algebras,

$$(End_{Vect}(\omega[\mathbf{V}]_a), \circ) \cong (\mathbb{C}(\mathbf{V}), \star_{\mathbf{V}})$$

(for subgroups $G \subseteq Sp(\mathbf{V})$, the endomorphism algebra of $\omega[\mathbf{V}]$ over G is again isomorphic to the subalgebra generated by G-orbits on $\mathbf{V}$). We put, for $u, v \in \mathbf{V}$,

$$(u) \star_{\mathbf{V}} (v) = \psi_a(\frac{1}{2}\mathbf{S}(u, v)) \cdot (u + v).$$

This can be considered as an "untwisted" variant of the algebra operation arising from the *Schrödinger model* of the oscillator representation:

Recall that for a decomposition of a symplectic space $\mathbf{V}$ into Lagrangians $\mathbf{\Lambda}_+ \oplus \mathbf{\Lambda}_-$, we may identify $\omega[\mathbf{V}]_a \cong \mathbb{C}\mathbf{\Lambda}_-$. The action of an element (v,c) of the Heisenberg group $\mathbb{H}$ (for $v \in \mathbf{V}$, $c \in \mathbb{F}_q$) on a generator $w \in \Lambda_-$ is given by

$$(v,c)(w) = \psi_a(c + \frac{1}{2}\mathbf{S}(v_+, w)) \cdot (v_- + w),$$

where $v = v_+ + v_-$ is the unique decomposition of a vector v into $v_+ \in \mathbf{\Lambda}_+$, $v_- \in \mathbf{\Lambda}_-$. This gives $\omega[\mathbf{V}]_a$ the structure of a *Weil-Shale representation*. The action of $Sp(\mathbf{V})$ making it an oscillator representation arises from the uniquess of the Weil-Shale representation for each central character. We then see a natural action of an algebra $(\mathbb{C}\mathbf{V}, *_\mathbf{V})$ for algebra operation $*$ given by, for $u = u_+ + u_-, v = v_+ + v_- \in \mathbf{V}$, $u_\pm, v_\pm \in \mathbf{\Lambda}_\pm$,

$$(u) * (v) = \psi_a(\mathbf{S}(u_+, v_-)) \cdot (u + v).$$

Applied to $w \in \mathbf{\Lambda}_-$, a vector $v \in (\mathbb{C}\mathbf{V}, *_\mathbf{V})$ acts by

$$(v)(w) = \psi_a(\mathbf{S}(v_+, w)) \cdot (v_- + w).$$

Now our choice of algebra $(\mathbb{C}\mathbf{V}, \star_\mathbf{V})$ is isomorphic to the algebra $(\mathbb{C}\mathbf{V}, *_\mathbf{V})$ along

$$(\mathbb{C}\mathbf{V}, \star_\mathbf{V}) \to (\mathbb{C}\mathbf{V}, *_\mathbf{V})$$
$$(v) \mapsto \psi_a(\frac{1}{2}\mathbf{S}(v_+, v_-)) \cdot (v)$$

for $v = v_+ + v_- \in \mathbf{V}$ with $v_\pm \in \mathbf{\Lambda}_\pm$. Therefore, an element $v \in (\mathbb{C}\mathbf{V}, \star)$ acts on $w \in \mathbf{\Lambda}_-$ by

$$(v)(w) = \psi_a(\mathbf{S}(v_+, w) + \frac{1}{2}\mathbf{S}(v_+, v_-))(v_+ + w) \tag{10}$$

for $v = v_+ + v_-$ with $v_\pm \in \mathbf{\Lambda}_\pm$.

Applying this to the present situation, where $\mathbf{V} = U \otimes V$, $\mathbf{S} = B \otimes S$, we have $(End_{Vect}(\omega[U \otimes V]), \circ) \cong (\mathbb{C}(U \otimes V), \star_{U\otimes V}$. For $u_1, u_2 \in U$, $v_1, v_2 \in V$

$$(u_1 \otimes v_1) \star_{U\otimes V} (u_2 \otimes v_2) =$$
$$\psi(\frac{B(u_1, u_2) \cdot S(v_1, v_2)}{2}) \cdot (u_1 \otimes v_1 + u_2 \otimes v_2).$$

When the ground space is clear, we omit the subscript in $\star$. To prove Theorem 1, we will consider the subalgebra of $O(U)$-equivariant endomoprhisms

$$(End_{O(U)}(\omega[U \otimes V]), \circ) \cong ((\mathbb{C}(U \otimes V))^{O(U)}, \star). \tag{11}$$

We reframe Theorem 1 as the following

Theorem 2. *For $m(U,B) \geq dim(V)$, we have*

$$End_{O(U)}(\omega[U \otimes V]) \cong \bigoplus_{k=0}^{N} M_{|Sp(V)/P_{V,k}|}(\mathbb{C}Sp(V[2N-2k])). \tag{12}$$

The proofs of Theorems 1 and 2 naturally progress together in a single argument. The first step is to enumerate the basis vectors of $End_{O(U)}(\omega[U \otimes V])$.

