TYPE I HOWE DUALITY OVER FINITE FIELDS

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Abstract. In this paper, we consider higher tensor powers of oscillator representations over finite fields. We find a decomposition of their endomorphism algebras into group algebras of orthogonal groups, giving a new version of Howe duality for a certain range of Type I reductive dual pairs. Specifically, we find that all occurring terms arise from parabolic inductions and the eta correspondence of R. Howe and S. Gurevich. This also leads to a version of Howe duality in a certain category "interpolating" the oscillator representations.

1. Introduction

The goal of this paper is to approach the problem of Howe duality over finite fields from the perspective of tensor category theory. The key point of this paper's results is specifically the study the endomorphism algebra of tensor powers of the oscillator representation of the symplectic group over a finite field, which can be combined to give results about the restriction of the Weil representation to the product of a reductive dual pair, giving new information about Howe duality and the eta correspondence [13, 14].

1.1. **The problem of Howe duality.** For a field k, the problem of Howe duality concerns the *Weil-Shale* or *oscillator representation* of the symplectic (in the case when k is a finite field, which is the focus of the present paper, or $\mathbb{C}$) or metaplectic (when k is $\mathbb{R}$ or $\mathbb{Q}_p$) group of k (see for example [40, 1, 12, 38]). One may then consider a reductive dual pair of subgroups G, G' in this group (see e.g. [11]). The restriction of the oscillator representation to $G \times G'$ then decomposes as a direct sum of tensor products of irreducible representations of G and G'. The Howe duality conjecture [17, 11, 37] proposes the existence of certain sub-collections of irreducible representations

$$\mathcal{S}(G) \subseteq \widehat{G},\ \mathcal{S}(G') \subseteq \widehat{G'}$$

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and a bijection

$$\Phi : \mathcal{S}(G) \to \mathcal{S}(G') \tag{1}$$

such that the restriction of the oscillator representation to $G \times G'$ can be written as a direct sum

$$\bigoplus_{\rho \in \mathcal{S}(G)} \rho \otimes \Phi(\rho). \tag{2}$$

This conjecture, and more generally, the question of what can be said about the pairings of irreducible representations of a reductive dual pair occuring in the restriction of an oscillator representation, have been the subject of great interest and study for the various cases of fields k (see, for example, see [10, 19, 29, 30, 34, 39]).

In this paper, we will primarily focus on the case when k is a finite field $\mathbb{F}_q$ (and the oscillator representation is then a representation of the symplectic group). Further, we will also restrict attention to the case of Type I reductive dual pairs, which are of the form $(Sp_{2N}(\mathbb{F}_q), O^{\pm}_n(\mathbb{F}_q))$ (see, for example, [16]). In this case, it turns out that a decomposition of the form (2) arising from a bijection (1) is not possible. This is related to the fact that symmetric bilinear forms over a finite field have large isotropic summands, while the representation spectrum of a finite group is discrete, and hence the corresponding representations are visible in the Howe decomposition. However, patterns of what pairs of representations of symplectic and orthogonal groups can occur have been studied extensively, for example, see [2, 3, 13, 14]. Certain combinatorial descriptions of the decomposition of the restriction of the oscillator representation have been found by S.-Y. Pan [35, 36] involving G. Lusztig's classification of irreducible representations of algebraic groups [32]. However, questions about conceptual interpretations of these results are still the subject of current research [8, 31, 41].

In [13, 14], S. Gurevich and R. Howe introduced the *eta correspondence*, which describes a process of matching an irreducible representation ρ of an orthogonal group $O^{\pm}(\mathbb{F}_q^n)$ with an irreducible representation $\eta(\rho)$ of $Sp(\mathbb{F}_q^{2N})$. The tensor products $\rho \otimes \eta(\rho)$ all occur with multiplicity 1 in the appropriate restriction of the oscillator representation of $Sp(\mathbb{F}_q^{2Nn})$, and the images of η for all choices of $n \leq N$ and all choices of orthogonal group remain disjoint. The representation $\eta(\rho)$ can in fact be considered to be maximal with respect to a certain sense of "rank" introduced by Howe and Gurevich within the choices of irreducible representations of $Sp(\mathbb{F}_q^{2N})$ paired with ρ.

The results of the present paper in fact show that the restriction of the oscillator representation to $Sp(\mathbb{F}_q^{2N}) \times O(\mathbb{F}_q^n, Q)$ (for Q a quadratic form) is the sum, over all irreducible representations π of orthogonal groups obtained by "deleting hyperbolic summands" from Q, of terms of the form

$$Ind_{O(Q)}^{P}(\pi \otimes \epsilon) \otimes \eta(\pi)$$

where $\eta(\pi)$ denotes the eta correspondence for this lower-dimensional orthogonal group with $Sp(\mathbb{F}_q^{2N})$ and $Ind_{O(Q)}^{P}$ denotes a parabolic induction functor and ϵ denotes a certain quadratic character of a general linear group (see Theorem 1 and (11) below). This offers new information towards conjectures proposed in [13], particularly Conjecture 0.4.12.

1.2. **Interpolation and tensor category theory.** The methods of this paper were also motivated by the theory of *interpolation* in tensor categories first introduced by P. Deligne [4, 5, 6] and heavily studied since (for example, [9, 15, 23, 24, 26, 28]). Specifically, an additive, $\mathbb{C}$-linear category with an associative commutative unital tensor product and strong duality generated by a basic object turns out to be describable using only universal algebra data on the vector spaces of homomorphisms between tensor powers of this basic generating object (see [4, 26]). In the case of the present paper, we apply this philosophy and the methods it suggests to the case of taking the basic generating object to be (the sum of) the oscillator representations.

The study of endomorphisms of tensor powers of a basic object can also be interpreted from N. M. Katz and P. H. Tiep's point of view of *moments* with respect to Haar measure on a locally compact group, see [20, 21, 22].

The method of proof of Theorem 1 relies on calculations of the endomorphisms of tensor powers of oscillator representations in the category of representations of a large enough symplectic group, and therefore one obtains applications to the interpolated settings as well. We exhibit one concrete statement in Theorem 2 and discuss it further in Section 6 of the present paper (see also [28, 27]).

1.3. **The set-up.** To state our results, let us recall the basic concepts involved. For further details, see e.g. R. Howe [18]. Fix a power q of an odd prime p. Let V_N be a $2N$-dimensional vector space over a field $\mathbb{F}_q$ with a symplectic form S_N. To simplify notation, we shall omit the N and S_N from the notation of V_N and its symplectic group, writing V

and $Sp(V)$. In this paper, we consider the oscillator representations of $Sp(V)$, corresponding to non-trivial additive characters $\psi : \mathbb{F}_q \to \mathbb{C}^\times$.

Given an identification of the Pontrjagin dual of $\mathbb{F}_q$ with itself

$$\widehat{\mathbb{F}_q} \cong \mathbb{F}_q, \tag{3}$$

we denote the oscillator representation of $Sp(V)$ defined by the non-trivial additive character of $\mathbb{F}_q$ corresponding via (3) to $a \in \mathbb{F}_q^\times$ by ω_a. For example, for $a \in \mathbb{F}_q^\times$, let $\psi_a : \mathbb{F}_q \to \mathbb{C}^\times$ be the character

$$x \mapsto e^{\frac{2\pi i}{p} \cdot \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a\cdot x)}.$$

We emphasize however the non-canonicity of the notation ω_a, since it depends on the underlying choice of identification (3), since the main result of this paper yields different results depending on this choice.

Recall that, as representations of $Sp(V)$,

$$\omega_a \cong \omega_b$$

if and only if $a = b$ in the quotient $(\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$ (see [16]). In particular, there are two choices of non-isomorphic oscillator representations.

Let Q be a non-degenerate symmetric bilinear form (equivalently, quadratic form) on $\mathbb{F}_q^n$. (We shall pass between the points of views of symmetric bilinear and quadratic forms fluidly.) Let us consider the orthogonal group $O(Q)$ of $n \times n$ matrices over $\mathbb{F}_q$ preserving Q. We have a natural inclusion map

$$f : O(Q) \times Sp(V) \xrightarrow{\subseteq} Sp(\mathbb{F}_q^{2nN}) \tag{4}$$

given by, in the target, considering the symplectic form $Q \otimes S$ on $\mathbb{F}_q^n \otimes V = \mathbb{F}_q^{2nN}$ (f can be described then as the tensor product of matrices). Then $O(Q)$, $Sp(V)$ form a reductive dual pair of type I in $Sp(\mathbb{F}_q^{2nN})$.

Now choose $\boldsymbol{\omega}$ to be one of the two oscillator representations of $Sp(\mathbb{F}_q^{2nN})$. Let us choose an identification (3) so that the character of $\mathbb{F}_q$ defining $\boldsymbol{\omega}$ is identified with $1 \in \mathbb{F}_q$, i.e.

$$\boldsymbol{\omega} \cong \omega_1$$

as representations of $Sp(\mathbb{F}_q^{2nN})$. For the remainder of this paper, we will be using this choice of (3).

Let $(V_i^Q)_{i=1}^{m_Q}$ be representatives of isomorphism classes of irreducible $\mathbb{C}$-representations of $O(Q)$ (letting m_Q denote the number of conjugacy classes of $O(Q)$). Let us denote the hyperbolic quadratic form on $\mathbb{F}_q^2$ by

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Recall that a quadratic form Q on $\mathbb{F}_q^n$ can be decomposed as

$$Q \cong \underbrace{H \oplus \cdots \oplus H}_{h_Q} \oplus Z \tag{5}$$

where h_Q denotes the number of hyperbolics in Q and Z denotes an anisotropic quadratic form of dimension 0, 1, or 2. Further, the decomposition (5) is unique, i.e. the isomorphism class of Z is independent of the choice of decomposition. In particular, therefore, there are only two non-isomorphic orthogonal groups on $\mathbb{F}_q^n$ when n is even (corresponding to $n = 2h_q, 2h_q + 2$) and a single orthogonal group when n is odd (corresponding to $n = 2h_q + 1$). These groups are denoted by $O^+(n)$ and $O^-(n)$ when n is even and $dim(Z) = 0$ and 2, respectively, and $O(n)$ when n is odd.

We define a sequence of quadratic forms $Q[n-2k]$ of dimensions $n - 2k$ for $k = 0, \dots, h_Q$ so that

$$Q = Q[n-2k] \oplus H^{\oplus k}.$$

(By Witt's Theorem, the quadratic forms $Q[n-2k]$ are unique up to isomorphism.) We can visualize $Q[n-2k]$ by deleting hyperbolics in (5) so that

$$Q[n-2k] \cong H^{\oplus h_q - k} \oplus Z.$$

We have

$$O(Q) \supseteq O(Q[n-2]) \supseteq \cdots \supseteq O(Q[n-2h_q]).$$

Next, for $k = 0, \dots, h_q$ let us consider the set of k-dimensional subspaces of the solutions of Q, i.e.

$$\{W \subseteq V \mid dim(W) = k, \ \forall x \in W \ \ Q(x) = 0\}. \tag{6}$$

The number of elements in (6) can also be computed as the cardinality $|O(Q)/P_{Q,k}|$ of the quotient of $O(Q)$ by the maximal parabolic $P_{Q,k} \subseteq O(Q)$ with Levi factor

$$O(Q[n-2k]) \times GL_k(\mathbb{F}_q), \tag{7}$$

corresponding to the decomposition

$$Q = (Q[n-2k]) \oplus (H^{\oplus k}) \tag{8}$$

where the first factor of (7) acts by orthogonal transformation on the first summand of (8) and the second factor of (7) acts by linear transformation of a maximal isotropic subspace on the second summand of (8).

