ON COMPLETION AND THE EVENNESS CONJECTURE FOR HOMOTOPICAL EQUIVARIANT COBORDISM

SOPHIE KRIZ

ABSTRACT. We give counterexamples to the evenness conjecture for homotopical equivariant cobordism. To this end, we prove a completion theorem for certain complex cobordism modules which does not involve higher derived functors. A key step in the proof is provided by a certain new relation between Mackey and Borel cohomology.

1. INTRODUCTION

In 2018, B. Uribe [42] called attention to the evenness conjecture [37] for G-equivariant complex cobordism for a finite (or more generally compact Lie) group G. This statement concerns the equivariant cobordism groups of G-equivariant manifolds M whose tangent bundle, after adding a finite-dimensional real bundle (with trivial G-action) has a given structure of a G-equivariant complex bundle. The conjecture stated that the G-equivariant complex cobordism groups form a flat module over the non-equivariant complex cobordism ring MU_*, which is concentrated in even degrees. (Recall that MU_* is isomorphic to the Lazard ring placed in even degrees, by the classical result of Milnor and Novikov [32, 34, 35].)

The conjecture was disproved by Samperton [38] and Ángel, Samperton, Segovia, Uribe [1] by a highly surprising geometric calculation showing that failures of the evenness conjecture can be produced from cobordism of surfaces when the group G has non-zero Bogomolov multiplier (see [6]). Therefore, these counterexamples also automatically give examples to the Noether problem over $\mathbb{C}$, [33].

There is a variant of G-equivariant complex cobordism known as *stable* or *homotopical cobordism* $(MU_G)_*$. For formal geometrical reasons,

The author was supported by a 2023 National Science Foundation Graduate Research Fellowship, no. 2023350430.

equivariant cobordism $(\Omega_G)_*$ can be made into (at least a $\mathbb{Z}$-graded) generalized homology theory, and *homotopical* (or *stable*) G*-equivariant cobordism* is formed by taking the colimit of $(\widetilde{\Omega}_G)_{|V|}S^V$ where S^V is the 1-point compactification of a complex representation V (in this paper, our convention is to always use $|V|$ to denote the real dimension of a representation V, i.e., in this case, $|V| = 2 \cdot dim_{\mathbb{C}}(V)$). Thus, we obtain a map

$$(\Omega_G)_* \to (MU_G)_*. \tag{1}$$

Homotopical cobordism $(MU_G)_*$ was first considered by tom Dieck [40, 41] and its characterization as a localization of the ring

$$\bigoplus_V \widetilde{\Omega}^G_*(S^V). \tag{2}$$

was formally stated and proved by Bröcker and Hook [7].

A homotopical variant of the evenness conjecture, i.e. the statement that $(MU_G)_*$ is a flat MU_*-module concentrated in even degrees, has also been around since the 1970's, and has been proved for abelian (compact Lie) groups by Löffler in [30] and Comezaña in [31], Chapter XXVIII.

Our main result is

Theorem A. *Let G be the p-Sylow subgroup $G \subset GL_4(\mathbb{F}_p)$ for any prime $p > 2$. Then the G-equivariant homotopical complex cobordism groups $(MU_G)_*$ do not form a flat MU_*-module concentrated in even degrees.*

Notably, there is no direct implication between Theorem A and the results of [1, 38]. While Theorem A does disprove the evenness conjecture for the ring (2), [1, 38] disprove it for its $V = 0$ part, which is a stronger result. On the other hand, no implication in the opposite direction is known either, since, a priori, Bogomolov multipliers could disappear in the stabilization process.

In fact, the p-Sylow subgroup $G \subseteq GL_4(\mathbb{F}_p)$ has Bogomolov multiplier 0. (Due to the fact that it has $p, p^2-1, p^3+p^2-2p, p^3-p^2-p+1$ conjugacy classes of sizes $1, p, p^2, p^3$, respectively, it is of type Φ_{31} or Φ_{32} according to the classification of James [21], and thus has Bogomolov multiplier 0 by the results of Chen and Ma [8].)

As far as we know, however, the Noether problem remains open for G. This prompts the following

Question: *Does a positive solution to the Noether problem over $\mathbb{C}$ for a finite group G imply (any version of) the evenness conjecture for stable G-equivariant complex cobordism?*

Theorem A is proved by very different methods than those of [1, 38]. Our strategy is to prove a suitable *completion theorem* (Theorem B below), i.e. a theorem linking generalized equivariant cohomology to a suitable completion of non-equivariant generalized cohomology.

Equivariant completion theorems for complex cobordism are known (see Greenlees and May [17] for finite extensions of tori and recently La Vecchia [26] for every compact Lie group). Our new contribution (Theorem B below) is to prove, using a different method, for suitable modules over $(MU_G)_*$, a completion theorem *without higher derived functors*. The key condition on the module (Assumption A of Section 6) is the existence of an upper bound on the length of differentials in the Atiyah-Hirzebruch spectral sequence of a G-space. This applies in particular to Morava K-theory which allows us to deduce Theorem A from the results of I. Kriz and Lee [22, 23].

The key step toward proving Theorem B is to show that, given suitable "orientability" assumptions, Borel cohomology is, in fact, a localization of ordinary equivariant cohomology by inverting a "weak orientation class" e_V. This is precisely stated in Theorem C below. Filtering by "e_V-suspensions" of the Postnikov-Whitehead tower gives a direct system of spectral sequences, from which our completion theorem arises by taking colimits; this can be done when differentials are limited in length.

Theorem C, in turn, is a result of a discussion of "weak orientations" of equivariant cohomology over a finite group, which is stated in Theorem D, and in its most basic form, is proved on chain level.

The present paper is organized as follows: In Section 2 below, we state Theorems B, C, D precisely and outline the proof in more technical terms. We prove Theorem D in Section 3. In Section 4, we restate Theorem C in more detail (Theorem 1, Corollaries 2, 3), and give a proof. Section 5 contains some background material on equivariant Whitehead towers of ring and module spectra. In Section 6, we give a proof of Theorem B (and a counterexample to its statement after omitting Assumption A). In Section 7, we prove Theorem A.

2. The main steps in the proof of Theorem A

A basic method for studying equivariant spectra is to use *completion theorems*. A completion theorem for equivariant K-theory was proved by Atiyah and Segal [2]. A completion theorem for MU_G was proved by Greenlees-May [17] (recently extended by La Vecchia [26]). A completion theorem for HM is false (which will be important to our discussion).

A key step in the proof of Theorem A is the following completion theorem. We formulate a technical assumption (Section 6, Assumption A) on a G-spectrum, which guarantees convergence.

Theorem B. *Let E be a G-equivariant commutative ring spectrum for a finite group G such that for some faithful real G-representation V, there exists a unit of the $RO(G)$-graded E-coefficient ring*

$$\widetilde{e}_V \in E_{|V|-V}. \tag{3}$$

If E satisfies Assumption A (see Section 6), then there exists a decreasing ring filtration F^i on the $\mathbb{Z}$-graded coefficient ring E_* such that $F^0E_* = E_*(= E_*^G)$, F^1E_* is the augmentation ideal

$$\mathrm{Ker}(E_* = E_*^G \to E_*^{\{e\}}),$$

and

$$F(EG_+, E)_* \cong \lim_n E_*/F^n E_*.$$

For any E for which a unit (3) exists, an analogous statement also holds for E-module spectra which satisfy Assumption A.

We will show that Theorem B applies to the MU_G-module spectrum $MU_G \wedge_{MU} K(n)$, which can be applied to reducing Theorem A to a known statement about Morava $K(n)$-theory.