By (11), as a vector space, $End_{O(U)}(\omega[U \otimes V])$ has a basis corresponding to $O(U)$-orbits in

$$U \otimes V = \underbrace{U \oplus \cdots \oplus U}_{2N} \tag{13}$$

(with respect to the diagonal action). We claim that in the orthogonal stable range of dimensions, we have the following enumeration of orbits:

Lemma 3. *For $dim(V) \leq m(U,B)$, the orbits of $O(U)$ acting on (13) are indexed by reduced row echelon form matrices with $2N$ columns, and a choice of a constant in $\mathbb{F}_q$ for each unordered pair of pivot columns (including a pivot column with itself). Hence, we have*

$$dim(End_{O(U)}(\omega[U \otimes V])) = (q+1)\dots(q^{2N}+1).$$

Proof. To enumerate the number of $O(U)$-orbits in (13), note that, for a $2N$-tuple $(u_1, \dots, u_{2N})$ of vectors $u_i \in U$, the action of $O(U)$ must precisely preserve 1. the linear independence/dependence relations of $u_1, \dots, u_{2N}$ and 2. the values of the bilinear form applied to the u_i's.

First, the linear independence/dependence relations of the u_i's correspond exactly to the data of which $GL(U)$-orbit $(u_1, \dots, u_{2N})$ is an element of. Recall that the $GL(U)$-orbits of (13) can be identified as precisely the images of $d \times 2N$ matrices M in reduced row echelon form for

$$0 \leq d = dim(\langle u_1, \dots, u_{2N} \rangle) \leq 2N,$$

i.e. the sets of $2N$-tuples

$$\{M \cdot (u'_1, \dots, u'_d) \mid u'_1, \dots, u'_d \in U \text{ linearly independent}\}$$

($u'_1, \dots, u'_d$ are renamings of the vectors $u_{i_1}, \dots, u_{i_d}$ where the indices $1 \leq i_1 < \dots < i_d \leq 2N$ are the pivot columns of M). By our assumption on the dimension of U, all d and choices of M are possible.

Now, given a choice of a $d \times 2N$ matrix M in reduced row echelon form, if

$$(u_1, \dots, u_{2N}) = M \cdot (u'_1, \dots, u'_d)$$

for some linearly independent $u'_1, \dots, u'_d \in U$, then the data of the form B's values on any pair of u_i, u_j, is determined by (and equivalent to) the data of constants

$$\nu_{i \leq j} = B(u'_i, u'_j) \in \mathbb{F}_q,$$

for $i \leq j \in \{1, \dots, d\}$. Since we assumed that $dim(U) > 2 \cdot dim(V)$, there indeeed must exist an isotropic subspace of U with respect to B of dimension $2N = dim(V)$. In fact, then, all values of B on linearly independent $u'_1, \dots, u'_d$ are possible and therefore give non-empty orbits (the cardinality of each orbit will depend on the choice of symmetric bilinear form B on U).

Hence, the orbits are indexed by the data of, for $d = 0, \dots, 2N$, a $d \times 2N$ matrix M in reduced row echelon form, together with a choice of constants $\nu_{i \leq j} \in \mathbb{F}_q$ for $i, j \in \{1, \dots, d\}$. Therefore, the number of orbits is

$$\sum_{d=0}^{2N} q^{\binom{d}{2}+d} \cdot \left(\sum_{1 \leq i_1 < \cdots < i_d \leq 2N} q^{2dN - (i_1 + \cdots + i_d) - \binom{d}{2}} \right) =$$

$$\sum_{1 \leq i_1 < \cdots < i_d \leq 2N} \prod_{j=1}^{d} q^{(2N+1)-i_j} = (q+1) \dots (q^{2N}+1).$$

□

4. The Top Subalgebra

Now that we have identified a set of basis elements of $End_{O(U)}(\omega[U \otimes V])$, the next step is to find a subalgebra isomorphic to the group algebra of $Sp(V)$. The goal of this section is to prove the following

Proposition 4. *If $dim(V) \leq m(U, B)$, then the subalgebra of*

$$End_{O(U)}(\omega[U \otimes V])$$

consisting of endomorphisms that cannot be factored as a composition

$$\omega[U \otimes V] \to \omega[U \otimes V[2N - 2k]] \to \omega[U \otimes V] \tag{14}$$

for any $k = 1, \dots, N$ contains the group algebra $\mathbb{C}Sp(V)$.

Proof of Proposition 4. We begin by recalling how an $O(U)$-orbit in (13), considered as a linear combination of vectors in $U \otimes V$, can be applied to an element of $\omega[U \otimes V]$. We will again use the Schrödinger model of $\omega[U \otimes V]$. Let us write $V = \Lambda^+ \oplus \Lambda^-$ for V's decomposition into complementary Lagrangians with respect to S. According to (10), writing an element of the algebra $(\mathbb{C}(U \otimes V), \star)$ as $(v_1^+ + v_1^-, \dots, v_n^+ + v_n^-)$ for $v_i^\pm \in \Lambda^\pm$, and writing an element of $\omega[U \otimes V] = \mathbb{C}U \otimes \Lambda^-$ as $(w_1, \dots, w_n)$ for $w_i \in \Lambda^-$, we have

$$(15)\qquad \begin{gathered}(v_1^+ + v_1^-, \dots, v_n^+ + v_n^-) \cdot (w_1, \dots, w_n) = \\ \psi(\sum_{j=1}^n a_i \cdot (S(v_i^+, w_i) + \frac{S(v_i^+, v_i^-)}{2})) \cdot (v_1^- + w_1, \dots, v_n^- + w_n)\end{gathered}$$

(where ψ denotes the non-trivial additive character corresponding to our choice of oscillator representation $\boldsymbol{\omega}$).