For an $O(Q[n-2k])$-representation V, we denote by V_ϵ the $P_{Q,K}$-representation given by projection from $P_{Q,K}$ to the Levi factor (7) where the first factor acts by V and the second factor acts by the composition

$$\epsilon(det) : GL_k(\mathbb{F}_q) \xrightarrow{det} \mathbb{F}_q^\times \xrightarrow{\epsilon} \{\pm 1\}$$

where $\epsilon : \mathbb{F}_q^\times \to \{\pm 1\}$ is the non-trivial character of order 2 of $\mathbb{F}_q^\times$.

1.4. **Statement of results.** The main result of this paper is the following

Theorem 1. *Let q be a power of an odd prime. For any $n \in \mathbb{N}$, for all $N \geq n$, we have an explicitly defined isomorphism of $O(Q) \times Sp(V)$-representations*

$$f^*(\boldsymbol{\omega}) \cong$$

$$\bigoplus_{k\geq 0}^{h_Q} \bigoplus_{i=1}^{m_{Q[n-2k]}} Ind_{O(Q)}^{P_{Q,K}}(V_{i,\epsilon}^{Q[n-2k]}) \otimes W_i^{Q[n-2k]} \tag{9}$$

for non-isomorphic irreducible $Sp(V)$-representations $W_i^{Q[n-2k]}$ for $0 \leq k \leq h_Q$, $i = 1, \ldots, m_{Q[n-2k]}$. The representations $W_i^{Q[n-2k]}$ depend on i, the isomorphism class of $Q[n-2k]$, and the choice of $\boldsymbol{\omega}$, but the dimensions of the representations $W_i^{Q[n-2k]}$ are independent of $\boldsymbol{\omega}$.

Furthermore, one has

$$|O(Q)/P_{Q,k}| = \begin{cases} \binom{m}{k}_q \prod_{j=m-k+1}^{m} (q^j+1) & \text{if } n = 2m+1 \\ \binom{m}{k}_q \prod_{j=m-k}^{m-1} (q^j+1) & \text{if } n = 2m,\ dim(Z) = 0 \\ \binom{m-1}{k}_q \prod_{j=m-k+1}^{m} (q^j+1) & \text{if } n = 2m,\ dim(Z) = 2 \end{cases} \tag{10}$$

where $\binom{m}{k}_q$ denote the Gaussian binomial coefficients.

(Note that the cardinality (10) can also be calculated as

$$|O(Q)/P_{Q,k}| = \frac{|O(Q)|_{p'}}{|GL_k(\mathbb{F}_q)|_{p'} \cdot |O(Q[n-2k])|_{p'}}$$

where $|?|_{p'}$ denotes the part of the order which is prime to p.)

The "top" pairing of V_i^Q with W_i^Q recovers the eta correspondence, and therefore, in the notation of Howe and Gurevich, the decomposition (9) can be re-written as

$$\bigoplus_{k\geq 0}^{h_Q} Ind_{O(Q)}^{P_{Q,k}}(V_{i,\epsilon}^{Q[n-2k]}) \otimes \eta(V_i^{Q[n-2k]}). \tag{11}$$

Comment: The existence of the isomorphism (9) was established from the point of view of mathematical physics (probabilistic theory of quantum codes) by F. Montealegre-Mora and D. Gross [33]. In this paper, we follow a different approach, giving an explicit combinatorial formula for the decomposition.

Our result can be further interpreted and generalized in the language of tensor categories as follows: P. Deligne [7, 27, 28] introduced a variant $\mathscr{R}_{q,t}$ of the category of $Sp_t(\mathbb{F}_q)$-representations for $t \in \mathbb{C}$ (interpolating the case $t = 2N$ for $N \in \mathbb{N}$) whose objects consist of direct summands of interpolated oscillator representations. We briefly introduce these concepts in Section 6 below. The category $\mathscr{R}_{q,t}$ is a semisimple pre-Tannakian category, i.e. a locally finite semisimple $\mathbb{C}$-linear tensor category with strong duality. What is remarkable is that the category $\mathscr{R}_{q,t}$ is $W(\mathbb{F}_q)$-graded where $W(\mathbb{F}_q)$ is the Witt ring of quadratic forms over $\mathbb{F}_q$. (For our purposes, we only use the additive structure.) As an additive group, $W(\mathbb{F}_q)$ is $\mathbb{Z}/4$ for $q \equiv 3 \bmod 4$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$ for $q \equiv 1 \bmod 4$. This category $\mathscr{R}_{q,t}$ has particular objects ω^Q where Q is a quadratic form over $\mathbb{F}_q$. One can define ([4], Section 3) the *top algebra* $End^{top}(\omega^Q)$ as the quotient of $End(\omega^Q)$ by the two-sided ideal of endomorphisms which factor through some $\omega^{Q'}$ with $dim(Q') < dim(Q)$. Howe duality for finite fields can then also be stated as follows:

Theorem 2. *One has* $End^{top}_{\mathscr{R}_{q,t}}(\omega^Q) \cong \mathbb{C}(O(Q))$ *for* $t \notin 2 \cdot \mathbb{Z}$ *and, for* $t = 2N$ *where* $N \in \mathbb{Z}$ *such that* $|N| > dim(Q)$.

Going further, we can even consider the case of a general field of characteristic not equal to 2. When F is not finite, the oscillator representation is infinite dimensional, and techniques of analysis are needed to specify the kind of representations we are considering. However, when Q is split, we can still describe a purely algebraic candidate for (a dense subspace of) the top algebra $End^{top}(\omega^Q)$ extending the case of F finite, and prove an isomorphism with $\mathbb{C}O(Q)$, thus giving an "abstract type I Howe duality" statement for Q split. We also discuss this in Section 6.

The present paper is organized as follows In Section 2, we recall some background about the endomorphism algebra structure of oscillator representations and their tensor products. In Section, 3 we reformulate the statement of Theorem 1 in terms of the endomorphism algebra structure of the restriction of $f^*(\boldsymbol{\omega})$ to an $Sp(V)$-representation, and prove some preliminary results of the algebra structure of its endomorphisms. In Section 4, we calculate that the dimension of the endomorphism algebra of $f^*(\boldsymbol{\omega})$ predicted by Theorem 1 matches its combinatorial interpretation, therefore implying that $W_i^{Q[n-2k]}$ are all non-isomorphic and irreducible. In Section 5, we conclude the proof of Theorem 1. Finally, the stability of Theorem 1 for high enough N gives an interpretation of its statement in the context of interpolated categories, which will be discussed in the Section 6.

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2. The endomorphism algebra over the symplectic group

In this section, we discuss the structure of the endomorphism algebra of a tensor products of oscillator representations.

2.1. **Dimensions.** Recall that the dual of ω_a is ω_{-a}, with

$$\omega_a \otimes \omega_{-a} \cong \mathbb{C}V$$

with its natural $Sp(V)$-action (see [16]). The trace map

$$tr : \mathbb{C}V \cong \omega_a \otimes \omega_{-a} \to \mathbb{C}$$

is given by setting, for $v \in V$,

$$tr((v)) = \begin{cases} q^N = dim(\omega_a) & \text{if } v = 0 \\ \\ 0 & \text{else.} \end{cases}$$

For a single oscillator representation ω_a corresponding to a $a \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$, as a vector space, its endomorphism algebra is isomorphic to the $Sp(V)$-fixed points on the $\mathbb{C}$-vector space freely generated by V

$$End_{Sp(V)}(\omega_a) \cong Hom_{Sp(V)}(\mathbb{C}, \mathbb{C}V) \cong (\mathbb{C}V)^{Sp(V)}.$$

For a tensor product of oscillator representations $\omega_{a_1} \otimes \cdots \otimes \omega_{a_n}$, then

$$End_{Sp(V)}(\omega_{a_1} \otimes \cdots \otimes \omega_{a_n}) \cong ((\mathbb{C}V)^{Sp(V)})^{\otimes n} = (\mathbb{C}(V \oplus \cdots \oplus V))^{Sp(V)}$$

(letting $Sp(V)$ act diagonally on the n copies of V). In particular, therefore, the dimension of $End_{Sp(V)}(\omega_{a_1} \otimes \cdots \otimes \omega_{a_n})$ can be calculated:

Lemma 3. *Suppose that $dim(V) = 2N \geq 2n$. The dimension of the endomorphism algebra of $\omega_{a_1} \otimes \cdots \otimes \omega_{a_n}$, as a vector space, is*

$$dim(End_{Sp(V)}(\omega_{a_1} \otimes \cdots \otimes \omega_{a_n})) = 2(q+1)\dots(q^{n-1}+1)$$

Proof. We have that the dimension of $End_{Sp(V)}(\omega_{a_1} \otimes \cdots \otimes \omega_{a_n})$ is equal to the number of $Sp(V)$-orbits in $V^{\oplus n}$.

Consider an n-tuple of vectors $v_1, \dots, v_n \in V$ spanning a d-dimensional subspace $W \subseteq V$ for $0 \leq d \leq n$, with a (non-degenerate) antisymmetric form induced by S. The orbit of $(v_1, \dots, v_n)$ is classified by the isomorphism class of $(W, \{v_1, \dots, v_n\}, A)$, i.e. the data of a subspace of V, a spanning set of n vectors, and an antisymmmetric form. Note that such a collection of data $(W, \{v_1, \dots, v_n\}, A)$ occurs if and only if

$$dim(V) \geq dim(W) + dim(ker(A)).$$

In particular, all $(W, \{v_1, \dots, v_n\}, A)$ occur if and only if

$$dim(V) \geq 2n.$$

Now given the data of a subspace W of dimension d spanned by $\{v_1, \dots, v_n\}$, there are $q^{\binom{d}{2}}$ choices of an antisymmetric form on W. Given W, the data of a spanning set of vectors $\{v_1, \dots, v_n\}$ is then equivalent to an onto linear map

$$\mathbb{F}_q^n \twoheadrightarrow W, \tag{12}$$

with the equivalence relation of isomorphisms on W acting by composition on (12). Therefore, the number of orbits of $Sp(V)$ acting on $V^{\oplus n}$ is

$$\sum_{d=0}^{n} |Gr_{n,n-d}(\mathbb{F}_q))| \cdot q^{\binom{d}{2}},$$

where $Gr_{n,n-d}(\mathbb{F}_q)$ denotes the Grassmanian of $(n-d)$-dimensional subspaces of $\mathbb{F}_q^n$.

To count the number of elements of $Gr_{n,n-d}(\mathbb{F}_q)$, or equivalently, the number of possible equivalence classes of maps (12) (under composition with elements of $GL(W)$), first note that this is precisely the number of $d \times n$ matrices in reduced row echelon form (which attains d pivots). This number can be calculated as

$$\sum_{1 \leq \ell_1 < \cdots < \ell_d \leq n} q^{dn-(\ell_1+\cdots+\ell_d)-\binom{d}{2}}$$

by taking ℓ_i to be the length of the ith row from the left up to and including the pivot (the $\binom{d}{2} = 1+2+\cdots+(d-1)$ term arising from the entries of the matrix above a pivot being 0). In other words, this follows from the decomposition of the Grassmanian into its Schubert cells.