Note that Theorem B does not immediately follow from the completion theorems [17, 26], since those theorems contain potentially higher derived terms. Its main feature is the absence of higher derived functors, which is key for our application.

As already mentioned, the ordinary cohomology spectrum HM fails the assumptions of Theorem B. It satisfies, however, a weaker property that is in fact a key step in the proof of Theorem B:

Theorem C. *Let M be a Mackey module over a Green functor R. Then there exists an orientable faithful G-equivariant representation V and a class*

$$e_V \in HR_{|V|-V}$$

such that

$$e_V^{-1}HM \sim F(EG_+, HM(G/\{e\})).$$

The following more specific statement, which is needed in the proof, is also of independent interest:

Theorem D. *Suppose V is an orientable finite-dimensional faithful G-representation and G is a finite group.*

(1) If $|V| = n$, then

$$H_n^G(S^V; \underline{\mathbb{Z}}) = \mathbb{Z},$$

where $\underline{\mathbb{Z}}$ denotes the constant G-Mackey functor (i.e. restrictions are $Id_{\mathbb{Z}}$).

(2) Let M be a Mackey functor where the G-action on $M(G/\{e\})$ is trivial, and let $m > 1$. Then we have

$$\widetilde{H}^G_{m|V|}(S^{mV}; M) = M(G/\{e\}).$$

In some sense, Theorems C and D together give an analogue of the main result of [7] with Ω_G replaced by ordinary G-equivariant cohomology.

3. Proof of Theorem D

We shall begin with a proof of Theorem D, which is done entirely on the chain level.

For a CW-complex X, we denote by $C_*(X)$ the cellular chain complex of X. If X is a G-CW-complex, then $C_*(X)$ becomes a chain complex of $\mathbb{Z}[G]$-modules (where $\mathbb{Z}[G]$ denotes the group ring of G), i.e. each $C_k(G)$ gets a $\mathbb{Z}$-linear G-action. We denote by $\mathscr{O}_G$ the orbit category of G. A G-coefficient system is a functor $\mathscr{O}_G^{Op} \to Ab$ (the category of abelian groups). Likewise, a G-co-coefficient system is a functor $\mathscr{O}_G \to Ab$. For a G-CW complex X, let

$$C_{G,*}(X)(G/H) := C_*(X^H)$$

denote the cellular coefficient-system-valued chain complex of X. On the other hand, let

$$C_G^*(X)(G/H) := Hom_{\mathbb{Z}}(C_{G,*}(X)(G/H), \mathbb{Z})$$

denote the dual of $C_{G,*}(X)$. Since $Hom_{\mathbb{Z}}$ is contravariant in the first variable, $C_G^*(X)$ is a chain complex of co-coefficient systems. (In this paper, we identify chain and cochain complexes by reversing the sign of the grading.)

A *Mackey functor* is a pair consisting of a coefficient system and a co-coefficient system which agree on objects. There is a compatibility condition. (For details, see [9, 25].)

Proof of Theorem D. First, we shall prove (1). By definition, we have

$$H_n^G(S^V;\underline{\mathbb{Z}}) = H_n(C_{G,*}(S^V)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}}).$$

Now, for any subgroup $\{e\} \neq H \subseteq G$, $V^H \subsetneq V$, since V is faithful. Thus, $dim(V^H) < dim(V)$. So, no top (i.e. n-dimensional) cell of S^V can be contained in such a V^H, by invariance of domain. So, every top cell must have isotropy $\{e\}$ and thus be free. Therefore, $C_n(S^V)$ must be of the form of a free $\mathbb{Z}[G]$-module, so we can write

$$C_n(S^V) = (\mathbb{Z}[G])^m,$$

for some m. In fact,

$$(C_{G,n}(S^V))(G/H) = \mathbb{Z}[G]^m$$

for $H = \{e\}$ and is 0 else. Therefore,

$$C_{G,n}(S^V)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} = C_{G,n}(S^V)(G/\{e\})\otimes_{\mathbb{Z}[G]}\mathbb{Z} = \mathbb{Z}^m,$$

since all other $C_{G,n}(S^V)(G/H)\otimes_{\mathbb{Z}[G]}\mathbb{Z} = 0$ for $\{e\}\neq H \subseteq G$ and there are no identifications introduced by morphisms of $\mathscr{O}_G$ coming from inclusions of subgroups.

Now, so far, we have

$$\begin{array}{ccc}
\mathbb{Z}^m = C_{G,n}(S^V)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} & & C_n(S^V) \\
\Big\downarrow{\scriptstyle d\otimes_{\mathscr{O}_G}\mathbb{Z}} & & \Big\downarrow{\scriptstyle d} \\
C_{G,n-1}(S^V)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} & & C_{n-1}(S^V)
\end{array}$$

The next step is to construct horizontal arrows completing the diagram. We shall define, for a functor $F : \mathscr{O}_G^{Op} \to Ab$, a map

$$F\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} \to F(G/\{e\}), \tag{4}$$

and then use it on $F = C_{G,i}(S^V)$. We attempt to define this map by the one induced by

$$x\otimes \ell \mapsto \ell\cdot \sum_{f:G/e\to G/H} f^*x. \tag{5}$$

However, we need to show that this definition would be consistent. Consistency under composition is immediate. Thus, we need, in particular, to show for subgroups $H' \subseteq H$, and a map $\phi : G/H' \to G/H$, for $x \in F(G/H)$, the map (5) sends $\phi^*(x)\otimes 1 \in F(G/H')\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}}(G/H')$ and $x\otimes |H|/|H'| \in F(G/H)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}}(G/H)$ to the same element. From the given definition, it sends each to

$$\sum_{f:G/e\to G/H'} f^*\circ\phi^*x = \frac{|G/H'|}{|G/H|}\cdot \sum_{f:G/e\to G/H} f^*x.$$

and

$$|H|/|H'|\cdot\sum_{f:G/e\to G/H} f^*x,$$

respectively, which are equal since $|G/H'| = |H/H'|\cdot|G/H|$.

So, we can define

$$\Psi_i : C_{G,i}(S^V)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} \to C_i(S^V)$$

by (5).

We will show that Ψ_i is injective. This amounts to considering (4) when F is the free abelian group F_S on the representable functor by a G-set S:

$$F_S(G/H) = \mathbb{Z}Map_G(G/H, S).$$

Then (4) takes the form

$$\eta : F_S\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} \to \mathbb{Z}S.$$

Now, we can assume that $S = G/H$ is an orbit. Let $W(H) = N(H)/H$ be the Weyl group of H. Then we have a surjective map

$$\mathbb{Z}((G/H)^H)\otimes_{\mathbb{Z}W(H)}\mathbb{Z} \twoheadrightarrow \mathbb{Z}F_{G/H}\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}}.$$

By composing with η, we get a map

$$\mathbb{Z}((G/H)^H)\otimes_{\mathbb{Z}W(H)}\mathbb{Z} \to \mathbb{Z}G/H.$$

Then we have $(G/H)^H = W(H)$, so this map is

$$\mathbb{Z} \to \mathbb{Z}(G/H)$$

$$1 \mapsto (1,\dots,1).$$

Therefore, the map must be injective. So, η is injective, too. So, in particular, the Ψ_k's must also be injective.

So, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^m & \xrightarrow{\Psi_n} & (\mathbb{Z}[G])^m \\ \downarrow{\scriptstyle d\otimes_{\mathscr{O}_G}\mathbb{Z}} & & \downarrow{\scriptstyle d} \\ C_{G,n-1}(S^V)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} & \xrightarrow{\Psi_{n-1}} & C_{n-1}(S^V) \end{array} \tag{6}$$

where the horizontal arrows are injective. By Poincaré duality, we have that $Ker(d) = \{\mu\}$, where

$$\mu \in C_n(S^V)^G = (\mathbb{Z}[G]^m)^G$$

since V is G-orientable. Moreover, μ is the sum of all the top cells with appropriate signs, which are G-compatible since V is orientable. Thus, $\mu = \sum_{g\in G} g(\mu_0) \in \mathbb{Z}[G]^m$, for some $\mu_0 \in \mathbb{Z}^m$. Then, $d\circ\Psi_n(\mu_0) = d(\mu) = 0$. So $x \in Ker(d\otimes_{\mathscr{O}_G}\mathbb{Z})$ if and only if $\Psi_{n-1}\circ(d\otimes_{\mathscr{O}_G}\mathbb{Z})(x) = 0$, which happens if and only if $d \circ \Psi_n(x) = 0$. So, $Ker(d \otimes_{\mathscr{O}_G} \mathbb{Z}) = Ker(d \circ \Psi_n) = \langle \mu_0 \rangle \cong \mathbb{Z}$. So,

$$H_n^G(S^V; \underline{\mathbb{Z}}) = Ker(d \otimes_{\mathscr{O}_G} \mathbb{Z})/Im(d \otimes_{\mathscr{O}_G} \mathbb{Z}) = \mathbb{Z}/0 = \mathbb{Z}.$$

This concludes the proof of (1).

Now we will prove (2). Since V is faithful, for $|V| = n$ and $m > 1$, the sets of $(m \cdot n)$-cells and $(m \cdot n - 1)$-cells of S^{mV} are both free G-sets (using the CW-decomposition of a smash product and the fact that a product of a free G-set with any G-set is free).

Now for a Mackey functor M where $M(G/\{e\})$ is G-fixed, we have

$$\begin{array}{ccc} M(G/\{e\}) & \longrightarrow & M(G/\{e\})[G] \\ x & \mapsto & \sum_{g \in G} gx \end{array} \tag{7}$$

is injective, and thus, (7) can be used to define injective Ψ_{mn}, Ψ_{mn-1} so that the diagram (6) commutes. Thus, the same argument applies. □

Comments:

1. Note that the same argument works with $\underline{\mathbb{Z}}$ replaced by $\underline{R}$ for any commutative ring R and an R-oriented representation V, which in particular gives new information for $R = \mathbb{Z}/2$, since every representation is $\mathbb{Z}/2$-orientable.

2. Note that part (1) of the statement of Theorem D also holds for a representation V which is not faithful since if H is the isotropy group, then
$$H_n^G(S^V; \underline{R}) \cong H_n^{G/H}(S^V; \underline{R}).$$
Similarly, part (2) of the statement remains valid when we replace $M(G/\{e\})$ with $M(G/H)$.

4. Proof of Theorem C

In this section, we will state and prove a more precise version of Theorem C.

Let $\mathscr{A}$ denote the universal Mackey functor, i.e. the Burnside ring Green functor, where $\mathscr{A}(G/H)$ is the Burnside ring of H (the group completion of the set of isomorphism classes of finite H-sets with respect to disjoint union). For details, see [9, 15]. Note that Mackey functors are the same as Mackey $\mathscr{A}$-modules, and (commutative) Green functors are the same as (commutative) Mackey $\mathscr{A}$-algebras. Thus, for the purpose of constructing (weak) orientation classes, it suffices to work with coefficients in $\mathscr{A}$. Also note that we always have an orientable faithful G-representation for a finite group G (e.g. the complex regular representation). Also, for any G-representation V, $2V$ is orientable.

By applying Theorem D (2) to $M = \mathscr{A}$, we have

$$\mathbb{Z} \cong \mathscr{A}(G/\{e\}) = \widetilde{H}^G_{m|V|}(S^{mV}; \mathscr{A}) = H\mathscr{A}_{m|V|-mV}.$$

Then we have

$$1 \in \mathbb{Z} \mapsto e_V \in \widetilde{H}^G_{m|V|}(S^{mV}; \mathscr{A}). \tag{8}$$

We shall call e_V the *weak orientation class*. We will study the ring obtained by inverting the generator e_V in the $RO(G)$-graded coefficients of $H\mathscr{A}$. Clearly, as a spectrum, $H\mathscr{A}$ is an E_∞-ring spectrum, and thus in particular a commutative ring spectrum. So, it is possible to invert e_V.

This construction can be entirely described on chain level (see [25], the end of Section 3).

Theorem 1. *Using the notation of (8), for an orientable faithful representation V and a Mackey functor M, we have*

$$e_V^{-1}HM = \operatorname*{hocolim}_m(HM \to \Sigma^{m|V|-mV}HM \to \Sigma^{2m|V|-2mV}HM \to \dots)$$

$$\sim F(EG_+, H(M(G/\{e\}))).$$

Here $H(M(G/\{e\}))$ is considered as a naive G-spectrum, i.e. a non-equivariant spectrum with a G-action.

Proof. First, for a finite G-set X,

$$H_{k+nm|V|-nmV}(X; M) = \widetilde{H}_{k+mn|V|}(S^{nmV} \wedge X_+; M) =$$

$$= H^G_{k+mn|V|}(\widetilde{C}_{G,*}(S^{mV} \wedge X_+) \otimes_{\mathscr{O}_G} M).$$

We also have that $S^{nmV} \wedge X_+$ is free in degrees k where $mn(|V|-1) < k \leq mn|V|$ and 0 in degrees $> mn|V|$. Thus, for $k > -mn$,

$$H^G_{k+mn|V|}(\widetilde{C}_{G,*}(S^{mV} \wedge X_+) \otimes_{\mathscr{O}_G} M) =$$

$$= H^G_{k+mn|V|}(\widetilde{C}_*(S^{mV} \wedge X_+) \otimes_{\mathbb{Z}[G]} M(G/\{e\})) =$$

$$= H^G_{k+mn|V|}((\widetilde{C}_*(S^{mnV}) \otimes C_*(X)) \otimes_{\mathbb{Z}[G]} M(G/\{e\})).$$

Furthermore, in degrees $> mn(|V|-1)$, $\widetilde{C}_*(S^{mnV})$ coincides with

$$Hom_{\mathbb{Z}}(R, \mathbb{Z})[mn|V|] = Hom_{\mathbb{Z}[G]}(R, \mathbb{Z}[G])[mn|V|]$$

for a finite type free $\mathbb{Z}[G]$-resolution R of $\mathbb{Z}$. Therefore, for $k > -mn+1$,

$$\begin{aligned} & H^G_{k+mn|V|}((\widetilde{C}_*(S^{mnV}) \otimes C_*(X)) \otimes_{\mathbb{Z}[G]} M(G/\{e\})) = \\ (9) \qquad & = H_{k+mn|V|}((C_*(X) \otimes Hom_{\mathbb{Z}}(R,\mathbb{Z})) \otimes_{\mathbb{Z}[G]} M(G/\{e\})) = \\ & = H_{k+mn|V|}(Hom_{\mathbb{Z}[G]}(C^*X \otimes R, M(G/\{e\})). \end{aligned}$$

Mapping to higher n induces an isomorphism in homology in the given range.

Now on the level of spectra, we have a map

$$(10) \qquad HM \to F(EG_+, HM) \sim F(EG_+, HM(G/\{e\})).$$

By the fact that

$$\widetilde{C}_*(S^{mV}) \otimes Hom_{\mathbb{Z}}(R,\mathbb{Z}) = Hom_{\mathbb{Z}}(R,\mathbb{Z})[m|V|]$$

for another finite type free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$, we have an equivalence

$$F(EG_+, HM(G/\{e\})) \sim e_V^{-1} F(EG_+, HM(G/\{e\})),$$

so (10) induces a map

$$e_V^{-1} HM \to F(EG_+, HM(G/\{e\})),$$

which, by (9) applied to $X = G/H$, is an equivalence.