To write this in terms of the symmetric bilinear form B and vectors $u \in U$, let us fix bases of the Lagrangians Λ^+, Λ^- such that, with respect to the basis $\lambda_1^+, \dots, \lambda_N^+, \lambda_1^-, \dots, \lambda_N^-$ of V, the symplectic form S is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Then, alternatively, writing an element of $U \otimes V$ as $(z_1^+, z_1^-, \dots, z_N^+, z_N^-)$ for $z_i^\pm \in U \otimes \mathbb{F}_q\{\lambda_i^\pm\}$, and an element of $U \otimes \Lambda^-$ as $(u_1, \dots, u_N)$ for $u_i \in U \otimes \mathbb{F}_q\{\lambda_N^-\}$, we have

$$(16)\qquad \begin{gathered}(z_1^+, z_1^-, \dots, z_N^+, z_N^-) \cdot (u_1, \dots, u_N) = \\ \psi(\sum_{i=1}^N B(z_i^+, u_i) + \frac{B(z_i^+, z_i^-)}{2}) \cdot (u_1 + z_1^-, \dots, u_N + z_N^-)\end{gathered}$$

To prove the claimed statement, we will find elements of $\mathbb{C}(U \otimes V)^{O(U)}$ which act on elements of the Schrödinger model of $\omega[U \otimes V]$ by the representation action of group generators of $Sp(V)$.

Let us first consider the case when $dim(V) = 2$. In this case, we may reduce (16) to

$$(z^+, z^-) \cdot (u) = \psi(B(z^+, u) + \frac{B(z^+, z^-)}{2}) \cdot (u + z^-).$$

From this perspective, the action of the matrices in $SL_2(\mathbb{F}_q)$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

on the oscillator representation should correspond to transformations

$$(u) \mapsto \frac{1}{q^{n/2}} \cdot \sum_{v \in U} \psi(B(-u,v)) \cdot (v), \qquad (u) \mapsto \psi(\frac{t \cdot B(u,u)}{2}) \cdot (u) \tag{17}$$

respectively. The matrices

$$\begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix}$$

act by $(u) \mapsto (s \cdot u)$, for $s \in \mathbb{F}_q^\times$. Now consider, for example, the operators given by the action of

$$g_t = \frac{1}{q^{n/2}} \cdot \sum_{z \in U} \psi(\frac{tB(z,z)}{2}) \cdot (z, tz) \in \mathbb{C}(U \otimes V)^{O(U)}$$

for $t \in \mathbb{F}_q^\times$. Applied to $u \in U$, these endomorphisms give

$$g_t \cdot (u) = \frac{1}{q^{n/2}} \cdot \sum_{z \in U} \psi(B(z,u) + B(z,tz)) \cdot (u + tz),$$

which, replacing $v = u + tz$, can be simplified to

$$\frac{1}{q^{n/2}} \cdot \sum_{v \in U} \psi(B(\frac{v-u}{t}, v)) \cdot (v) = \frac{1}{q^{n/2}} \cdot \sum_{v \in U} \psi(\frac{B(v,v)}{t}) \cdot \psi(B(\frac{-u}{t}, v)) \cdot (v).$$

Therefore, each g_t corresponds to the group action of the composition of matrices

$$\begin{pmatrix} 1 & 0 \\ 2/t & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & t \\ -1/t & 2 \end{pmatrix} \tag{18}$$

on $\omega[U \otimes V]$. These matrices generate $SL_2(\mathbb{F}_q)$.

Now, for general V, $dim(V) = 2N$, we may find these generators for all choices of 1-dimensional subspaces in a Lagrangian (and its dual). This system of group algebras over $SL_2(\mathbb{F}_q)$ corresponding to choices of isotropic 1-dimensional subspace of V therefore generate $Sp(V)$. Hence, we get

$$\mathbb{C}Sp(V) \subseteq End_{O(U)}(\omega[U \otimes V]). \tag{19}$$

Additionally, since these endomorphisms encode the geometric action of $Sp(V) \subseteq Sp(U \otimes V)$ on $\omega[U \otimes V]$ and are, in particular, bijective, they are inexpressible as compositions of the form (14). □

The reason why we use the matrices (18) (instead of a more common set of generators of $SL_2(\mathbb{F}_q)$, such as (20) below) is due to the fact that their corresponding elements of $\mathbb{C}(U \otimes \mathbb{F}_q^2)^{O(U)}$ are fairly simple and easy to guess. While not directly necessary to the logic of the proof of the results of the above proposition, it is, however, instructive to write down the explicit formulae for the elements corresponding to the matricies

$$\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \tag{20}$$

for $t, s \in \mathbb{F}_q^\times$.