Therefore,

$$dim(End_{Sp(V)}(\omega_{a_1} \otimes \cdots \otimes \omega_{a_n})) =$$

$$\sum_{d=0}^{n} q^{\binom{d}{2}} \cdot \left(\sum_{1 \leq \ell_1 < \cdots < \ell_d \leq n} q^{dn-(\ell_1+\cdots+\ell_d)-\binom{d}{2}} \right) = \sum_{1 \leq \ell_1 < \cdots < \ell_d \leq n} \prod_{i=1}^{d} q^{n-\ell_i}. \tag{13}$$

□

2.2. **The composition operation.** Now we consider the algebra structure of $End_{Sp(V)}(\omega_{a_1} \otimes \cdots \otimes \omega_{a_n})$. First, let us again consider the $n=1$ case. Defining an algebra structure

$$(v) \star_a (w) = \psi_a(S(w,v)) \cdot (v+w)$$

on $\mathbb{C}V$, we have that as $\mathbb{C}$-algebras, the endomorphism algebra of ω_a is isomorphic to the subalgebra of $(\mathbb{C}V, \star_a)$ generated by $Sp(V)$-fixed points in $\mathbb{C}V$

$$(End_{Sp(V)}(\omega_a), \circ) \cong ((\mathbb{C}V)^{Sp(V)}, \star_a) \subseteq (\mathbb{C}V, \star_a). \tag{14}$$

Therefore, for a tensor product $\omega_{a_1} \otimes \cdots \otimes \omega_{a_n}$ we define an algebra operation $\star_{(a_1,\ldots,a_n)}$ on $\mathbb{C}(V^{\oplus n})$ by

$$(v_1, \ldots, v_n) \star_{(a_1,\ldots,a_n)} (w_1, \ldots, w_n) =$$
$$\psi_1(a_1 \cdot S(w_1, v_1) + \cdots + a_n \cdot S(w_n, v_n)) \cdot (v_1 + w_1, \ldots, v_n + w_n).$$

Then, again, we have that the endomorphism algebra of $\omega_{a_1} \otimes \cdots \otimes \omega_{a_n}$ is the sub-algebra of $\mathbb{C}(V^{\oplus n})$ on fixed points

$$(End_{Sp(V)}(\omega_{a_1} \otimes \cdots \otimes \omega_{a_n}), \circ) \cong$$
$$((\mathbb{C}(V^{\oplus n}))^{Sp(V)}, \star_{(a_1,\ldots,a_n)}) \subseteq (\mathbb{C}(V^{\oplus n}), \star_{(a_1,\ldots,a_n)}).$$

Remark: While it is nor directly necessary to the proof of Theorem 1, it is instructive to see how an element of $(\mathbb{C}V)^{Sp(V)}$, and more generally $\mathbb{C}V$, acts on an element of ω_a. Consider a decomposition of V into Lagrangians

$$V = \Lambda_+ \oplus \Lambda_- \tag{15}$$

Recall that the Heisenberg group $\mathbb{H} = (V \times \mathbb{F}_q, *)$,

$$(v_1, x) * (v_2, y) = (v_1 + v_2, x + y + \frac{S(v_2, v_1)}{2})$$

can be considered to act on $\mathbb{C}\Lambda_-$ by, for $(v_+ + v_-, x) \in \mathbb{H}$, with $v_\pm \in \Lambda_\pm$, and $w \in \Lambda_-$,

$$(v, x) \cdot w = \psi_a(x + \frac{S(v_+, w)}{2})(v_- + w)$$

defining the *Schrödinger model* of the Weil-Shale representation. The uniqueness of the Weil-Shale representation with a fixed central character of $\mathbb{F}_q = Z(\mathbb{H})$ determines an action of $Sp(V)$, giving the oscillator representation ω_a. There is a natural action of $\mathbb{C}V$ with a different algebra structure $\circ$ where, for $u_\pm, v_\pm \in \Lambda_\pm$,

$$(u_+ + u_-) \circ (v_+ + v_-) = \psi_a(S(u_+, v_-))(u_+ + v_+ + u_- + v_-).$$

The algebra $(\mathbb{C}V, \circ)$ acts on $\mathbb{C}\Lambda_-$ by putting, for a generator $w \in \Lambda_-$ of $\omega_a \cong \mathbb{C}\Lambda_-$,

$$(v_+ + v_-)w = \psi_a(S(v_+, w))(v_- + w).$$

To get the action of (14), we must apply the isomorphism

$$(\mathbb{C}V, \star) \to (\mathbb{C}V, \circ)$$

$$(v_+ + v_-) \mapsto \psi_a(\frac{S(v_+, v_-)}{2}) \cdot (v_+ + v_-).$$

Therefore, the action of (14) is given by putting, for $v_\pm \in \Lambda_\pm$,

$$(v_+ + v_-)(w) = \psi_a(S(v_+, w) + \frac{S(v_+, v_-)}{2})(v_+ + w) \tag{16}$$

(in fact defining an action of the whole algebra $(\mathbb{C}V, \star_a)$ on ω_a).

3. The structure theory of the endomorphism algebra over the symplectic group

To relate the statements in the previous section to Theorem 1, we consider the restriction of $f^*(\boldsymbol{\omega})$ to an $Sp(V)$-representation

$$\omega_Q := Res_{Sp(V)}^{O(Q)\times Sp(V)}(f^*(\boldsymbol{\omega})).$$

The statement of Theorem 1 can be restated as the following

Theorem 4. *For a quadratic form Q on $\mathbb{F}_q^n$, the endomorphism algebra of ω_Q decomposes as a direct sum of matrix algebras of the group algebras of lower orthogonal groups*

$$End_{Sp(V)}(\omega_Q) = \bigoplus_{k=0}^{h_Q} M_{|O(Q)/P_{Q,k}|}(\mathbb{C}O(Q[n-2k])). \tag{17}$$

3.1. **The generators.** It will be helpful to introduce specific generators of $End_{Sp(V_N)}(\omega_Q)$.

Take, for $\lambda = [\lambda_1 : \dots : \lambda_n] \in \mathbb{P}^{n-1}(\mathbb{F}_q)$,

$$f_\lambda = \sum_{v\in V}(\lambda_1 v, \dots, \lambda_n v).$$

More generally, for a $d = 0, \dots, n$, for a $d\times n$ matrix M in reduced row-echelon form and a choice of elements $\nu_{\{i<j\}} \in \mathbb{F}_q$ for $i < j \in \{1, \dots, d\}$, put

$$f_{M,\nu_{\{i,j\}}} = \sum_{\substack{v_1,\dots,v_d \in V,\\ S(v_j,v_i)=\nu_{\{i<j\}}}} (v_1, \dots, v_d)\cdot M.$$

Proposition 5. *As an algebra, $End_{Sp(V)}(\omega_Q)$ is $\star_Q$ generated by f_λ for $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$.*

Proof. Without loss of generality, suppose Q is of the form of a diagonal matrix with entries $a_1, \dots, a_n \in \mathbb{F}_q$. Define for $d = 0, \dots, n$, consider the subspace of $End_{Sp(V)}(\omega_Q)$ generated by orbits of $\mathbb{C}(V^{\oplus n})^{Sp(V)}$ consisting of n-tuples of vectors spanning a subspace of dimension $\leq d$:

$$F_d = \mathbb{C}\{[(v_1, \dots, v_n)] \mid dim(span(v_1, \dots, v_n)) \leq d\rangle\} \\ \subseteq End_{Sp(V)}(\omega_Q).$$

Equivalently, as $\mathbb{C}$-vector spaces,

$$F_d = \mathbb{C}\{f_{M,\nu_{\{i,j\}}} \mid M \text{ a matrix in reduced row echelon form with} \\ d' \leq d \text{ rows, } \nu_{\{i<j\}} \in \mathbb{F}_q \text{ for } i < j \in \{1, \dots, d'\}\} .$$

We claim first that

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots \subseteq F_n = End_{Sp(V)}(\omega_Q)$$

form a multiplicative filtration of $End_{Sp(V)}(\omega_Q)$. To see this, we must check that if, for $k = 1, 2$, M_k is a reduced row echelon form with d_k rows, $\nu^{(k)}_{\{i<j\}} \in \mathbb{F}_q$ for $i < j \in \{1, \dots, d_k\}$,

$$(18) \qquad f_{M_1,\nu^{(1)}_{\{i<j\}}} \star_Q f_{M_2,\nu^{(2)}_{\{i<j\}}} \in F_{d_1+d_2}.$$

Expanding $f_{M_1,\nu^{(1)}_{\{i<j\}}} \star_Q f_{M_2,\nu^{(2)}_{\{i<j\}}}$ is a sum over vectors

$$v_1, \dots, v_{d_1}, w_1, \dots, w_{d_2} \in V$$

such that

$$S(v_j, v_i) = \nu^{(1)}_{\{i<j\}}, \ S(w_j, w_i) = \nu^{(2)}_{\{i<j\}}$$

of terms

$$(19) \qquad ((v_1, \dots, v_{d_1}) \cdot M_1) \star_Q ((w_1, \dots, w_{d_2}) \cdot M_2),$$

and each term (19), by the definition of $\star_Q$, is a coefficient multiplied by the n-tuple of vectors

$$(v_1, \dots, v_{d_1}) \cdot M_1 + (w_1, \dots, w_{d_2}) \cdot M_2 = \\ (v_1, \dots, v_{d_1}, w_1, \dots, w_{d_2}) \cdot \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

where $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ denotes the $(d_1 + d_2) \times n$ matrix given by putting M_1 on top of M_2. Hence, we have (18).

Thus, it suffices to prove that for ever $d = 0, \dots, n-1$

$$(20) \qquad F_d \star_Q F_1 \supseteq F_{d+1}.$$

Now, for a $d \times n$ matrix in reduced row echelon form, $\nu_{\{i<j\}} \in \mathbb{F}_q$ for $i < j \in \{1, \dots, d\}$, and $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ (i.e. a $1 \times n$ matrix in reduced row echelon form), we may compute that

$$f_{M,\nu_{\{i<j\}}} \star_Q f_\lambda = \sum_{\nu=(\nu_{\{1,d+1\}},\dots,\nu_{\{d,d+1\}})\in\mathbb{F}_q^d} \psi_1\left(-det(A_{M,\lambda})\nu \cdot M \cdot Q \cdot \lambda^T\right) \cdot f_{RREF\binom{M}{\lambda},\nu_{\{i<j\}}} \tag{21}$$

where $RREF\binom{M}{\lambda}$ denotes the reduced row echelon form of $\begin{pmatrix} M \\ \lambda \end{pmatrix}$, and $A_{M,\lambda}$ denotes the matrix in $GL_{d+1}(\mathbb{F}_q)$ such that

$$RREF\begin{pmatrix} M \\ \lambda \end{pmatrix} = A_{M,\lambda} \cdot \begin{pmatrix} M \\ \lambda \end{pmatrix}.$$

Given (21), (20) follows, since all $f_{N,\nu_{\{i<j\}}}$ for N a $(d+1) \times n$ matrix of reduced row echelon form can be produced as a linear combination of (21) for choices of M, λ such that $N = RREF\binom{M}{\lambda}$ by independence of characters.