□

Corollary 2. *For a finite G-spectrum X, any orientable faithful G-representation V, and any G-Mackey functor M,*

$$\begin{aligned} \operatorname{colim}_n H^G_{k+nm|V|-nmV}(X;M) = H^{-k}_{G,Borel}(DX; M(G/\{e\})) = \\ = F(EG_+ \wedge DX, HM)_k \end{aligned}$$

where DX is the Spanier-Whitehead dual of X.

The proof of Theorem 1 also gives the following

Corollary 3. *Let M be a Green functor and let, for an orientable faithful G-representation V,*

$$e_V \in H_{|V|-V}(*;M) = H^{V-|V|}(*;M) = \widetilde{H}^G_{|V|}(S^V;M)$$

be a class which restricts to a unit in the ring

$$H^{V-|V|}(G/\{e\};M) = M(G/\{e\}).$$

Then the canonical morphism below is a G-equivalence

$$e_V^{-1} HM \xrightarrow{\sim} F(EG_+, HM) = F(EG_+, HM(G/\{e\})).$$

5. Whitehead Towers of Ring and Module Equivariant Spectra

In this section, we treat some preliminary material needed in the proof of Theorem B. We use the language of triangulated categories and t-structures (see [3], Section 1.3), and also discuss some multiplicative properties.

Recall that the derived category of G-spectra is a triangulated category (see [3], Section 1.1, [29], Chapter 1). Suppose G is a finite group and E is a G-spectrum. Then the k-th homotopy groups of E form a Mackey functor

$$\underline{\pi}_k(E)(G/H) = \pi_k(E^H).$$

This is the "homology theory" associated with a t-structure. Then we have

$$\underline{\pi}_k(\tau_{\geq n}E) = \underline{\pi}_k E$$

for $k \geq n$ and is 0 for $k < n$, and

$$\underline{\pi}_k(\tau_{<n}E) = \underline{\pi}_k E$$

for $k < n$ and is 0 for $k \geq n$.

We also have a distinguished triangle

$$\tau_{\geq 0}E \to E \to \tau_{<0}E \to \tau_{\geq 0}E[1].$$

Also, note that, while $[\tau_{<k}F, \tau_{\geq k}E]$ can be non-trivial,

$$[\tau_{\geq k}E, \tau_{<k}E] = 0, \tag{11}$$

(where $[?, ?]$ denotes the abelian group of morphisms in our triangulated category, i.e. the derived category of G-spectra). In addition, for $k > \ell$, we have

$$\tau_{<k}\tau_{<\ell} = \tau_{<\ell}. \tag{12}$$

Lemma 4. *(See also Dugger* [10]*, §4) If E is a ring spectrum, then $\tau_{\geq 0}E$ is also a ring spectrum and $\tau_{\geq n}E$ is a $\tau_{\geq 0}E$-module spectrum (in this paper, only commutative ring spectra are considered). If E is a ring spectrum and K is an E-module spectrum, then $\tau_{\geq n}K$ is a $\tau_{\geq 0}E$-module spectrum.*

Proof. In the derived category of G-spectra, we have a diagram

$$
\begin{array}{ccc}
\tau_{\geq 0}E \wedge \tau_{\geq 0}E & \longrightarrow & E \wedge E \\
& \searrow^{\mu_0} & \downarrow{\scriptstyle \nu} \\
& {\scriptstyle 0}\searrow & E \\
& & \downarrow \\
& & \tau_{<0}E
\end{array}
\tag{13}
$$

by (12), where ν denotes the operation of E. Thus, μ_0 lifts to a map

$$\mu : \tau_{\geq 0}E \wedge \tau_{\geq 0}E \to \tau_{\geq 0}E.$$

Note that μ is unique. This is because if there is another such lift μ', then we have the diagram

$$
\begin{array}{ccc}
\tau_{\geq 0}E \wedge \tau_{\geq 0}E & \overset{0}{\dashrightarrow} & \tau_{<0}E[-1] \\
& \searrow^{\mu-\mu'} & \downarrow \\
& & \tau_{\geq 0}E \\
& & \downarrow \\
& & E,
\end{array}
$$

by the distinguished triangle

$$\tau_{<0}E[-1] \to \tau_{\geq 0}E \to E \to \tau_{<0}E$$

and the fact that $[\tau_{\geq 0}E \wedge \tau_{\geq 0}E, \tau_{<0}E[-1]] = 0$. The associativity diagram

$$
\begin{array}{ccc}
\tau_{\geq 0}E \wedge \tau_{\geq 0}E \wedge \tau_{\geq 0}E & \longrightarrow & \tau_{\geq 0}E \wedge \tau_{\geq 0}E \\
\downarrow & & \downarrow \\
\tau_{\geq 0}E \wedge \tau_{\geq 0}E & \longrightarrow & \tau_{\geq 0}E
\end{array}
\tag{14}
$$

commutes by a similar reason using the diagram

$$
\begin{array}{ccc}
\tau_{\geq 0}E \wedge \tau_{\geq 0}E \wedge \tau_{\geq 0}E & \longrightarrow & \tau_{\geq 0}E \wedge \tau_{\geq 0}E \\
\downarrow & \overset{0}{\searrow} \; \tau_{<0}E[-1] \searrow & \downarrow \\
\tau_{\geq 0}E \wedge \tau_{\geq 0}E & \longrightarrow & \tau_{\geq 0}E.
\end{array}
$$

The cases of the unit and commutativity are analogous.

The case of $\tau_{\geq n}E$ is analogous.

The argument for modules precisely mimics the proof of Lemma 4 with one E-coordinates replaced by K. $\square$

Suppose that V is an orientable faithful representation of G.

Lemma 5. *Any morphism of G-spectra*

$$\widetilde{e}_V : S^0 \to S^{V-|V|} \wedge E$$

lifts to a morphism of G-spectra

$$S^0 \to (S^{V-|V|} \wedge \tau_{\geq 0}E).$$

Proof. We have a distinguished triangle

$$S^{V-|V|} \wedge \tau_{\geq 0}E \to S^{V-|V|} \wedge E \to S^{V-|V|} \wedge \tau_{<0}E \to S^{V-|V|}\tau_{\geq 0}E[1]. \tag{15}$$

Then we have

$$\begin{array}{ccc} & & S^{V-|V|} \wedge \tau_{\geq 0}E \\ & & \downarrow \\ S^0 & \xrightarrow{\widetilde{e}_V} & S^{V-|V|} \wedge E \\ & \searrow^{\varphi} & \downarrow \\ & & S^{V-|V|} \wedge \tau_{<0}E. \end{array}$$

Then $\varphi = 0$, by (11), since $S^{V-|V|} \wedge \tau_{<0}E$ only has cells in degrees ≤ 0.