To do this, we need to introduce certain constants, arising from the quadratic sums of characters. They will depend on our choice of charcter ψ. If we use a different character, the answer may differ by a sign. Let us write $q = p^\ell$ for p an odd prime, and $\ell \in \mathbb{N}$. To avoid confusion, we denote the quadratic multiplicative characters of $\mathbb{F}_q$ and $\mathbb{F}_p$ by

$$\epsilon_p : \mathbb{F}_p^\times \to \{\pm 1\}, \qquad \epsilon_q : \mathbb{F}_q^\times \to \{\pm 1\},$$

respectively. As usual, we extend these to 0 by $\epsilon_p(0) = \epsilon_q(0) = 0$. Denoting the norm of the field extension by $N_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q^\times \to \mathbb{F}_p^\times$, we have

$$\epsilon_p \circ N_{\mathbb{F}_q/\mathbb{F}_p} = \epsilon_q. \tag{21}$$

We may re-write the classical quadratic Gauss sum as

$$\sum_{n\in\mathbb{F}_p} e^{\frac{2\pi i}{p}n^2} = \sum_{m\in\mathbb{F}_p} (1+\epsilon_p(m)) \cdot e^{\frac{2\pi i}{p}n^2} = \sum_{m\in\mathbb{F}_p} \epsilon_p(m) \cdot e^{\frac{2\pi i}{p}n^2} \tag{22}$$

(since the linear sum of characters is 0, and for each $m \in \mathbb{F}_p$, there are exactly $1 + \epsilon_p(m)$ elements in $\mathbb{F}_p$ whose square is m), which is well-known to equal

$$\sum_{n\in\mathbb{F}_p} e^{\frac{2\pi i}{p}a\cdot n^2} = \epsilon_p(a) \cdot \sqrt{\epsilon_p(-1)\cdot p}. \tag{23}$$

Now the same argument as (22) can be applied to give

$$\sum_{x\in\mathbb{F}_q} \psi(x^2) = \sum_{x\in\mathbb{F}_q} e^{\frac{2\pi i}{p}Tr_{\mathbb{F}_q/\mathbb{F}_p}(x^2)} =$$

$$\sum_{y\in\mathbb{F}_q} \epsilon_q(y) \cdot e^{\frac{2\pi i}{p}Tr_{\mathbb{F}_q/\mathbb{F}_p}(y)}.$$

Applying the Hasse-Davenport relation for Gauss sums to (23), we get that

$$\sum_{y\in\mathbb{F}_q} \epsilon_q(y) \cdot e^{\frac{2\pi i}{p}Tr_{\mathbb{F}_q/\mathbb{F}_p}(y)} = (-1)^{\ell+1} \cdot \left(\sqrt{\epsilon_p(-1)\cdot p}\right)^\ell,$$

which simplifies to give

$$\sum_{x\in\mathbb{F}_q}\psi(x^2)=(-1)^{\ell+1}\sqrt{\epsilon_q(-1)\cdot q} \tag{24}$$

Now, for $c\in\mathbb{F}_q^\times$, again since a linear sum of characters vanishes, we have

$$\sum_{x\in\mathbb{F}_q}\psi(c\cdot x^2)=\epsilon_q(c)\cdot\sum_{x\in\mathbb{F}_q}\psi(x^2).$$

Combining this with (24), we find that

$$\sum_{u\in U}\psi(c\cdot B(u,u))=\sum_{u_1,\dots,u_n\in\mathbb{F}_q}\psi(\sum_{i=1}^{n}c\cdot a_i\cdot u_i^2)=$$

$$\prod_{i=1}^{n}\sum_{u_i\in U}\psi(c\cdot a_i\cdot u_i^2)=\epsilon_q(c^n\cdot a_1\dots a_n)\cdot(\sum_{x\in\mathbb{F}_q}\psi(x^2))^n=$$

$$(-1)^{n(\ell+1)}\cdot disc(B)\cdot q^{n/2}\cdot\epsilon_q(c)^n\cdot\epsilon_q(-1)^{n/2},$$

where $disc(B)$ denotes is discriminant, i.e. $\epsilon_q(det(B))$. For notational brevity, we denote these coefficients by

$$\begin{aligned}K(c):=\sum_{x\in\mathbb{F}_q}\psi(c\cdot x^2)=\\(-1)^{n(\ell+1)}\cdot disc(B)\cdot q^{n/2}\cdot\epsilon_q(c)^n\cdot\epsilon_q(-1)^{n/2}.\end{aligned} \tag{25}$$

Proposition 5. *Consider an $\mathbb{F}_q$-space U and a symmetric bilinear form B.*