To prove (21), writing

$$M = \begin{pmatrix} m_{1,1} & \dots & m_{1,n} \\ \vdots & & \vdots \\ m_{d,1} & \dots & m_{d,n} \end{pmatrix}, \ \lambda = (\lambda_1, \dots, \lambda_n),$$

the left hand side expands as a sum over $v_1, \dots, v_d, w \in V$ such that $S(v_j, v_i) = \nu_{\{i<j\}}$ for $i < j \in \{1, \dots, d\}$ of terms of the form

$$(v_1, \dots, v_d, w) \cdot \begin{pmatrix} M \\ \lambda \end{pmatrix}$$

multiplied by the coefficient

$$\psi_1\left(\sum_{i=1}^d a_i S(\lambda_i \cdot w,\ m_{1,i} \cdot v_1 + \dots + m_{d,i} \cdot v_d)\right).$$

We may rewrite

$$\sum_{i=1}^d a_i S(\lambda_i \cdot w,\ m_{1,i} \cdot v_1 + \dots + m_{d,i} \cdot v_d) =$$
$$(S(w, v_1), \dots, S(w, v_d)) \cdot \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

Now taking the matrix $A_{M,\lambda} \in GL_{d+1}(\mathbb{F}_q)$ such that

$$A_{M,\lambda} \cdot \begin{pmatrix} M \\ \lambda \end{pmatrix} = RREF \begin{pmatrix} M \\ \lambda \end{pmatrix},$$

put

$$(u_1, \ldots, u_{d+1}) = (v_1, \ldots, v_d, w) \cdot A_{M,\lambda}^{-1}.$$

Note that $v_i = u_i$ for $i = 1, \ldots, d$ since M is already in reduced row echelon form, and therefore $A_{M,\lambda}$ is of the form

$$A_{M,\lambda} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ * & * & \ldots & * & * \end{pmatrix}$$

Thus, for $i = 1, \ldots, d$,

$$S(u_{d+1}, u_i) = det(A_{M,\lambda})^{-1} \cdot S(w, v_i).$$

Hence, $f_{M,\nu_{\{i,j\}}} \star_Q f_\lambda$ is the sum over $u_1, \ldots, u_{d+1} \in V$ such that $S(u_j, u_i) = \nu_{\{i<j\}}$ for $i < j \in \{1, \ldots, d\}$ of terms

$$(u_1, \ldots, u_{d+1}) \cdot RREF \begin{pmatrix} M \\ \lambda \end{pmatrix}$$

with coefficient

$$\psi_1 \left(det(A_{M,\lambda}) \cdot (S(u_{d+1}, u_1), \ldots, S(u_{d+1}, u_d)) \cdot Q \cdot \lambda^T \right),$$

giving (21).

□

3.2. **The "regular" and "singular" generators.** Now, for $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$,

$$f_\lambda \star_Q f_\lambda =$$
$$\sum_{v,w \in V} \psi_a(\frac{Q(\lambda)}{a} S(w, v))(\lambda_1(v + w), \lambda_2(v + w), \ldots, \lambda_n(v + w)).$$

An immediate consequence of this formula is the following

Lemma 6. *For $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ such that applying the quadratic form Q to λ is non-zero gives a reflection in $End_{Sp(V)}(\omega_Q)$, i.e.*

$$\left(\frac{f_\lambda}{q^N}\right) \star_Q \left(\frac{f_\lambda}{q^N}\right) = (0). \tag{22}$$

For $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ such that applying the quadratic form Q to λ is 0 gives an idempotent

$$\left(\frac{f_\lambda}{q^{2N}}\right) \star_Q \left(\frac{f_\lambda}{q^{2N}}\right) = \left(\frac{f_\lambda}{q^{2N}}\right).$$

We find that the $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ which have $Q(\lambda) \neq 0$ generate $O(\mathbb{F}_q^n, Q)$. The remaining $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ which have $Q(\lambda) = 0$ are precisely the elements of $O(Q)/P_{Q,1}]$. Note also that their images have dimension

$$\frac{tr(f_\lambda)}{q^{2N}} = \frac{\sum_{v\in V} tr((\lambda_1 v, \dots, \lambda_n v))}{q^{2N}} = \frac{tr((0,\dots,0))}{q^{2N}} = q^{(n-2)N}.$$

Without loss of generality, Q is equivalent to a diagonal matrix with entries $a_1, \dots, a_n$ so that

$$\omega_Q \cong \omega_{a_1} \otimes \omega_{a_2} \otimes \cdots \otimes \omega_{a_n}$$

and $a_1 = -a_2$. Then, for example,

$$\lambda_0 = [1 : 1 : 0 : \cdots : 0]$$

satisfies

$$Q(\lambda_0) = a_1 \cdot 1^2 + a_2 \cdot 1^2 + a_3 \cdot 0^2 + \cdots + a_n \cdot 0^2 = 0$$

and thus defines an idempotent f_{λ_0}/q^{2N} in $End_{Sp(V)}(\omega_Q)$. Now

$$f_{\lambda_0} = \sum_{v\in V} (v, v, 0, \dots, 0) = (\sum_{v\in V} (v, v)) \otimes ((0, \dots, 0)),$$

considering

$$\sum_{v\in V} (v, v) \in End_{Sp(V)}(\omega_{a_1} \otimes \omega_{-a_1})$$
$$(0, \dots, 0) = Id_{\omega_{a_3}\otimes\cdots\otimes\omega_{a_n}} \in End_{Sp(V)}(\omega_{a_3} \otimes \cdots \otimes \omega_{a_n}).$$

The first factor $\sum_{v\in V}(v, v)$ can be understood from the analysis done in [27] of the structure of tensor products of two oscillator representations. This element corresponds to a "bad" element of Case I in the terminology of [27], and therefore its image is a copy of the trivial representation 1. Therefore,

$$Im(f_{\lambda_0}/q^{2N}) = Im(f_{\lambda_0}) \cong \omega_{a_3} \otimes \cdots \otimes \omega_{a_n}.$$

(Note also that, in the notation of the Introduction,

$$\omega_{Q[n-2]} = \omega_{a_3} \otimes \cdots \otimes \omega_{a_n}.)$$

In fact, all images of idempotents $Im(f_\lambda/q^{2N})$ for $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ with $Q(\lambda) = 0$ are isomorphic, since for two such λ, μ with $Q(\lambda) = Q(\mu) = 0$, there exists an element $\phi \in O(\mathbb{F}_q^n, Q)$ such that $\phi(\lambda) = \lambda'$ by Witt's Theorem.

Hence, all $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ with $Q(\lambda) = 0$ give idempotents f_λ/q^{2N} whose images are all isomorphic to

$$Im(f_\lambda/q^{2N}) \cong \omega_{Q[n-2]}. \tag{23}$$

4. Combinatorics

In this section, we shall make the combinatorial computations showing that both sides of formula (17) in Theorem 4 have equal dimension.

Let us write $Q_n = q^n + 1$. Recall the Gaussian binomial coefficients

$$\binom{a}{b}_q := \frac{(q^a - 1) \cdot (q^{a-1} - 1) \dots (q^{a-b+1} - 1)}{(q^b - 1) \cdot (q^{b-1} - 1) \dots (q - 1)}. \tag{24}$$

The purpose of this section is to prove the following

Lemma 7. *For every $p \in \mathbb{N}_0$, for every $r > b \in \mathbb{N}_0$,*

$$Q_{r+p} \cdot Q_{r-1+p} \dots Q_{b+p} =$$

$$\sum_{a=0}^{p} q^{a(b+a-1)} \cdot \binom{r-b+1}{a}_q \cdot \prod_{i=p-a+1}^{p} (q^i - 1) \cdot \prod_{j=b+a}^{r} Q_j \tag{25}$$

4.1. An important corollary.

Lemma 7 has the following

Corollary 1. *For a quadratic form Q on $\mathbb{F}_q^n$,*

$$dim(Hom_{Sp(V)}(\omega_{Q[n-2k]}, \omega_Q)) =$$

$$\sum_{\ell=0}^{h_Q - k} |O(Q[n-2k])/P_{Q[n-2k],\ell}| \cdot |O(Q)/P_{Q,k+\ell}| \cdot |O(Q[n-2k-2\ell])|. \tag{26}$$

In particular, for $k = 0$, this gives

$$dim(End_{Sp(V)}(\omega_Q)) = \sum_{k=0}^{h_Q} |O(Q)/P_{Q,k}|^2 \cdot |O(Q[n-2k])|. \tag{27}$$

Assuming Lemma 7, let us begin with the

Proof of Corollary 1. First, let us process the left hand side of (26). We have $\omega_Q = \mathbb{C}(V^{\oplus k}) \otimes \omega_{Q[n-2k]}$, and therefore

$$Hom_{Sp(V)}(\omega_{Q[n-2k]}, \omega_Q) \cong Hom_{Sp(V)}(1, \mathbb{C}(V^{\oplus(n-k)}))$$

Thus, we may rewrite the left hand side of (26) as

$$dim(Hom_{Sp(V)}(\omega_{Q[n-2k]}, \omega_Q)) = 2Q_{n-k-1} \dots Q_1 \tag{28}$$

by Lemma 3.

Now note that

$$|O(Q)/P_{Q,k}| = \frac{|O(Q)/P_{Q,1}| \cdot |O(Q[n-2])/P_{Q[n-2],k-1}|}{|\mathbb{P}^{k-1}(\mathbb{F}_q)|},$$

and therefore,

$$|O(Q)/P_{Q,k}| = \frac{\prod_{\ell=1}^{k-1} |O(Q[n-2\ell])/P_{Q[n-2\ell],1}|}{\prod_{\ell=1}^{k-1} |\mathbb{P}^{\ell}(\mathbb{F}_q)|}. \tag{29}$$

It is also well known that for any quadratic form Q on $\mathbb{F}_q^n$, the number of elements in $O(Q)/P_{Q,1}$, which is the set of points the quadric defined by Q in $\mathbb{P}^{n-1}(\mathbb{F}_q)$,

$$|O(Q)/P_{Q,1}| = \begin{cases} \dfrac{q^{n-1}-1}{q-1} & n \text{ odd} \\ \\ \dfrac{q^{n-1}-1}{q-1} \pm q^{(n-2)/2} & n \text{ even.} \end{cases} \tag{30}$$

Case 1: n is odd. Let us write $n = 2m+1$. Note that m is the number of hyperbolics h_Q in Q. In this case, for $\ell = 0, \dots m-k$, combining (29) and (30),

$$|O(Q)/P_{Q,k+\ell}| = \frac{(q^{2m}-1) \cdot (q^{2m-2}-1) \dots (q^{2m-2k-2\ell+2}-1)}{(q^{k+\ell}-1) \cdot (q^{k+\ell-1}-1) \dots (q-1)}.$$

We may rewrite this as

$$|O(Q)/P_{Q,k+\ell}| = \binom{m}{k+\ell}_q \cdot \prod_{i=m-k-\ell+1}^{m} (q^i+1).$$

Similarly,

$$|O(Q[n-2k])/P_{Q[n-2k],\ell}| = \frac{(q^{2m-2k}-1) \dots (q^{2m-2k-2\ell+2}-1)}{(q^{\ell}-1) \dots (q-1)}.$$

We also have

$$|O(Q[n-2k-2\ell])| = 2q^{(m-k-\ell)^2} \prod_{j=1}^{m-k-\ell} (q^{2k}-1) =$$

$$2q^{(m-k-\ell)^2} \prod_{j=1}^{m-k-\ell} (q^j-1)(q^j+1).$$

Combining this with (28), the statement reduces to

$$2Q_{2m-k}\dots Q_1 =$$

(31)
$$\sum_{\ell=0}^{m-k} 2q^{(m-k-\ell)^2} \cdot \binom{m}{k+\ell}_q \cdot \prod_{j=m-k-\ell+1}^{m} Q_j \cdot \frac{\prod_{i=1}^{m-k} (q^i-1)\cdot Q_i}{(q^\ell-1)\dots(q-1)}$$

We may divide both sides of (31) by $2Q_1 \dots Q_{m-k}$, giving

$$Q_{2m-k}\dots Q_{m-k+1} =$$

(32)
$$\sum_{\ell=0}^{m-k} q^{(m-k-\ell)^2} \binom{m}{k+\ell}_q \cdot \prod_{i=m-k-\ell+1}^{m} Q_j \cdot \prod_{j=\ell+1}^{m-k} (q^j-1).$$

Replacing

$$\binom{m}{k+\ell}_q = \binom{m}{m-k-\ell}_q,$$

this follows exactly from Lemma 7 by putting

$$r=m,\ p=m-k,\ b=1$$

and substituting $a=m-k-\ell$.