So, by the long exact sequence in homotopy classes corresponding to (15), we get a lifting

$$\begin{array}{ccc} & & S^{V-|V|} \wedge \tau_{\geq 0}E \\ & \nearrow & \downarrow \\ S^0 & \xrightarrow{\widetilde{e}_V} & S^{V-|V|} \wedge E. \end{array}$$

$\square$

Remark: J. P. C. Greenlees [12] pointed out that if G is a (finite) group acting freely on a sphere $S(V)$ and $a_V : S^0 \to S^V$ is the map defined by sending the non-base point to the point at infinity, we can write

$$F(EG_+, HM) \simeq HM^{\wedge}_{(a_V)} \tag{16}$$

which, in view of Corollary 3, leads to the curious formula

$$e_V^{-1}HM \simeq F(EG_+, HM) \simeq HM^{\wedge}_{(a_V)}. \tag{17}$$

The right hand side of (16) is more precisely defined to be the homotopy limit over n of the mapping spectrum of the homotopy fiber of

$$a_V^n : S^0 \to S^{nV},$$

which is $S(nV)_+$, into HM, so (16) follows from the fact that

$$EG_+ = \operatorname*{hocolim}_n S(nV)_+.$$

On the chain level, since $C_*S(nV)$ is a chain complex of free $\mathbb{Z}[G]$-modules, we obtain a chain map

$$C_*(S(nV)) \to C_*(EG) = C_*(S(\infty V))$$

and hence a map

$$C^*(EG) \to C^*(S(nV)) \tag{18}$$

which is, in fact, (using this model of EG) an isomorphism in degrees $> -n|V|$, and thus will be an isomorphism after taking limits. On the other hand, the chain complex $\widetilde{C}_*S^{nV}$ is $\mathbb{Z}[G]$-free in degrees > 0 and we have also constructed a map

$$\widetilde{C}_*(S^{nV})[-nV] \to C^*(EG) \tag{19}$$

which then becomes an equivalence after taking (homotopy) colimits.

To see why in (17) we have an equivalence between a homotopy limit and homotopy colimit, recall that for a general chain complex C, we have chain maps

$$\tau_{\geq m}C \to C \to C_{\geq m} \tag{20}$$

where $C_{\geq m}$ is the "stupid filtration," i.e. taking terms of degree $\geq m$ and 0 elsewhere, while $\tau_{\geq m}C$ is the "Whitehead filtration" which induces an isomorphism in homology in degrees $\geq m$. We will discuss the Whitehead filtration in more detail in the next section. In (20), C becomes a quasi-isomorphism when taking the homotopy colimit of the left hand term $\tau_{\geq m}C$ as well as the homotopy limit of the right hand term $C_{\geq m}$ with $m \to -\infty$. In the current setting, (19) represents (up to quasi-isomorphism) the Whitehead filtration $\tau_{\geq -n|V|+1}C^*(EG)$ while (20) represents the "stupid filtration" $C^*(EG)_{\geq -n|V|+1}$.

6. Proof of Theorem B

We shall now restate Theorem B in more detail and give a proof. First, let us state our assumption. Let L be a G-spectrum and let X be a finite G-CW-spectrum. Then we can consider a G-equivariant Atiyah-Hirzebruch spectral sequence that can arise either from the

CW-filtration on X (with cells of the form $G/H_+ \wedge S^n$) or from the Postnikov-Whitehead filtration on L (taking ordinary (co)homology with Mackey coefficients $\underline{\pi}_n L$). These spectral sequences are essentially the same (by considering another variant using both filtrations simultaneously); this is treated in [16], Appendix B. One notes a shift in the indexing, as the E^2-page of the cell-based Atiyah-Hirzebruch spectral sequence becomes the E^1-page of the Postnikov-Whitehead version.

Assumption A: There exists a faithful finite-dimensional G-representation V and an orientation class

$$\widetilde{e}_V \in \widetilde{E}_{|V|}(S^V)$$

such that for every $N \in \mathbb{N}$ and every class u in the E^2-term of the L-Atiyah-Hirzebruch spectral sequence for $\widetilde{L}_* S^{NV}$ there exists an $r_0 \in \mathbb{N}$ such that for each $\ell \in \mathbb{N}$, either $\widetilde{e}_V^{N(\ell-1)} u$ is a permanent cycle or there exists an $r \leq r_0$ such that $d^r \widetilde{e}_V^{N(\ell-1)} u \neq 0$ in the Atiyah-Hirzebruch spectral sequence for $\widetilde{L}_* S^{\ell NV}$.

Theorem 6. *Suppose there exists an* $\widetilde{e}_V \in \widetilde{E}_{|V|}(S^V) = E_{|V|-V}(*)$ *such that*

$$(\nu \wedge S^{V-|V|}) \circ (E \wedge \widetilde{e}_V) : E \xrightarrow{\sim} E \wedge S^{V-|V|} \tag{21}$$

is an equivalence. If $L = E$ *satisfies Assumption A, then there exists a decreasing filtration* F^i *on* $\pi_*^G E$ *such that*

(a)
$$\lim_i \pi_*^G E / F^i \pi_*^G E \cong \pi_*^G F(EG_+, E).$$

(b)
$$F^0 \pi_*^G E = \pi_*^G E$$

and $F^1 \pi_*^G E$ *is the augmentation ideal of* $\pi_*^G E$ *(i.e. the kernel of the restriction* $\pi_*^G E \to \pi_*^{\{e\}} E$*).*

(c) *F is a filtration of rings (i.e.* $F^i \cdot F^j \subseteq F^{i+j}$*).*

Suppose K *is an* E*-module spectrum. If* $L = K$ *satisfies Assumption A (without necessarily assuming Assumption A for* $L = E$*), then there exists a decreasing filtration* F^i *on* $\pi_*^G K$ *such that*

(d)
$$\lim_i \pi_*^G K / F^i \pi_*^G K \cong \pi_*^G F(EG_+, K).$$

(e)
$$F^0 \pi_*^G K = \pi_*^G K$$

(f) *If E also satisfies Assumption A, F is a filtration of E-modules (i.e.*
$$F^i(\pi_* E) \cdot F^j(\pi_* K) \subseteq F^{i+j}(\pi_* K)).$$

We will first discuss the ring case. Observe that we have a homotopy decreasing filtration

$$\cdots \to \tau_{\geq n+1}E \to \tau_{\geq n}E \to \tau_{\geq n-1}E \to \dots \tag{22}$$

Its homotopy limit is 0 because, by taking cofibers of the canonical maps to E, we obtain a sequence

$$\to \tau_{<n+1}E \to \tau_{<n}E \to \tau_{<n-1}E \to \dots$$

whose homotopy limit is E. Put

$${}_{(0)}\mathscr{F}^i = \tau_{<i}E.$$

Then by applying $\widetilde{e}_V$ to the filtration (22) repeatedly we get a sequence of filtrations

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\geq n+1}E & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n+1}E) \wedge S^{V-|V|} & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n+1}E) \wedge S^{2(V-|V|)} \to \dots \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\geq n}E & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n}E) \wedge S^{V-|V|} & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n}E) \wedge S^{2(V-|V|)} \to \dots \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\geq n-1}E & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n-1}E) \wedge S^{V-|V|} & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n-1}E) \wedge S^{2(V-|V|)} \to \dots \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

Now each vertical homotopy filtration individually has homotopy limit 0, since again by taking cofibers of the canonical maps into $E = E \wedge S^{j(V-|V|)}$ (see (21)), we obtain a sequence

$${}_{(j)}\mathscr{F}^i = (\tau_{<i}E) \wedge S^{j(V-|V|)}$$

whose homotopy limit in i is E.

Lemma 7. *We have*

$$\begin{aligned} {}_{(\infty)}\mathscr{F}^i &= \operatorname{hocolim}_j((\tau_{<i}E) \wedge S^{j(V-|V|)}) \\ &\sim F(EG_+, \tau_{<i}E). \end{aligned} \tag{23}$$

Proof. The statement is true with $\tau_{<i}E$ replaced by $\tau_{\geq k}\tau_{<i}E$ by Theorem 1. We have

$$\tau_{<i}E = \operatorname*{hocolim}_{k\to-\infty} \tau_{\geq k}\tau_{<i}E.$$

Thus, we are done if we can conclude

$$F(EG_+, \tau_{<i}E) = \operatorname*{hocolim}_{k\to-\infty} F(EG_+, \tau_{\geq k}\tau_{<i}E). \tag{24}$$

However, this follows from the fact that homotopy groups in each degree on the right hand side of (24) are eventually constant, since EG_+ is of finite type. □

Proof of Theorem 6. Denote by ${}_{(j)}E$ the spectral sequence in homotopy groups based on the homotopy filtrations ${}_{(j)}\mathscr{F}$. Then by Assumption A,

$${}_{(\infty)}E = \operatorname*{colim}_{j}\, {}_{(j)}E$$

for a suitable representation V. In particular, ${}_{(\infty)}E$ converges algebraically.