(1) For $t\in\mathbb{F}_q^\times$, the element

$$\alpha_t:=\frac{1}{q^n}\cdot\sum_{y^+,y^-\in U}\psi(-\frac{t+1}{2(t-1)}\cdot B(y^+,y^-))\cdot(y^+,y^-)$$

acts as the matrix

$$\begin{pmatrix}t&0\\0&1/t\end{pmatrix}.$$

(2) The element

$$\beta:=\frac{1}{K(1)\cdot q^{n/2}}\sum_{y^+,y^-\in U}\psi\left(\frac{1}{4}(B(y^+,y^+)+B(y^-,y^-))\right)\cdot(y^+,y^-)$$

acts as the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(3) *For* $s \in \mathbb{F}_q^\times$*, the element*

$$\gamma_s := \frac{1}{K(-1/2s)} \sum_{z \in U} \psi(-\frac{1}{2s} B(z,z)) \cdot (z, 0)$$

acts as the matrix

$$\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

Proof. We begin with the proof of (1): For an element $u \in U \cong U \otimes \Lambda^-$, we can calculate that f_t acts as follows:

$$\begin{aligned} &\alpha_t(u) = \\ &\frac{1}{q^n} \cdot \sum_{y^+, y^- \in U} \psi\left((\frac{1}{2} - \frac{t+1}{2(t-1)}) \cdot B(y^+, y^-) + B(y^+, u) \right) \cdot (y^- + u) \\ &= \frac{1}{q^n} \cdot \sum_{y^+, y^- \in U} \psi(B(y^+, -\frac{y^-}{t-1} + u)) \cdot (y^- + u). \end{aligned} \tag{26}$$

The sum runs over arbitrary choices of y^+, meaning that for fixed $u \in U$ and chosen $y^- \in U$, the coefficient sum

$$\sum_{y^+ \in U} \psi(B(y^+, -\frac{y^-}{t-1} + u)) \tag{27}$$

of the vector $(y^- + u)$ is a linear sum of characters, and is therefore 0, unless

$$-\frac{y^-}{t-1} + u = 0,$$

in which case (27) is q^n. Hence, the only contributing choice of y^- is $y^- = (t-1) \cdot u$. Therefore, (26) simplifies as

$$\alpha_t(u) = \frac{q^n}{q^n} \cdot ((t-1) \cdot u + u) = (t \cdot u),$$

agreeing with the action of the proposed matrix on the oscillator representation $\mathbb{C}U$.

Now we prove (2): For $u \in U$, at each choice of $y^+, y^- \in U$, applying the corresponding term of the sum in β (disregarding the coefficient, for now) gives

$$\psi\left(\frac{1}{4}(B(y^+,y^+)+B(y^-,y^-))\right)\cdot(y^+,y^-)(u)=$$

$$\psi\left(\frac{1}{4}(B(y^+,y^+)+2B(y^+,y^-)+B(y^-,y^-))+B(y^+,u)\right)\cdot(y^-+u)=$$

$$\psi(B(\frac{y^++y^-}{2},\frac{y^++y^-}{2})+B(y^+,u))\cdot(y^-+u)$$

Therefore, we have

$$\beta(u)=$$

$$\frac{1}{K(1)\cdot q^{n/2}}\sum_{y^+,y^-\in U}\psi(B(\frac{y^++y^-}{2},\frac{y^++y^-}{2})+B(y^+,u))\cdot(y^-+u). \tag{28}$$

Renaming variables using $z=y^-+u$, we may rewrite this as

$$\frac{1}{K(1)\cdot q^{n/2}}\sum_{y^+,z\in U}\psi(B(\frac{y^++z-u}{2},\frac{y^++z-u}{2})+B(y^+,u))\cdot(z). \tag{29}$$

Now we may also notice that

$$B(\frac{y^++z-u}{2},\frac{y^++z-u}{2})+B(y^+,u)=$$

$$B(\frac{y^++z+u}{2},\frac{y^++z+u}{2})-B(z,u),$$

allowing us to rewrite (29) as

$$\frac{1}{K(1)\cdot q^{n/2}}\sum_{z,y^+\in U}\psi(B(\frac{y^++z+u}{2},\frac{y^++z+u}{2})-B(z,u))\cdot(z).$$

Renaming variables using $w=(y^++z+u)/2$ gives

$$\frac{1}{K(1)\cdot q^{n/2}}\sum_{z,w\in U}\psi(B(w,w))\psi(-B(z,u))\cdot(z),$$

which, applying (25), reduces to

$$\beta(u)=\frac{1}{q^{n/2}}\sum_{z\in U}\psi(-B(z,u))\cdot(z),$$

which is precisely the action (17).

Finally, we prove (3): For $u \in U$,

$$\gamma_s(u) =$$

$$\frac{1}{K(-1/2s)} \sum_{z \in U} \psi(-\frac{1}{2s}B(z,z)) \cdot (z,0)(u) = \tag{30}$$

$$\frac{1}{K(-1/2s)} \sum_{z \in U} \psi(-\frac{1}{2s}B(z,z) + B(z,u)) \cdot (u).$$

Now we may notice that

$$B(z,u) - \frac{1}{2s}B(z,z) = -\frac{1}{2s}(-2s \cdot B(z,u) + B(z,z)) =$$

$$-\frac{1}{2s}\left(B(su-z, su-z) - s^2 \cdot B(u,u)\right) =$$

$$-\frac{1}{2s} \cdot B(su-z, su-z) + \frac{s}{2} \cdot B(u,u).$$

Therefore, substituting $w = su - z$, we can rewrite (30) as

$$\gamma_s(u) = \frac{1}{K(-1/2s)} \sum_{w \in U} \psi(-\frac{1}{2s} \cdot B(w,w)) \cdot \psi(\frac{s}{2} \cdot B(u,u)) \cdot (u). \tag{31}$$

Since, by definition,

$$\sum_{w \in U} \psi(-\frac{1}{2s} \cdot B(w,w)) = K(-1/2s),$$

(31) then reduces to

$$\gamma_s(u) = \psi(\frac{s}{2} \cdot B(u,u)) \cdot (u),$$

agreeing precisely with (17).