Case 2: n is even. Let us write $n=2m$.

Case 2A: Suppose Q decomposes completely into hyperbolics in (5), i.e. $h_Q = m$. In this case, combining (29) and (30), noting first that we may first re-write this case of (30) as

$$|O(Q[n-2\ell])/P_{Q[n-2\ell],1}| = \frac{q^{2m-2\ell-1} - q^{m-\ell+1} + q^{m-\ell} - 1}{q-1} =$$
$$\frac{(q^{m-\ell}-1)(q^{m-\ell+1}+1)}{q-1},$$

we have

$$|O(Q)/P_{Q,k+\ell}| = \frac{\prod_{i=m-k-\ell}^{m-1}(q^{i+1}-1)\cdot Q_i}{(q^{k+\ell}-1)\cdot(q^{k+\ell-1}-1)\dots(q-1)} =$$

$$\binom{m}{k+\ell}_q \cdot \prod_{i=m-k-\ell}^{m-1} Q_i.$$

Similarly, we have

$$|O(Q[n-2k])/P_{Q[n-2k],\ell}| = \frac{\prod_{i=m-k-\ell}^{m-k-1}(q^{i+1}-1)Q_i}{(q^{\ell}-1)\dots(q-1)}.$$

We also have

$$|O(Q[n-2k-2\ell])| =$$

$$2q^{(m-k-\ell)(m-k-\ell-1)}\cdot(q^{m-k-\ell}-1)\prod_{j=1}^{m-k-\ell-1}(q^{2j}-1) =$$

$$2q^{(m-k-\ell)(m-k-\ell-1)}\cdot(q^{m-k-\ell}-1)\prod_{j=1}^{m-k-\ell-1}(q^{j}-1)(q^{j}+1).$$

Again, combining with (28), we can reduce the right hand side of (27), giving the statement

$$2Q_{2m-k-1}\dots Q_1 =$$

(33)
$$\sum_{\ell=0}^{m-k} q^{(m-k-\ell)(m-k-\ell-1)}\binom{m}{k+\ell}_q \prod_{i=m-k-\ell}^{m-1} Q_i \frac{(q^{m-k}-1)\prod_{j=1}^{m-k-1}(q^j-1)Q_j}{(q^{\ell}-1)\dots(q-1)}$$

Again, we may divide both sides of (35) by $2Q_{m-k-1}\dots Q_1$, giving

$$Q_{2m-k-1}\dots Q_{m-k} =$$

(34)
$$\sum_{\ell=0}^{m-k} q^{(m-k-\ell)(m-k-\ell-1)}\cdot\binom{m}{k+\ell}_q\cdot\prod_{i=m-k-\ell}^{m-1} Q_i\cdot\prod_{j=\ell+1}^{m-k}(q^j-1).$$

Rewriting $\binom{m}{k+\ell}_q = \binom{m}{m-k-\ell}_q$, this statement follows directly from applying Lemma 7 to

$$r = m-1,\ p = m-k,\ b = 0$$

and substituting $a = m - k - \ell$.

Case 2B: Suppose Q has a non-trivial anisotropic part in (5), i.e. $h_Q = m - 1$. In this case, combining (29) and (30), noting first that we may first re-write this case of (30) as

$$|O(Q[n-2\ell])/P_{Q[n-2\ell],1}| = \frac{q^{2m-2\ell-1} + q^{m-\ell+1} - q^{m-\ell} - 1}{q-1} = \frac{(q^{m-\ell}+1)(q^{m-\ell+1}-1)}{q-1},$$

we have

$$|O(Q)/P_{Q,k+\ell}| = \frac{\prod_{i=m-k-\ell+1}^{m} (q^{i-1}-1)\cdot Q_i}{(q^{k+\ell}-1)\cdot(q^{k+\ell-1}-1)\dots(q-1)} =$$

$$\binom{m-1}{k+\ell}_q \cdot \prod_{i=m-k-\ell+1}^{m} Q_i.$$

Similarly, we have

$$|O(Q[n-2k])/P_{Q[n-2k],\ell}| = \frac{\prod_{i=m-k-\ell+1}^{m-k} (q^{i-1}-1)Q_i}{(q^{\ell}-1)\dots(q-1)}.$$

We also have

$$|O(Q[n-2k-2\ell])| =$$

$$2q^{(m-k-\ell)(m-k-\ell-1)}\cdot(q^{m-k-\ell}+1)\prod_{j=1}^{m-k-\ell-1}(q^{2j}-1) =$$

$$2q^{(m-k-\ell)(m-k-\ell-1)}\cdot(q^{m-k-\ell}+1)\prod_{j=1}^{m-k-\ell-1}(q^{j}-1)(q^{j}+1).$$

Again, combining with (28), we can reduce the right hand side of (27), giving the statement

$$2Q_{2m-k-1}\dots Q_1 =$$

(35) $$\sum_{\ell=0}^{m-k-1} q^{(m-k-\ell)(m-k-\ell-1)}\cdot\binom{m-1}{k+\ell}_q\prod_{i=m-k-\ell+1}^{m} Q_i\frac{Q_{m-k}\prod_{j=1}^{m-k-1}(q^j-1)\cdot Q_j}{(q^{\ell}-1)\dots(q-1)}$$

Similarly as in the previous cases, we may divide both sides of (35) by $2Q_{m-k}\dots Q_1$, giving

$$Q_{2m-k-1}\dots Q_{m-k+1} = \sum_{\ell=0}^{m-k-1} q^{(m-k-\ell)(m-k-\ell-1)} \cdot \binom{m-1}{k+\ell}_q \cdot \prod_{i=m-k-\ell+1}^{m} Q_i \cdot \prod_{j=\ell+1}^{m-k-1} (q^j-1). \tag{36}$$

Rewriting $\binom{m-1}{k+\ell}_q = \binom{m-1}{m-k-\ell-1}_q$, this statement follows directly from applying Lemma 7 to

$$r=m,\ p=m-k-1,\ b=2$$

and substituting $a = m-k-\ell-1$.

□

4.2. **The proof of Lemma 7.** The proof of Lemma 7 proceeds by induction. The argument is perhaps slightly unusual due to the fact that our formula does not reduce well at $q\to 1$ and is therefore not a "quantization" of a classical formula.

Proof of Lemma 7. To prove (25), we begin by rewriting

$$Q_{r+p}\dots Q_{b+p} = Q_r\dots Q_b + \sum_{k=b}^{r}(Q_{p+k}-Q_k)\cdot \prod_{j=k+1}^{r} Q_j \cdot \prod_{j'=b}^{k-1} Q_{j'+p}. \tag{37}$$

Now, for each $b\le k\le r$, we have

$$Q_{p+k}-Q_k = q^k\cdot(q^p-1),$$

so we may rewrite (37) as

$$Q_{r+p}\dots Q_{b+p} = Q_r\dots Q_b + \sum_{k=b}^{r} q^k(q^p-1)\prod_{j=k+1}^{r} Q_j \cdot \prod_{j'=b}^{k-1} Q_{j'+p}. \tag{38}$$

Our goal is now to process each term of the right hand side of (38) to convert, step by step, the highest appearing $Q_{j'+p}$ factor into the next smallest Q_j not yet appearing. In the statement of Lemma 7, all terms consist of multiples of products of the form

$$Q_r\cdot Q_{r-1}\dots Q_{b+a+1}\cdot Q_{b+a},$$

so we cannot skip any Q_j's in the process. We make the following

Claim 1. *For $b \leq k \leq r$, we have*

$$q^k(q^p-1)\prod_{j=k+1}^{r} Q_j \cdot \prod_{j'=b}^{k-1} Q_{j'+p} = \tag{39}$$
$$\sum_{a=1}^{k-b+1}\left(\sum_{a+b-1\leq \ell_1\leq\cdots\leq\ell_a=k} q^{\ell_1+\cdots+\ell_a}\right)\cdot\prod_{i=p-a+1}^{p}(q^i-1)\cdot\prod_{j=b+a}^{r} Q_j$$

Proof of Claim 1. We will proceed inductively, step by step, converting each factor $Q_{j'+p}$ in a term of the previous step's reduction of (39), starting with the largest appearing j', into a sum of the next lower Q_j not yet appearing, with the appropriate error term of q^j multiplied by a factor $(q^{j'+p-j}-1)$. This process will terminate in $k-b$ steps (we are already done when $k=b$).

The induction hypothesis is that after n steps, we will have reduced (39) to

$$\sum_{a=1}^{n+1}\left(\sum_{a-n+k-1\leq \ell_1\leq\cdots\leq\ell_a=k} q^{\ell_1+\cdots+\ell_a}\right)\cdot\prod_{i=p-a+1}^{p}(q^i-1)\cdot\prod_{j=k-n+a}^{r} Q_j\prod_{j'=b}^{k-n-1} Q_{j'+p} \tag{40}$$

Let us describe the first step of this process for (39). The largest appearing j' is $j'=k-1$. The next lower Q_j factor not yet appearing is for $j=k$. Therefore, this step uses the replacement

$$Q_{k-1+p} = Q_k + q^k(q^{p-1}-1).$$

This gives

$$q^k(q^p-1)\prod_{j=k}^{r} Q_j \cdot \prod_{j'=b}^{k-2} Q_{j'+p} +$$
$$q^{k+k}(q^p-1)(q^{p-1}-1)\prod_{j=k+1}^{r} Q_j \cdot \prod_{j'=b}^{k-2} Q_{j'+p}$$

proving (40) at $n=1$.

Suppose (40) holds at Step n. We now need to perform Step $(n+1)$. For $1 \leq a \leq n+1$, consider the term

$$\left(\sum_{a-n+k-1 \leq \ell_1 \leq \cdots \leq \ell_a = k} q^{\ell_1 + \cdots + \ell_a} \right) \cdot \prod_{i=p-a+1}^{p} (q^i - 1) \cdot \prod_{j=k-n+a}^{r} Q_j \cdot \prod_{j'=b}^{k-n-1} Q_{j'+p}, \tag{41}$$

of (40).