By Lemma 7, however, ${}_{(\infty)}E$ converges conditionally to $\pi_*F(EG_+, E)$, and thus converges strongly to $\pi_*F(EG_+, E)$ (see [5]).

In other words,

$$\pi_n^G F(EG_+, E) = \lim_{i} \operatorname{Im}(\pi_n^G F(EG_+, E) \to \pi_n^G({}_{(\infty)}\mathscr{F}^i)). \tag{25}$$

On the other hand, $\operatorname{Im}(\pi_n^G F(EG_+, E) \to \pi_n^G({}_{(\infty)}\mathscr{F}^i))$ is an extension of finitely many terms ${}_{(\infty)}E^\infty_{p,q}$ and thus is equal to

$$\operatorname*{colim}_{j}(\operatorname{Im}(\pi_n^G E \to \pi_n^G({}_{(j)}\mathscr{F}))) = \operatorname{Im}(\pi_n^G E \to \pi_n^G({}_{(\infty)}\mathscr{F})). \tag{26}$$

The multiplicativity on the level of spectral sequences follows from the arguments of Blumberg-Mandell [4], Section 4. See also Dugger [10], §4. The paper [18] by Hedenlund, Krause, Nikolaus contains a treatment in the language of ∞-categories.

This concludes the proof of (a), (b), (c). The proof of (d), (e) is analogous by replacing E by K in formulas (23), (24), (25), (26). In (f), Assumption A is in effect for both E, K, so we can again use the applicable results of [4, 10, 18].

□

Example: Now we will present a counterexample to the statement of Theorem 6 when we omit Assumption A. Let $G = \mathbb{Z}/2$. By the results of Löffler [30] and Comezaña [31], the coefficient ring $(MU_{\mathbb{Z}/2})_*$ is a flat MU_*-module concentrated in even degrees. Hence

$$(MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)_*^{\mathbb{Z}/2}$$

is concentrated in even degrees, while

$$F(E\mathbb{Z}/2_+, MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)_* = H^*(B\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2[a], \tag{27}$$

where the degree of a is 1. In particular, (27) is non-zero in odd degrees and hence the conclusion of Theorem 6 is false.

To explain precisely what is happening, we have ([39], Theorem 1, [19])

$$(MU_{\mathbb{Z}/2})_* = MU_*[b_{i,j}, q_j | i, j \in \mathbb{N}_0]/\sim$$

where, writing $u = b_{0,0}$, the relations are

$$\begin{gathered} q_0 = 0 \\ b_{0,1} = 1, b_{0,j} = 0 \text{ for all } j \geq 2 \\ q_j - c_j = uq_{j+1} \\ b_{i,j} - a_{i,j} = ub_{i,j+1}. \end{gathered}$$

Here we write

$$x +_F y = \sum_{i,j \geq 0} a_{i,j} x^i y^j$$

$$c_k = \sum_{i+j=k} a_{i,j}.$$

(so $[2]_F x = \sum_{j \geq 0} c_j x^j$), where F is the universal formal group law.

Hence,

$$(MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)_* = \mathbb{Z}/2[u, q_j, b_{i,j} | i \geq 1, j \geq 0]/\sim$$

where the relations are

$$\begin{gathered} q_0 = 0, \; q_j = uq_{j+1} \\ b_{i,j} = ub_{i,j+1}. \end{gathered}$$

We also have (for example by the arguments of [16], Theorem 22.5)

$$\begin{gathered} F(E\mathbb{Z}/2_+, MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2) = (MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)^\wedge_u = \\ = \operatorname*{holim}_s (MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)/u^s \end{gathered} \tag{28}$$

The generators $b_{i,j}$, $i \geq 1$ are uniquely divisible by u, and thus will not contribute to (28); for this reason, we shall ignore them. (This does not apply to the generators q_j which are infinitely u-divisible, but not uniquely.) Factoring the $b_{i,j}$ out (which we could also do on the spectral level), we obtain the ring

$$R = \mathbb{Z}/2[u, q_j | j \geq 0]/(q_0 = 0, \; uq_{j+1} = q_j)$$

which additively is

$$\mathbb{Z}/2[u] \oplus \mathbb{Z}/2\{q_j | j > 0\}. \tag{29}$$

(We have degrees $|u| = -2$, $|q_j| = 2j$.)

Non-equivariantly, we just have $\mathbb{Z}/2$ in degree 0, so on the Mackey functor level, we have (29) corresponds to

$$\underline{\mathbb{Z}/2} \oplus \Phi\{u^j | j \geq 0\} \oplus \Phi\{q_j | j > 0\} \tag{30}$$

where Φ is the Mackey functor given by

$$\Phi(\mathbb{Z}/2/\mathbb{Z}/2) = \mathbb{Z}/2,\ \Phi(\mathbb{Z}/2/\{e\}) = 0.$$

Let α be the real sign representation, $V = 2\alpha$. We will use the grading of the shifted Atiyah-Hirzebruch spectral sequence ${}_{(j)}E$ based on the Postnikov-Whitehead filtration as described in [16], Appendix B. Recalling that the $\tau_{\geq s}$ filtration is decreasing, if we want to write our spectral sequence homologically (so the total degree is the dimension degree), we therefore have the cofiber of $\tau_{\geq s+1} \to \tau_{\geq s}$ in filtration degree $-s$.

Thus, in ${}_{(j)}E$, (30) gives $(H\underline{\mathbb{Z}/2} \wedge S^{2j\alpha - 2j})_\ell$ in homological filtration degree $(0, \ell)$ and $(H\Phi \wedge S^{2j\alpha - \overline{2j}})_\ell$ in filtration degree $(-2s, 4s + \ell)$ for $s \in \mathbb{Z}$. We have ([20], Section 6)

$$\begin{gathered}(H\underline{\mathbb{Z}/2} \wedge S^{2j\alpha-2j})_\ell = \begin{cases} \mathbb{Z}/2, & \text{if } -2j \leq \ell \leq 0 \\ 0, & \text{else} \end{cases} \\ (H\Phi \wedge S^{2j\alpha-2j})_\ell = \begin{cases} \mathbb{Z}/2, & \text{if } \ell = -2j \\ 0, & \text{else} \end{cases}\end{gathered} \tag{31}$$