□

It may also be helpful to compute some examples of $\star$ applied to these elements, and see how it recovers matrix multiplication (especially to see the relationship between g_t, α_t, β, and $\gamma_{2/t}$). We do an example of such a computation in the Appendix.

Proposition 4 corresponds to the $k = 0$ term of the claimed decomposition (11), and would ensure that every irreducible $Sp(V)$-representation appears with multiplicity at least one in the restriction of $f^*(\omega[U \otimes V])$ to an $Sp(V)$-representation. From the point of view of Theorem 1, this implies that there exist some $O(U)$-representations $\zeta_{U,V}(\rho)$ for each irreducible representation $\rho \in \widehat{Sp(V)}$ such that $\zeta_{U,V}(\rho) \otimes \rho$ are summands of $f^*(\omega[U \otimes V])$.

5. The Proofs of Theorem 2 and Theorem 1

In this section, we conclude the remainder of the proofs of Theorem 2 and Theorem 1, which follow by an inductive argument combined with certain combinatorial observations, similarly as in [12]. Suppose that the claim (12) holds for (V, S) replaced by $(V[2N-2k], S[2N-2k])$. Let us denote by Y_{2N-2k} the top part of the restriction of $\omega[U \otimes V[2N-2k]]$ to $O(U) \times Sp(V[2N-2k])$, i.e.

$$Y_{2N-2k} = \bigoplus_{\pi \in \widehat{Sp(V[2N-2k])}} \pi \otimes \zeta_{U,V[2N-2k]}(\pi).$$

By the induction hypothesis, restricting the oscillator representation of $V[2N-2k] \otimes U$ to $O(U)$,

$$\bigoplus_{\ell=0}^{N-k} |Sp(V[2N-2k])/P_{V[2N-2k],\ell}| \cdot Y_{2N-2(k+\ell)}$$

For clarity, let us write

$$g : Sp(V[2N-2k]) \times O(U) \hookrightarrow Sp(V) \times O(U)$$

for the natural inclusion of groups.

Then first note that by the induction hypothesis, restricting $\omega[V \otimes U]$ to an $Sp(V[2N-2k]) \otimes O(U)$ representation,

$$Hom_{O(U)}(g^*(\omega[V \otimes U]), \omega[V[2N-2k] \otimes U]) =$$
$$\bigoplus_{\ell=1}^{N} |Sp(V[2N-2k])/P_{V[2N-2k],\ell}| \cdot Hom_{O(U)}(Y_{2N-2(k+\ell)}, \omega[V \otimes U]) \; .$$

Now, again by the duality of oscillator representations, we have

$$\begin{gathered} dim(Hom_{O(U)}(g^*(\omega[V \otimes U]), \omega[V[2N-2k] \otimes U]) = \\ dim(Hom_{O(U)}(1, \mathbb{C}(U^{\oplus 2N-k})) \end{gathered} \tag{32}$$

By the proof of Lemma 3, we find that (32) is

$$(q+1) \dots (q^{2N-k}+1).$$

We claim that

$$(33)\qquad (q+1)\dots(q^{2N-k}+1) = \sum_{\ell=0}^{m(U,B)-k} |Sp(V[2N-2k])/P_{V[2N-2k],\ell}|\cdot|Sp(V)/P_{V,k+\ell}|\cdot|Sp(V[2N-2k-2\ell])|$$

It is well-known that, for every $N\in\mathbb{N}$, the order of the symplectic group on $\mathbb{F}_q^{2N}$ is

$$(34)\qquad |Sp_{2N}(\mathbb{F}_q)| = q^{N^2}\prod_{i=1}^{N}(q^{2i}-1),$$

and, writing the Gaussian binomial coefficients

$$\binom{x}{y}_q = \frac{(q^y-1)\cdot(q^{y-1}-1)\dots(q^{y-x+1}-1)}{(q^x-1)\cdot(q^{x-1}-1)\dots(q-1)}$$

the cardinality of its quotient by a maximal paraolic $P_{\mathbb{F}_q^{2N},k}$ (i.e. the number of k-dimensional isotropic subspaces of $\mathbb{F}_q^{2N}$) is

$$(35)\qquad |Sp_{2N}(\mathbb{F}_q)/P_{\mathbb{F}_q^{2N},k}| = \binom{N}{k}_q\cdot\prod_{i=N-k+1}^{N}(q^i+1).$$

The argument for (33) then follows by dividing the left and right hand side by the shared factors of (q^i+1), then redistributing the remaining factors on the left hand side to approximate the factors , as in the proof of Lemma 7, [12].