The highest occuring $Q_{j'+p}$ is at $j' = k-n-1$. The next lower Q_j factor not yet appearing is for $j = k-(n+1)+a$. Therefore, in this term, we must use the replacement

$$Q_{k-n-1+p} = Q_{k-(n+1)+a} + q^{k-(n+1)+a} \cdot (q^{p-a} - 1).$$

This reduces (41) to

$$\begin{aligned} &\left(\sum_{a-n+k-1 \leq \ell_1 \leq \cdots \leq \ell_a = k} q^{\ell_1 + \cdots + \ell_a} \right) \cdot \prod_{i=p-a+1}^{p} (q^i - 1) \cdot \prod_{j=k-(n+1)+a}^{r} Q_j \cdot \prod_{j'=b}^{k-n-2} Q_{j'+p} + \\ &+ \left(\sum_{a-n+k-1 = \ell_0 \leq \ell_1 \leq \cdots \leq \ell_a = k} q^{\ell_0 + \ell_1 + \cdots + \ell_a} \right) \cdot \prod_{i=p-a}^{p} (q^i - 1) \cdot \prod_{j=k-n+a}^{r} Q_j \cdot \prod_{j'=b}^{k-n-2} Q_{j'+p} \end{aligned} \tag{42}$$

These terms appear in the $(n+1)$th inductive step; the first one occurs in the expression (40) with n replaced by $n+1$ with no reindexing of a or ℓ_i, and the second one occurs after replacing a by $a+1$ and shifting $\ell_0 \leq \cdots \leq \ell_a$ to $\ell_1 \leq \cdots \leq \ell_{a+1}$.

Therefore, we may proceed inductively, and at Step $n = k-b$, we obtain the reduction (39).

□

Recombining the terms (39) according to (38), we get

$$\begin{aligned} &Q_{r+p} \dots Q_{b+p} = \\ &\sum_{a=0}^{p} \left(\sum_{a+b-1 \leq \ell_1 \leq \cdots \leq \ell_a \leq r} q^{\ell_1 + \cdots + \ell_a} \right) \cdot \prod_{i=p-a+1}^{p} (q^k - 1) \cdot \prod_{j=b+a}^{r} Q_j \end{aligned} \tag{43}$$

(note that the $a=0$ term arises from the single term $Q_r \dots Q_b$ in (38)).

Finally, we compute

$$\sum_{a+b-1\le \ell_1\le\cdots\le\ell_a\le r} q^{\ell_1+\cdots+\ell_a} =$$

$$q^{a(a+b-1)}\cdot \sum_{0\le\ell_1\le\cdots\le\ell_a\le r-a-b+1} q^{\ell_1+\cdots+\ell_q} =$$

$$q^{a(a+b-1)}\cdot \binom{r-b+1}{a}_q,$$

by the Gaussian binomial coefficient theorem. Plugging this into (43) gives (25).

□

5. The Proof of Theorems 1 and 4 and an Explicit Endomorphism Computation

The main purpose of this section is to present an inductive argument which, combined with Proposition 5, Lemma 6, and Corollary 1, prove Theorems 1 and 4. We do this in Subsection 5.1. In Subsection 5.2, we use the Schrödinger model for the oscillator representation to explicitly describe how an endomorphism in the top subalgebra $\mathbb{C}O(Q)$ (i.e. the $k=0$ term of (17)) of $End_{Sp(V)\times O(Q)}(f^*(\boldsymbol{\omega}))$ acts on an element of $\boldsymbol{\omega}$. Strikingly, we find that it agrees exactly with the representation action of $O(Q)$ as a subgroup of $Sp(V)\times O(Q)\subseteq Sp(\mathbb{F}_q^{2nN})$ on $\boldsymbol{\omega}$.

5.1. **The Proof of Theorems 1 and 4.** Consider a quadratic form $Q:\mathbb{F}_q^n\to\mathbb{F}_q$. We will proceed by induction on h_Q. In other words, suppose that Theorem 1 holds for $Q[n-2k]$ for every $k=1,\dots,h_Q$. Concretely, writing

$$f^*: O(Q[n-2k])\times Sp(V)\to Sp(\mathbb{F}_q^{2(n-2k)N}),$$

suppose there exist distinct irreducible $Sp(V)$-representations $W_i^{Q[n-2k]}$ for $i=1,\dots,m_{Q[n-2k]}$ indexing the distinct irreducible representations $V_i^{Q[n-2k]}$ of $O(Q[n-2k])$ such that

$$Res_{Sp(V)}(f[n-2k])^*(\boldsymbol{\omega})\cong$$

$$\bigoplus_{j=0}^{h_Q-k}\bigoplus_{i=1}^{m_{Q[n-2k-2j]}} |O(Q[n-2k])/P_{Q[n-2k],j}|\cdot (V_i^{Q[n-2k-2j]}\otimes W_i^{Q[n-2k-2j]}), \quad (44)$$

Let us consider, for $k = 1, \dots, h_Q$, the sum of the representations attained first at "level $n-2k$" of the orthogonal group:

$$Z_{n-2k} := \bigoplus_{j=1}^{m_{Q[n-2k]}} V_j^{Q[n-2k-2i]} \otimes W_j^{Q[n-2k-2i]}.$$

We claim that (as in the introduction, taking $f = f[n]$), there exist distinct irreducible $Sp(V)$-representations W_i^Q for $i = 1, \dots, m_Q$ indexing the distinct irreducible representations V_i^Q of $O(Q)$, which are also distinct from all $W_i^{Q[n-2k]}$ for $k = 1, \dots, m_{Q[n-2k]}$

$$\begin{gathered} Res_{Sp(V)}(f^*(\boldsymbol{\omega})) \cong \\ \bigoplus_{i=1}^{m_Q}(V_i^Q \otimes W_i^Q) \oplus \bigoplus_{j=1}^{h_Q} |O(Q)/P_{Q,j}| \cdot Z_{n-2j} \end{gathered} \tag{45}$$

Adjunction of restriction with induction gives the final claim (9).

First, note that, by the induction hypothesis (44), restricting to $Sp(V)$, we in particular have that

$$\begin{gathered} Hom_{Sp(V)}(\omega_{Q[n-2k]}, \omega_Q) = \\ \bigoplus_{\ell}^{h_Q-k} |O(Q[n-2k])/P_{Q[n-2k],\ell}| \cdot Hom_{Sp(V)}(Z_{n-2k}, \omega_Q). \end{gathered}$$

We claim that this gives

$$dim(Hom_{Sp(V)}(Z_{n-2k}, \omega_Q)) = |O(Q)/P_{Q,k}|. \tag{46}$$

To prove this claim, we proceed recursively. For $k = h_Q$, this claim reduced to the calculation that

$$\begin{gathered} |O(Q)/P_{Q,h_Q}| = 2(q^{h_Q-1}+1)(q^{h_Q-2}+1)\dots(q+1) = \\ dim(Hom_{Sp(V)}(1, \mathbb{C}(V^{\oplus h_Q}))) \end{gathered}.$$

Then given (46) for k replaced by any $\ell \leq k$, (46) for k itself follows by Corollary 1. This proves that the multiplicities of Z_{n-2j} for all $j = 1, \dots h_Q$ in (45) are correct.

It remains to prove the form of the top level, i.e. that there exist W_i^Q such that

$$\bigoplus_{i=1}^{m_Q} V_i^Q \otimes W_i^Q \subseteq f^*(\boldsymbol{\omega}). \tag{47}$$

Taking the endomorphism algebras of both sides of (45) (or, writing out the right hand side of (4)), we need to show

$$\begin{gathered} End_{Sp(V)}(\omega_Q) = \mathbb{C}O(Q) \oplus \\ M_{|O(Q)/P_{Q,1}|}(\mathbb{C}O(Q[n-2])) \oplus \cdots \oplus M_{|O(Q)/P_{Q,h_Q}|}(\mathbb{C}O(Q[n-2h_Q])). \end{gathered} \tag{48}$$

(The endomorphism algebra of (47), restricted to an $Sp(V)$-representation is precisely $\mathbb{C}O(Q)$, and, similarly, for the lower Z_{n-2j} $j = 1, \ldots, h_Q$,

$$End_{Sp(V)}(Z_{n-2j}) = \mathbb{C}O(Q[n-2j]).)$$

Now, recalling (23), and noting further that we may compute that if linearly independent $\lambda_1, \ldots, \lambda_k \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ satsify the condition that for every $v \in span(\lambda_1, \ldots, \lambda_k)$, $Q(v) = 0$, then $f_{\lambda_1}, \ldots, f_{\lambda_k}$ commute with respect to $\star_Q$, and hence

$$\frac{1}{q^{2Nk}} f_{\lambda_1} \star_Q \cdots \star_Q f_{\lambda_k} \tag{49}$$

is an idempotent. A similar argument by Witt's Theorem as in the end of Section 3 gives that

$$Im(\frac{1}{q^{2Nk}} f_{\lambda_1} \star_Q \cdots \star_Q f_{\lambda_k}) \cong \omega_{Q[2n-k]}.$$

There are precisely $|O(Q)/P_{Q,k}|$ idempotents (49). Therefore, all the Z_{n-2j} summands observed above are accounted for in the images of the endomorphism subalgebra generated by f_λ where $Q(\lambda) = 0$.

Now the subalgebra of $End_{Sp(V)}(\omega_Q)$ generated with respect to $\star_Q$ by f_λ for $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ such that $Q(\lambda) \neq 0$ is isomorphic to the group algebra $\mathbb{C}O(Q)$, by (22) and (21). In Subsection 5.2 below, the explicit action of the group structure generated by the action of these f_λ is described, giving precisely the group action of $O(Q)$.

Therefore we have proved that $End_{O(Q)}(\omega_Q)$ is at least the right hand side of (48) (the case in which it would be larger would occur if some W_i^Q were not irreducible, or were isomorphic to some lower $W_j^{Q[n-2k]}$, which would lead to a larger matrix algebra). However, by (27), the dimensions of both sides of (48) match, and therefore both sides are equal. This proves (48) and (9), giving Theorem 4 and Theorem 1.

5.2. **The action of the top subalgebra.**

Using (16), we may explicitly describe the action of elements of the top subalgebra $\mathbb{C}O(Q) \subseteq End_{Sp(V)}(\omega_Q)$ on the Schrödinger model of ω_Q. Remarkably, we find that the group algebra $\mathbb{C}O(Q)$ acts according to the representation action of

$$O(Q) \subseteq O(Q) \times Sp(V) \subseteq Sp(\mathbb{F}_q^{2nN})$$

on $\boldsymbol{\omega} = \omega_Q$. Since they are algebra generators, it is enough to check that the reflections f_λ/q^N, for λ a point in the quadric defined by Q, act geometrically as the reflection across the hyperplane perpendicular to the line generated by λ.

Fix a choice of $\lambda \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ (interpreted as a $1\times n$ matrix $(\lambda_1, \dots, \lambda_n)$ in reduced row echelon form for scalars $\lambda_i \in \mathbb{F}_q$) such that $Q(\lambda) \neq 0$. For an n-tuple of vectors $(w_1, \dots, w_n) \in \Lambda_-^{\oplus n}$ generating

$$\omega_a^{\otimes n} = (\mathbb{C}\Lambda_-)^{\otimes n} = \mathbb{C}(\Lambda_-^{\oplus n}),$$

we may ask what $f_\lambda(w_1, \dots, w_n)$ explicitly is.