(note that $H\Phi$ is α-periodic since $H\Phi_{\{e\}} = 0$). Thus, the Atiyah-Hirzebruch spectral sequence (indexed using the Postnikov tower) for $j > 0$ is

$$E^1_{-2s,4s-2j} = \mathbb{Z}/2,\ \text{for } s \neq 0 \in \mathbb{Z}$$

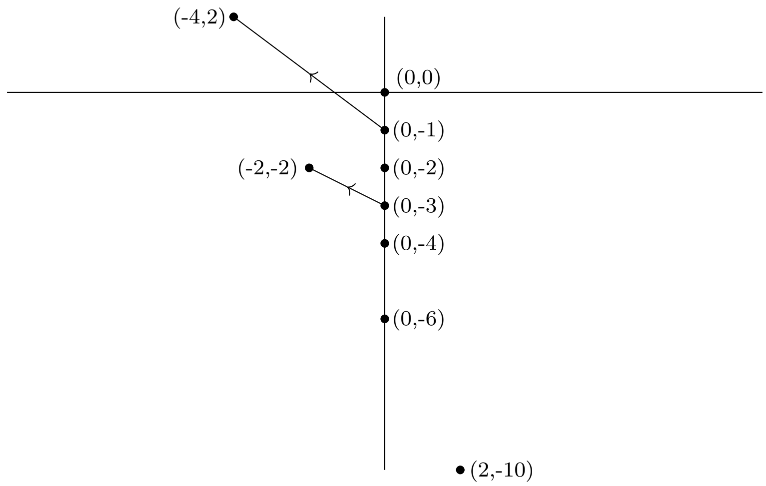
$$E^1_{0,q} = \mathbb{Z}/2,\ \text{for } 2 - 2j \leq q \leq 0 \text{ or } q = -2j$$

$$E^1_{p,q} = 0 \text{ else.}$$

(Note: The exception $E^1_{0,1-2j} = 0$ arises from a d^0 differential.) Knowing the answer forces differentials

$$d^{2j-2s} : E^{2j-2s}_{0,1-2s} \to E^{2j-2s}_{2s-2j,2j-4s},$$

for $1 \leq s \leq j$. In the case of $j = 3$, $E^1_{p,q}$ looks like



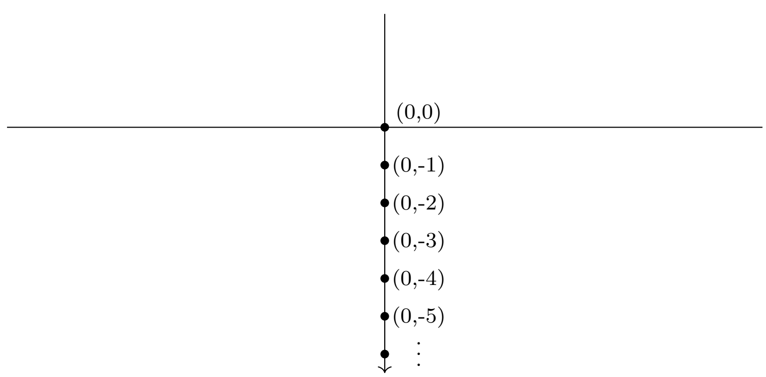
where q is the vertical axis and p is the horizontal one.

We see that in fact the colimit ${}_{(\infty)}\mathscr{F}$ has

$$E^1_{0,\ell} = \mathbb{Z}/2 \text{ for } \ell \leq 0$$

$$E^1_{p,q} = 0 \text{ else}$$

which collapses to $H(B\mathbb{Z}/2; \mathbb{Z}/2)$. $E^1_{p,q}$ looks like



Due to the increasing lengths of differentials in this example, colimit of a sequence of spectral sequences does not commute with abutment.

Comment: The Example well illuminates the fact that Theorem B without Assumption A only fails due to the fact that the passage from ${}_{(j)}\mathscr{F}$ to ${}_{(\infty)}\mathscr{F}$ fails to preserve convergence and abutment. The odd-degree elements in the above example seem like a derived functor of the process. Thus, we expect that some version of Theorem B without Assumption A holds where higher derived functors are taken into account, but we do not pursue that here.

7. The Proof of Theorem A

To prove Theorem A, the main step in the following

Proposition 8. *The MU_G-module spectrum $MU_G \wedge_{MU} K(n)$ satisfies Assumption A for any prime p and any natural number $n \geq 1$.*

Proof. It is well know (see [11]) that, up to weak equivalence, a G-space X can be modelled as a (finite) homotopy colimit of a diagram of G-spaces of the form

$$G_+ \wedge_{N(H)} (EW(H)_+ \wedge X^H) \tag{32}$$

where H runs through conjugacy classes of subgroups of G. For a G-representation sphere

$$X = S^V, \tag{33}$$

we have

$$X^H = S^{V^H}.$$

Applying $\widetilde{K(n)}_*$ to the space (32) (in the sense of taking orbits and then applying the non-equivariant $\widetilde{K(n)}_*$) then gives

$$\widetilde{K(n)}_*(EW(H)_+ \wedge_{W(H)} S^{V^H}). \tag{34}$$

If V is (say) a complex G-representation, then (34) is further isomorphic to

$$K(n)_* BW(H)[|V^H|] \tag{35}$$

which is a finitely generated $K(n)_*$-module by [36]. Hence (and also directly by the proof in [36]), the Atiyah-Hirzebruch spectral sequence for (35) collapses to a suitable finite E^r-term, with finitely many surviving filtration degrees.

To relate the Atiyah-Hirzebruch spectral sequence for the realization of the finite homotopy colimit to the Atiyah-Hirzebruch spectral sequence of its terms (35), we note that if we choose V to be a sufficiently high multiple of the complex regular representation $\mathbb{C}[G]$, the dimensions (and hence the filtration degrees) $|V^H|$, $|V^{H'}|$ for H' properly subconjugate to H are sufficiently far apart that the differentials in the Atiyah-Hirzebruch spectral sequence (35) happen first in the

Atiyah-Hirzebruch spectral sequence of the homotopy colimit, and we are left with differentials between two of the terms (35) from H to a properly subconjugate group H' (here we are considering and grading as in the equivariant Atiyah-Hirzebruch spectral sequence from a cell filtration).

When replacing V by its ℓ-multiple, the distance between the filtration degree of the two terms (35) for H, H' increases. While it is possible in this situation for a class to support differentials of length increasing with ℓ (as in the Example above), for a map of spectral sequence $E \to E'$, a class cannot be the target of a longer differential in E' than in E.

Observe that in the Example, under the comparison map induced by the inclusion

$$S^{m\alpha} \xrightarrow{\subset} S^{m'\alpha}$$

for $m' > m$, the image of a class is hit by the same length differential as in the source, while in the periodicity comparison map, classes in the target of the periodicity map support longer differentials than in the source.

More generally, to put the terms (35) into the same dimension in the Atiyah-Hirzebruch spectral sequence for S^{mV} and $S^{m'V}$ for $m < m'$, we need to obtain a comparison map which raises dimension by

$$(m' - m)V - (m' - m)|V^H|.$$

This can be accomplished by expressing

$$(m'-m)V-(m'-m)|V^H| = (m'-m)(V-V^H)+(m'-m)(V^H-|V^H|),$$

using the inclusion of fixed points

$$S^0 \to S^{V-V^H}$$

(note that $V - V^H$ is a true G-representation), composed with the $(V^H - |V^H|)$-dimensional periodicity class (after restricting to $N(H)$).

Now if we were computing the $\widetilde{K(n)}_*$-Atiyah-Hirzebruch spectral sequence, we could argue that since the $K(n)_*$-dimension of the sum of all the terms (35) is finite, only finitely many lengths of differentials are possible (by counting the possible targets), and hence our conclusion follows.

Instead of $K(n)$, however, we are considering $L = K(n) \wedge_{MU} MU_G$. By complex orientation, (35) is then replaced by

$$(MU_H \wedge_{MU} K(n)_*) \otimes_{K(n)_*} K(n)_*(BW(H))[|V^H|]. \tag{36}$$

While $MU_H \wedge_{MU} K(n)_*$ is infinite-dimensional, under the specification made above, the target of Atiyah-Hirzebruch spectral sequence differentials of a class

$$u = x \otimes u'$$

for $x \in MU_H \wedge_{MU} K(n)_*$ can only be of the form

$$y \otimes w$$

where y is a restriction of x to a subconjugate group, which still leaves only finitely many choices of possible targets, and thus, our argument applies.