However, we may also compare (34) and (35) to the numbers of elements in $O_{2N+1}(\mathbb{F}_q)$, and its parabolic quotients (see (10) of [12]), to notice that

$$|Sp_{2N}(\mathbb{F}_q)| = \frac{1}{2}\cdot|O_{2N+1}(\mathbb{F}_q)|,$$

and

$$|Sp_{2N}(\mathbb{F}_q)/P_{\mathbb{F}_q^{2N},k}| = |O_{2N+1}(\mathbb{F}_q)/P_{\mathbb{F}_q^{2N},k}|.$$

Therefore, the expression (33), combinatorially, is exactly formula (25) of Corollary 1, [12] at $n=2N+1$, divided on both sides by 2.

Now, as in [12], we may recursively use Lemma 3 and the induction hypothesis to solve for the dimensions of the Hom-spaces from each Y_{2N-2k} piece to $\omega[U\otimes V]$ to get

$$dim(Hom_{O(U)}(Y_{2N-2k},\omega[U\otimes V])) = |Sp(V)/P_{V,k}|,$$

verifying (after adjunction) that the parabolic induction summands appear with multiplicity 1. This argument again follows from [12], (45), where n is replace by $2N+1$, and both sides are divided by 2, by Lemma 3.

Finally, it remains to check that the dimensions of both sides of (12) are in fact equal. This implies that, in Proposition 4, the group algebra $\mathbb{C}Sp(V)$ makes up the entire top sub-algebra of $End_{O(U)}(\omega[U \otimes V])$. From the point of view of Theorem 1, this gives that the $\zeta_{U,V}(\rho)$ are all distinct irreducible $O(U)$-representations, and that no other summands may appear. By Lemma 3, this statement reduces to checking that

$$(q+1)\dots(q^{2N}+1) = \sum_{k=0}^{N} |Sp(V)/P_{V,k}|^2 \cdot |Sp(V[2N-2k])|.$$

Appendix A. An Explicit Composition Computation

Fix $t \in \mathbb{F}_q^\times$. In this appendix, we complete the calculation that

$$g_t \star \alpha_t = \gamma_{2/t} \star \beta. \tag{36}$$

The composition $g_t \star \alpha_t$ is $1/q^{3n/2}$ times the sum over all choices of $y^+, y^-, z \in U$ of terms

$$\psi(\frac{t}{2}B(z,z) - \frac{t+1}{2(t-1)}B(y^+,y^-)) \cdot (z,tz) \star (y^+,y^-). \tag{37}$$

Writing out

$$(z,tz) \star (y^+,y^-) = \psi(\frac{1}{2}(B(z,y^-) - t \cdot B(z,y^+))),$$

each term (37) can be simplified to the pair of vector $(y^+ + z, y^- + tz)$ multiplied by the coefficient

$$\psi(\frac{t}{2}B(z,z) - \frac{t+1}{2(t-1)}B(y^+,y^-) + \frac{1}{2}B(z,y^-) - \frac{t}{2}B(z,y^+)).$$

By considering

$$-\frac{t+1}{2(t-1)} = \frac{1}{2} - \frac{t}{t-1}, \qquad -\frac{t}{2} = \frac{t}{2} - t,$$

this can be rewritten as

$$\psi(\frac{1}{2}B(y^+ + z, y^- + tz) - \frac{t}{t-1}B(y^+, y^- + (t-1)z)).$$

Substituting $u = y^+ + z$, $v = y^- + tz$ gives

$$\psi(\frac{1}{2}B(u,v) - \frac{t}{t-1}B(u-z, v-z)).$$

Therefore, we have reduced $g_t \star \alpha_t$ to

$$\frac{1}{q^{3n/2}} \sum_{z,u,v \in U} \psi(\frac{1}{2}B(u,v) - \frac{t}{t-1}B(u-z, v-z)) \cdot (u,v). \tag{38}$$

Writing

$$B(u-z, v-z) = B(u,v) - B(u+v, z) + B(z,z),$$

we may "complete the square" by noticing that

$$-B(u+v,z) + B(z,z) = \\ B(z - \frac{u+v}{2}, z - \frac{u+v}{2}) - B(\frac{u+v}{2}, \frac{u+v}{2}).$$

Substituting variables using $w = z - (u+v)/2$, putting the terms together, we get

$$B(u-z, v-z) = B(u,v) + B(w,w) - B(\frac{u+v}{2}, \frac{u+v}{2}) =$$

$$B(w,w) - B(\frac{u-v}{2}, \frac{u-v}{2}).$$

Therefore, (38) reduces to

$$\begin{gathered} g_t \star \alpha_t = \\ \frac{1}{q^{3n/2}} \sum_{w,u,v \in U} \psi(\frac{1}{2}B(u,v) - \frac{t}{t-1}(B(w,w) - B(\frac{u-v}{2}, \frac{u-v}{2}))) \cdot (u,v) = \\ \frac{K(-t/(t-1))}{q^{3n/2}} \sum_{u,v \in U} \psi(\frac{1}{2}B(u,v) + \frac{t}{t-1}B(\frac{u-v}{2}, \frac{u-v}{2})) \cdot (u,v). \end{gathered} \tag{39}$$