Writing out f_λ, we find that

$$\begin{gathered} f_\lambda(w_1, \dots, w_n) = \\ \sum_{v_\pm \in \Lambda_\pm} (\lambda_1 v_+ + \lambda_1 v_-, \dots, \lambda_n v_+ + \lambda_n v_-)(w_1, \dots, w_n) \end{gathered} \tag{50}$$

Recalling (16), for each $i = 1, \dots, n$, applying an element $\lambda_i v_+ + \lambda_i v_-$ of V acts on $w_i \in \Lambda_-$, according to the structure of ω_{a_i}, by

$$(\lambda_i v_+ + \lambda_i v_-)(w_i) = \psi_{a_i}(\lambda_i \cdot S(v_+, w_i) + \lambda_i^2 \frac{S(v_+, v_-)}{2})(\lambda_i v_- + w_i).$$

Therefore, in the tensor product $\omega_{a_1} \otimes \dots \otimes \omega_{a_n}$, a term of (50) for a choice of $v_+ \in \Lambda_+$, $v_- \in \Lambda_-$ reduces to the n-tuple of vectors

$$(\lambda_1 v_- + w_1, \dots, \lambda_n v_- + w_n) = w + \lambda \otimes v_-, \tag{51}$$

(where we write $\lambda \otimes v_- = (\lambda_1 v_-, \dots, \lambda_n v_-)$, considering λ as a $1 \times n$ matrix and v_- as a 1×1 matrix with entries in Λ_-), multiplied by the coefficient

$$\begin{gathered} \psi_1(\sum_{i=1}^n a_i(\lambda_i \cdot S(v_+, w_i) + \lambda_i^2 \frac{S(v_+, v_-)}{2})) = \\ \psi_1(S(v_+, \sum_{i=1}^n a_i(\lambda_i w_i + \frac{\lambda_i^2 v_-}{2}))). \end{gathered} \tag{52}$$

Now, since the term (51) does not depend on v_+, for a fixed v_-, its coefficient in (50) is the sum over all $v_+ \in \Lambda_+$ of terms (52). Again, since linear sums of characters are 0, this gives that the coefficient of (51) vanishes unless

$$\sum_{i=1}^n a_i(\lambda_i w_i + \frac{\lambda_i^2 v_-}{2}) = 0, \tag{53}$$

in which case for every choice of v_+ (52) is 1, and therefore the coefficient is q^N. Now we may rewrite (53) as

$$\frac{(a_1\lambda_1^2 + \cdots + a_n\lambda_n^2)v_-}{2} + \sum_{i=1}^{n} a_i\lambda_i w_i = \frac{Q(\lambda)}{2}v_- + \lambda \cdot Q \cdot w^T,$$

where we write $\lambda \cdot Q \cdot w^T = \sum_{i=1}^n a_i\lambda_i w_i$, considering λ as a $1\times n$ matrix, Q as an $n \times n$ matrix, and $w^T = (w_1, \dots, w_n)^T$ as an $n \times 1$ matrix, with entries in Λ_-. Therefore, the only surviving term (51) occurs with coefficient q^N for

$$v_- = \frac{2}{Q(\lambda)} \cdot \lambda \cdot Q \cdot w^T.$$

Writing this out, we find that

$$\frac{1}{q^N} f_\lambda(w_1, \dots, w_n) = ((w_1, \dots, w_n) - \frac{2}{Q(\lambda)} \cdot \lambda \otimes (\lambda \cdot Q \cdot w^T)) \tag{54}$$

where on the right hand side, we compute the matrix operations by considering $\lambda = (\lambda_1, \dots, \lambda_n)$ as a $1 \times n$ matrix, Q as an $n \times n$ matrix, and $w = (w_1, \dots, w_n)$ as a $1 \times n$ matrix.

Let us write $w = w_0 \otimes v$ for an n-tuple $v \in \mathbb{F}_q^n$. Then we find that (54) gives

$$f_\lambda(w_0 \otimes v) = w_0 \otimes (v - 2 \cdot \frac{vQ\lambda^T}{Q(\lambda)}\lambda),$$

the second factor of which is precisely the action of the matrix corresponding to the reflection across the hyperplane orthogonal to λ in $O(Q)$. Since these reflections generate $O(Q)$, this therefore implies that for any $\phi \in O(Q)$, it acts on an element $w_0 \otimes v \in \Lambda_-^{\oplus n}$ by

$$w_0 \otimes v \ \mapsto \ w_0 \otimes \phi(v).$$

6. Interpolation and the "Top Algebra"

6.1. **The interpolation of tensor categories.** One may note that the statement of Theorem 1 are stable over N, as long as it is large enough compared to the degree of the tensor product of the oscillator representations. This effect suggests that Theorem 1 should correspond to a result in *interpolated categories* [4, 5, 9, 15, 23, 24, 28]. Interpolated categories were first introduced by P. Deligne [4, 5], generalizing an example given in [6]. The idea of these categories is to model stable effects of certain categories of representations, for example those of a classical group on a vector space of increasing dimension N with a new additive $\mathbb{C}$-linear category with associative, commutative, unital tensor product and strong duality. The value of N is taken to be arbitrarily large to define the new category's spaces of morphisms, and is "interpolated" and replaced by a general constant in $\mathbb{C}$ to define the composition and trace operations. In general, interpolated categories encode deep representation-theoretical structure and exemplify interesting results of category theory, algebraic geometry, and combinatorics. In this section, we introduce a relevant interpolation of the representations of $Sp(V_N)$ generated by the oscillator representations, which gives a semisimple pre-Tannakian category (for generic values of $t \in \mathbb{C}$) [28], and give an interpretation of Theorem 1.

In [23, 24], F. Knop introduced a category $Rep(Sp_t(\mathbb{F}_q))$, defined by being tensor-generated by a single object X with morphism structure of its tensor powers defined by the morphism structure of the tensor powers of $V_N \in Obj(Rep(Sp_{2N}(\mathbb{F}_q)))$ for a large enough N compared to the tensor powers appearing, and putting $dim(X) = tr(Id_X) = q^t$. (In this notation, t interpolates the value of $2N$.) This category is semisimple for generic values of t. However, while X corresponds to V_N, there is no object of $Rep(Sp_t(\mathbb{F}_q))$ corresponding to an oscillator representation. The purpose of this section is to discuss an augmented variant of this category which is generated by the oscillator representations introduced by P. Deligne in [7, 27, 28], and give a statement corresponding to Theorem 1 in the interpolated context.

Now fix a prime power q, and write $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 = \{1, a\}$. In [28], we discuss Deligne's category, formally defined by taking basic objects to be modeled as the two non-isomorphic oscillator representations, giving a "graded version" of $Rep(Sp_t(\mathbb{F}_q))$. (Note that this description will differ slightly from the one given in [28] using the *T-algebra formalism* [25, 26, 27, 28], where it was more beneficial to consider the category to be defined to be generated by a single formal object corresponding to a direct sum of the two oscillator representations.) In this paper, we will denote this category by $\mathscr{R}_{q,t}$. Specifically, writing X_1 and X_a for two formal objects, we first consider a category $\widetilde{\mathscr{R}_{q,t}}$ with objects of the form

$$(X_1)^{\otimes m} \otimes (X_a)^{\otimes n}$$

and morphisms defined by

$$Hom_{\widetilde{\mathscr{R}_{q,t}}}((X_1)^{\otimes m_1} \otimes (X_a)^{\otimes n_1}, (X_1)^{\otimes m_2} \otimes (X_a)^{\otimes n_2}) =$$

$$Hom_{Sp(V_N)}(\omega_1^{\otimes m_1} \otimes \omega_a^{\otimes n_1}, \omega_1^{\otimes m_2} \otimes \omega_a^{\otimes n_2})$$

for a choice of N sufficiently large compared to m_1, n_1, m_2, n_2 (the structure is stable for N large enough). We define the trace morphisms so that the categorical dimension of X_1 or X_a is $q^{t/2}$. Note that, depending on q mod 4, ω_1 is either self-dual or dual to ω_a in every category of $Sp(V_N)$-representations for $N \in \mathbb{N}$. Therefore, $\widetilde{\mathscr{R}_{q,t}}$ forms a $\mathbb{C}$-linear category with a symmetric monoidal tensor product which is rigid and is generated by X_1 and X_a. Again, write $X^{\otimes Q}$ for the tensor product of copies of X_1's and X_a's, where Q is the symmetric bilinear form represented by the matrix with diagonal entries 1 or a corresponding to the factors. We call these objects the *basic tensor powers*.

6.2. **The Deligne model and the proof of Theorem 2.** It will be useful to consider a more concrete description of the spaces of morphisms between the basic tensor powers

$$Hom_{\mathscr{R}_{q,t}}(X^{\otimes Q}, X^{\otimes Q'}) \tag{55}$$

for quadratic forms Q, Q'. First note that (55) is 0 unless Q and Q' are the space after deleting hyperbolics, i.e.

$$Q = Q' \in W(\mathbb{F}_q)$$

(see [28]).

Now note that for fixed $m, \ell \in \mathbb{N}$, for N large enough,

$$Hom_{Sp(V_N)}((\mathbb{C}V_N)^{\otimes m}, (\mathbb{C}V_N)^{\otimes \ell}) \tag{56}$$

can also be considered to be the $\mathbb{C}$-vector space freely generated by choices of the data of an equivalence class of a surjection

$$(\varphi, \psi) : \mathbb{F}_q^m \oplus \mathbb{F}_q^\ell \twoheadrightarrow W \tag{57}$$

(over composition with an element of $GL(W)$) for some vector space W endowed with an antisymmetric (possible degenerate) bilinear form $\mathscr{S}$ such that its pullback $\varphi^*(\mathscr{S})$, resp. $\psi^*(\mathscr{S})$, to $\mathbb{F}_q^m$, resp. $\mathbb{F}_q^\ell$, is 0. The partial trace of such a choice of data, along an identification of

$$\sigma : \mathcal{S} \xrightarrow{\cong} \mathcal{T}$$

for choices of coordinates $\mathcal{S} \subseteq \{1, \dots, m\}$, $\mathcal{T} \subseteq \{1, \dots \ell\}$, identifying whole factors of $\mathbb{C}V_N$ in the source and target of an element of (56) is taken to be 0 unless for every $s \in \mathcal{S}$, the restrictions

$$\varphi|_{\mathbb{F}_q^{\{s\}}} = \psi|_{\mathbb{F}_q^{\{\sigma(s)\}}}$$

exactly coincide. In this case, the partial trace is the new choice of data achieved by deleting all factors corresponding to elements of $\mathcal{S}$, $\mathcal{T}$ in the source of (57) and in its image, restricting $\mathscr{S}$, multiplied by the number of such choices, which will be a power of q. We also require that the elements to be deleted be in the kernel of the antisymmetric bilinear form $\mathscr{S}$. Otherwise, the trace is also declared to be 0.