Now let us consider the other side of (36). The composition $\gamma_{2/t} \star \beta$ is $1/q^{n/2}K(-t/4)K(1)$ times the sum over all choices of $y^+, y^-, z \in U$ of terms

$$\psi(-\frac{t}{4}B(z,z) + \frac{1}{4}(B(y^+, y^+) + B(y^-, y^-))) \cdot (z, 0) \star (y^+, y^-)$$

Writing out

$$(z, 0) \star (y^+, y^-) = \psi(\frac{1}{2}B(z, y^-)) \cdot (z + y^+, y^-),$$

this term is the pair of vectors $(z+y^+, y^-)$ multiplied by the coefficient

$$\psi(-\frac{t}{4}B(z,z)+\frac{1}{4}(B(y^+,y^+)+B(y^-,y^-))+\frac{1}{2}B(z,y^-)) =$$

$$\psi(-t\cdot B(\frac{z}{2},\frac{z}{2})+B(\frac{y^+-y^-}{2},\frac{y^+-y^-}{2})+\frac{1}{2}B(z+y^+,y^-))$$

Substituting variables $u=z+y^+$, $v=y^-$, $w=z/2$ we reduce $\gamma_s\star\beta$ to

$$\sum_{u,v,w\in U}\psi(-tB(w,w)+B(\frac{u-v}{2}-w,\frac{u-v}{2}-w)+\frac{1}{2}B(u,v))\cdot(u,v).$$

Writing

$$B(\frac{u-v}{2}-w,\frac{u-v}{2}-w) =$$

$$B(\frac{u-v}{2},\frac{u-v}{2})-2B(\frac{u-v}{2},w)+B(w,w),$$

we have

$$-tB(w,w)+B(\frac{u-v}{2}-w,\frac{u-v}{2}-w) =$$

$$-(t-1)\cdot B(w,w)-2\cdot B(\frac{u-v}{2},w)+B(\frac{u-v}{2},\frac{u-v}{2}).$$

Completing the square gives

$$B(w,w)+\frac{2}{t-1}B(\frac{u-v}{2},w) =$$

$$B(w+\frac{u-v}{2(t-1)},w+\frac{u-v}{2(t-1)})-\frac{1}{(t-1)^2}B(\frac{u-v}{2},\frac{u-v}{2}).$$

Replacing variables $x=w+\frac{1}{2(t-1)}(u-v)$ gives

$$-tB(w,w)+B(\frac{u-v}{2}-w,\frac{u-v}{2}-w)+\frac{1}{2}B(u,v) =$$

$$-(t-1)B(x,x)+(1+\frac{t-1}{(t-1)^2})B(\frac{u-v}{2},\frac{u-v}{2})+\frac{1}{2}B(u,v) =$$

$$-(t-1)B(x,x)+\frac{t}{t-1}B(\frac{u-v}{2},\frac{u-v}{2})+\frac{1}{2}B(u,v)$$

Thus, $\gamma_t \star \beta$ is the factor $\frac{1}{q^{n/2}K(-t/4)K(1)}$ times

$$\sum_{u,v,x\in U} \psi(-(t-1)B(x,x) + \frac{t}{t-1}B(\frac{u-v}{2}, \frac{u-v}{2}) + \frac{1}{2}B(u,v)) \cdot (u,v) =$$

$$K(-(t-1)) \cdot \sum_{u,v\in U} \psi(\frac{t}{t-1}B(\frac{u-v}{2}, \frac{u-v}{2}) + \frac{1}{2}B(u,v)) \cdot (u,v).$$

This agrees with our above calculation of $g_t \star \alpha_t$ in (39), up to a constant. It remains to check that the constants precisely agree, i.e.

$$\frac{K(-(t-1))}{q^{n/2}K(-t/4)K(1)} = \frac{K(-t/(t-1))}{q^{3n/2}}. \tag{40}$$

Recalling (25), first note that since

$$K(c) = (-1)^{n(\ell+1)} disc(B) \cdot q^{n/2} \cdot \epsilon_q(c)^n \cdot \epsilon_q(-1)^{n/2}$$

only depends on $\epsilon_q(c)$, we have $K(-t/4) = K(-t)$. We can therefore simplify (40) to

$$q^n \cdot K(-(t-1)) = K(-t/(t-1)) \cdot K(-t) \cdot K(1).$$

Next, the signs, i.e. the factors $(-1)^{n(\ell+1)}disc(B)$ in each K factor will cancel, since both the left and right hand side have and odd number of K factors. Further, collect the powers of q, both sides have a factor of $q^{3n/2}$, which we may factor out. This reduces the claim to

$$\epsilon_q(-(t-1))^n \cdot \epsilon(-1)^{n/2} = \epsilon_q(-t/(t-1))^n \epsilon_q(-t)^n \epsilon_q(-1)^{3n/2}.$$

Dividing both sides by $\epsilon_q(-1)^{n/2}$ and collecting terms gives

$$\epsilon_q(-(t-1))^n = \epsilon_q(\frac{-t}{t-1} \cdot (-t) \cdot (-1))^n,$$

which holds, since $\epsilon_q(-(t-1)) = \epsilon_q(-1/(t-1))$.

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