This precisely describes the space (55) and its partial traces in the case when Q and Q' completely split as direct sums of hyperbolics, where $dim(Q) = 2m$, $dim(Q') = 2\ell$. To define the category, one also needs to describe the partial trace when identifying possibly single factors of an oscillator representation in the source and target of (56). First, this reduces completely to the structure of

$$End_{Sp(V_N)}(\omega_\alpha),\ End_{Sp(V_N)}(\omega_\alpha \otimes \omega_\beta)$$

where $\alpha \neq -\beta \in (\mathbb{F}_q^\times)/(\mathbb{F}_q^\times)^2$ (i.e. ω_α, ω_β are not dual), and the partial trace of such morphisms. This can be deduced from the work done in Section 2 (see [28] for a full explicit description).

Now we define $\mathscr{R}_{q,t}$ by formally adding direct sums to $\widetilde{\mathscr{R}_{q,t}}$ add taking its pseudo-abelian envelope. Again, this forms a semisimple pre-Tannakian category for generic values of t (a proof is given in [28]). To understand the structure of this category, we may consider the "top algebra" of the endomorphism algebras of the basic tensor powers, i.e. the subalgebra of $End_{\mathscr{R}_{q,t}}(X^{\otimes Q})$ of morphisms f such that, if Q is symmetric bilinear form of dimension n, there does not exist a symmetric bilinear form Q' of dimension $n' < n$ such that f factors as a composition of morphisms

$$X^{\otimes Q} \to X^{\otimes Q'} \to X^{\otimes Q}. \tag{58}$$

We denote the top algebra by

$$End^{top}_{\mathscr{R}_{q,t}}(X^{\otimes Q}).$$

Proof of Theorem 2. The statement for Theorem 2 for $t = 2N$, $N \in \mathbb{Z}$ is directly equivalent to Theorem 1. Therefore, let us suppose that $t \notin 2 \cdot \mathbb{Z}$.

We shall proceed by induction. Suppose for every symmetric bilinear form Q' of dimension $< n$

$$End_{\mathscr{R}_{q,t}}(X^{\otimes Q'}) \cong \mathbb{C}O(Q').$$

First, note that by the above remark, for symmetric bilinear forms Q, Q' of dimensions n, n', as a $\mathbb{C}$-vector space,

$$Hom_{\mathscr{R}_{q,t}}(X^{\otimes Q}, X^{\otimes Q'})$$

can be identified with the free $\mathbb{C}$-vector space generated by choices of the data of surjective linear maps

$$U_1 \oplus U_2 \to W$$

where U_1, U_2 are n-dimensional $\mathbb{F}_q$-vector spaces, and W is an $\mathbb{F}_q$-vector space endowed with a (not necessarily non-degenerate) symplectic form S which is 0 when applied to two vectors in U_1 or U_2.

To find the structure of the top algebra of $End_{\mathscr{R}_{q,t}}(X^{\otimes Q})$, we begin by considering the "lower morphisms" of the form (58). For such a lower morphism f, one may choose a symmetric bilinear form Q' of minimal dimension n' such that f can be expressed as a composition (58). Call such f's the lower morphisms of degree n'. First note that any possible Q' giving a non-zero lower morphism f is of the form $Q[n-2j]$ where $n-2j = n'$. The dimension of each subspace of lower morphisms of degree n' in $End_{\mathscr{R}_{q,t}}(X^{\otimes Q})$ is precisely

$$|O(Q)/P_{Q,j}|^2 \cdot |O(Q[n-2j])|, \tag{59}$$

since elements of the basis of this subspace are indexed by choices of j-dimensional subspaces of the source of the first map in the composition (58) and the target of the second map which can be considered as j-dimensional subspaces of the quadric defined by when Q vanishes, along with a basis element of the top algebra of $End_{\mathscr{R}_{q,t}}(X^{\otimes Q'})$, of which, by the induction hypothesis, there are $|O(Q[n-2j])|$ choices. (Recall that the number of choices of isotropic j-dimensional subspaces is the number of elements of the quotient

$$|O(Q)/P_{Q,j}|.)$$

Now by the definition of $\mathscr{R}_{q,t}$, we have

$$End_{\mathscr{R}_{q,t}}(X^{\otimes Q}) = End_{Sp(V_N)}(\omega_Q)$$

for a sufficiently large N, which, by Theorem 4 is isomorphic to

$$\bigoplus_{j=0}^{h_Q} M_{|O(Q)/P_{Q,j}|}(\mathbb{C}O(Q[n-2j])).$$

By the above counts, all terms for $j = 1, \dots, h_Q$ correspond precisely to the lower morphisms of degree $n-2j$. This leaves precisely the term for $j = 0$, i.e.

$$\mathbb{C}O(Q)$$

for the top part of (4).

□

6.3. **The abstract top algebra.** We can be more explicit about how $O(Q)$ corresponds to the top algebra $End^{top}_{\mathscr{R}_{q,t}}(X^{\otimes Q})$ as described above in the case when Q is split (meaning isomorphic to a sum of hyperbolic forms).

In fact, in this case, a candidate for $End^{top}_{\mathscr{R}_{q,t}}(X^{\otimes Q})$ can be described over an arbitrary field F of characteristic not equal to 2.

In this case, writing $n = dim(Q) = 2m$, we have, for any $N \in \mathbb{N}$,

$$\omega^Q \cong (\mathbb{C}V_N)^{\otimes m}$$

(where $V_N \cong F^{2N}$). Therefore, for $N > m$,

$$End(X^{\otimes Q}) = End_{Sp(V_N)}((\mathbb{C}V_N)^{\otimes m}).$$

Recalling the description of generators of this space given earlier in this section, we find that the top algebra $End^{top}(X^{\otimes Q})$ is generated by choices of data of a surjection (57), an antisymmetric form $\mathscr{S}$ on the target space W, such that $\varphi^*(\mathscr{S}), \psi^*(\mathscr{S}) = 0$, with the additional properties that

(1) φ, ψ are injections.

(2) Considering the pullback

$$\begin{array}{ccc} P & \longrightarrow & F^m \\ \downarrow & & \downarrow \varphi \\ F^m & \xrightarrow{\psi} & W \end{array}$$

the form $\mathscr{S}$ is non-degenerate on the kernel $Ker(P \to W)$.

To understand the algebra structure of $End^{top}(X^{\otimes Q})$, we need to compose two such choices of data. Suppose we are given surjections

$$(\varphi_1, \psi_1) : F^m \oplus F^m \twoheadrightarrow W_1$$

$$(\varphi_2, \psi_2) : F^m \oplus F^m \twoheadrightarrow W_2,$$

with antisymmetric bilinear forms $\mathscr{S}_i$ on W_i such that $\varphi_i^*(\mathscr{S}_i), \psi_i^*(\mathscr{S}_i) = 0$ for $i = 1, 2$, satisfying the above two conditions. First note that, without loss of generality (since the data of the endomorphisms is only defined up to equivalence over isomorphisms on the target of (57)) we may assume that the source of φ_1 is a fixed maximal Q-isotropic space in F^n, which we will denote by U_0:

$$\varphi_1 : U_0 \hookrightarrow W_1,$$

Let us denote the sources of ψ_1 by U_1. We may then also similarly assume U_1 is the source of

$$\varphi_2 : U_1 \hookrightarrow W_2.$$

Denote the source of ψ_2 by U_2. In other words, we have

$$(60)\qquad \begin{aligned} (\varphi_1, \psi_1) &: U_0 \oplus U_1 \twoheadrightarrow W_1 \\ (\varphi_2, \psi_2) &: U_1 \oplus U_2 \twoheadrightarrow W_2 \end{aligned}$$

Also, without loss of generality, by composing with elements of $GL(W_1)$, $GL(W_2)$, we may assume the injections φ_i, ψ_i are in fact identity on vectors and inclusions of U_0, U_1, U_2 as subspaces of W_1, W_2, and therefore

$$W_1 = U_0 + U_1,\ W_2 = U_1 + U_2$$

(where $+$ denotes the span, not direct sum).

We may calculate the composition of two endomorphisms by taking their tensor product and then applying a partial trace matching the set of coordinates in the source corresponding to the first endomorphism with the first set of coordinates in the target corresponding to the second endomorphism. In this case, we take a product of (60) and take trace by identifying the two copies of U_1. First note that without loss of generality, we may assume that there is no intersection of all three subspaces

$$U_0 \cap U_1 \cap U_2 = 0,$$

since if there is a non-trivial intersection, it simply passes through to the composition map

$$(61)\qquad U_0 \oplus U_2 \twoheadrightarrow W,$$

where by definition $W := U_0 + U_2$. Let us write

$$dim(U_0 \cap U_1) = k,\ dim(U_1 \cap U_2) = \ell.$$

Now, let us consider the spaces

$$\begin{aligned} \overline{(U_1 \cap U_2)} &\subseteq U_0 \\ \overline{(U_0 \cap U_1)} &\subseteq U_2 \end{aligned}$$

which are the unique subspaces such that applying the restriction of the forms

$$\begin{aligned} \mathscr{S}_1 &: (U_1 \cap U_2) \otimes \overline{(U_1 \cap U_2)} \to F \\ \mathscr{S}_2 &: (U_0 \cap U_1) \otimes \overline{(U_0 \cap U_1)} \to F \end{aligned}$$

form a perfect pairing.

Let us then write

$$\begin{aligned} U_0 &= \overline{(U_1 \cap U_2)} \oplus (U_0 \cap U_1) \oplus \widetilde{U}_0 \\ U_1 &= (U_0 \cap U_1) \oplus (U_1 \cap U_2) \oplus \widetilde{U}_1 \\ U_2 &= \overline{(U_0 \cap U_1)} \oplus (U_1 \cap U_2) \oplus \widetilde{U}_2. \end{aligned}$$

We have

$$dim(\widetilde{U}_0) = dim(\widetilde{U}_1) = dim(\widetilde{U}_2) = n - k - \ell.$$

Then in the composition the surjection (61), the subspaces $\overline{(U_1 \cap U_2)}$, $U_1 \cap U_2$ and $\overline{(U_0 \cap U_1)}$, $U_0 \cap U_1$ are matched by functoriality. In the composition, since the coordinates corresponding to $\widetilde{U}_1$ will be deleted, it must be in the kernel of any possible antisymmetric bilinear form on W appearing in the composition. This precisely determines the matching of the remaining subspaces $\widetilde{U}_0$, $\widetilde{U}_2$, combined with their pairing with the other basis elements (which is already determined).

To see the correspondence of this data with elements of $O(Q)$, fix a maximal Q-isotropic space in F^n. Call this U_0. For an element $g \in O(Q)$, we have

$$U_0, g(U_0) \cong F^m.$$

We consider φ, ψ to be the natural inclusions of U_0, $g(U_0)$ in their span

$$W = U_0 + g(U_0),$$

and for $u \in U_0$, $v \in g(U_0)$, we put

$$\mathscr{S}(u, v) = Q(u, v).$$

We have that $P = U_0 \cap (g(U_0))^\perp \cong g(U_0) \cap (U_0)^\perp$, giving both conditions of being an element of the top algebra.

Note that the above described description of composition also agrees with composition in $O(Q)$, since for $g, h \in O(Q)$, considering the corresponding endomorphisms $(\varphi_1, \psi_1), (\varphi_2, \psi_2)$, in (60), we have

$$U_1 = g(U_0),\ U_2 = h(U_1) = hg(U_0),$$

giving that the composition exactly is the endomorphism corresponding to hg.

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