□

Then Theorem A results as follows:

Proof of Theorem A. We will proceed by contradiction. Assume that the G-equivariant coefficients $(MU_G)_*$ form a flat MU_*-module concentrated in even degrees. Consider the ring spectrum MU_G and the MU_G-module spectrum $K(n) \wedge_{MU} MU_G$, where $K(n)$ denotes Morava K-theory. (This can be formed since MU_G is an E_∞-algebra over the pushforward of the E_∞-ring spectrum MU, over which $K(n)$ is an E_∞-module.) We then obtain by our flatness assumption that

$$(K(n) \wedge_{MU} MU_G)_* = K(n)_* \otimes_{MU_*} (MU_G)_*$$

is concentrated in even degrees.

On the other hand, by Proposition 8, $K(n) \wedge_{MU} MU_G$ satisfies Assumption A, and hence, we can apply Theorem 6 (d), to conclude that there exists a suitable completion satisfying

$$((K(n) \wedge_{MU} MU_G)_*)^\wedge \cong K(n)^* BG$$

Therefore, $K(n)^*BG$ is also concentrated in even degrees. However, this is a contradiction with the results of [22, 23], which state that $K(2)^*BG$ contains non-zero elements in odd degrees for G the p-Sylow subgroup of $GL_4(\mathbb{F}_p)$ for $p > 2$.

□

8. Statements and Declarations

The author has no conflict of interest.

References

[1] A. Ángel, E. Samperton, C. Segovia, B. Uribe: Oriented and unitary equivariant bordism of surfaces, *Alg. Geom. Topol.* 24 (2024) 3, 1623-1654.

[2] M. F. Atiyah, G. B. Segal: Equivariant K-theory and completion, *J. Diff. Geom.* 3 (1969), 1-18.

[3] A. A. Beilinson, J. Bernstein, P. Deligne: Analyse et topologie sur les espaces singuliers (I), *Astérisque*, no. 100 (1982), 180 p.

[4] A. Blumberg, M. Mandell: K-theoretic Tate-Poitou duality and the fiber of the cyclotomic trace, *Invent. math.* 221 (2020), 397-419.

[5] J. M. Boardman: *Conditionally convergent spectral sequences.* Homotopy invariant algebraic structures (Baltimore, MD, 1998), 49-84, Contemp. Math., 239, Amer. Math. Soc., Providence, RI, 1999.

[6] F. A. Bogomolov: The Brauer group of quotient spaces of linear representations, *Math. USSR Izv.* 30 (1988), 455-485.

[7] T. Bröcker, E. C. Hook: Stable equivariant bordism, *Math. Z.* 129 (1972), 269-277.

[8] Y. Chen, R. Ma: Bogomolov multipliers of some groups of order p^6, *Comm. Algebra* 49 (2021) 242-255.

[9] A. W. M. Dress: Contributions to the theory of induced representations, in: *"Classical" Algebraic K-Theory, and Connections with Arithmetic*, Lecture Notes in Mathematics, Vol. 342 Springer, Berlin, (1973) 181-240.

[10] D. Dugger: Multiplicative structures on homotopy spectral sequences II, arXiv:math/0305187.

[11] A. D. Elmendorf: Systems of fixed point sets, *Trans. Amer. Math. Soc.* 277 (1983) 275-284.

[12] J. P. C. Greenlees: Private communication, 2021.

[13] J. P. C. Greenlees: Equivariant connective K-theory for compact Lie groups, *J. Pure Appl. Algebra* 187 (2004), no. 1-3, 129-152.

[14] J. P. C. Greenlees: Equivariant forms of connective K-theory, *Topology* 38 (1999), no. 5, 1075-1092.

[15] J. P. C. Greenlees: Some remarks on projective Mackey functors, *Journal of Pure and Applied Algebra* 81 (1992) 17-38.

[16] J. P. C. Greenlees, J. P. May: *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 113 (1995), no. 543, viii+178 pp.

[17] J. P. C. Greenlees, J. P. May: Localization and completion theorems for MU-module spectra, *Ann. of Math.* (2) 146 (1997), no. 3, 509-544.

[18] A. Hedenlund, A. Krause, T. Nikolaus: Convergence of spectral sequences revisited, to appear, 2019.

[19] P. Hu: The coefficients of equivariant complex cobordism for primary cyclic groups, preprint, 2021

[20] P. Hu, I. Kriz: Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence, *Topology* 40 (2001), 317-399.

[21] R. James: The groups of order p^6 (p an odd prime), *Mathematics of Computation* 34 (1980) 613-637.

[22] I. Kriz: Morava K-theorem of classifying spaces: some calculations, *Topology* 36 (1997), no. 6, 1247-1273.

[23] I. Kriz, K. P. Lee: Odd-degree elements in the Morava $K(n)$ cohomology of finite groups, *Topology and its Applications* 103 (2000) 229-241.

[24] S. Kriz: Notes on equivariant homology with constant coefficients, *Pacific J. Math.* 309, no. 2, (2020) 381-399

[25] S. Kriz: Some remarks on Mackey functors, arXiv:2205.12192.

[26] M. La Vecchia: The completion and local cohomology theorems for complex cobordism for all compact Lie groups, arXiv:2107.03093v1.

[27] M. La Vecchia: The local (co)homology theorem for equivariant bordism, *Geom. Topol.* 28 (2024) 627-639.

[28] L. G. Jr. Lewis, J. P. May, J. E. McClure: Ordinary RO(G)-graded cohomology, *Bull. Amer. Math. Soc.* (N.S.) 4 (1981), no. 2, 208-212.

[29] L. G. Jr. Lewis; J. P. May; M. Steinberger; J. E. McClure *Equivariant stable homotopy theory* With contributions by J. E. McClure. Lecture Notes in Mathematics, Vol. 1213. Springer-Verlag, Berlin, (1986), x+538 pp.

[30] P. Löffler: Bordismengruppen Unitärer Torusmannigfaltigkeiten, *Manuscripta Math.* 12 (1974), 307-327.

[31] J. P. May: *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[32] J. W. Milnor: On the cobordism ring Ω^* and a complex analogue. I. *Amer. J. Math.* 82 (1960), 505-521.

[33] E. Noether: Gleichungen mit vorgeschriebener Gruppe. *Math. Ann.* 78 (1917) 221-229.

[34] S. P. Novikov: Some problems in the topology of manifolds connected with the theory of Thom spaces, *Dokl. Akad. Nauk SSSR* 132 (1960) 717-720.

[35] S. P. Novikov: Homotopy properties of Thom complexes, *Mat. Sb. (N.S.)* 57 (1962) 407-442.

[36] D. Ravenel: Morava K- theories and finite groups, Symposium on Algebraic Topology in honor of José Adem (Oaxtepec, 1981) *Contemp. Math.*, 12 (1982), 289-292, 12; by the American Mathematical Society, Providence, RI.

[37] R. J. Rowlett: Bordism of metacyclic group actions, *Michigan Math. J.* 27 (1980), 223-233.

[38] E. Samperton: Free actions on surfaces that do not extend to arbitrary actions on 3-manifolds, *C. R. Math. Acad. Sci. Paris* 360 (2022), 161-167.

[39] N. P. Strickland: Complex cobordism of involutions, *Geom. Top.* 5 (2001), 335-345.

[40] T. tom Dieck: Bordism of G-manifolds and integrality theorems, *Topology* Vol. 9, (1970), no. 4, 345-358.

[41] T. tom Dieck: Kobordismentheorie und Transformationsgruppen, *Aaarhus University Preprint Series* (1968/69) No. 30

[42] B. Uribe: The evenness conjecture in equivariant unitary bordism, *Proc. International Congress of Mathematicians (ICM 2018)* (2019), 1217-1239.

[43] A. Wasserman: Equivariant differential topology, *Topology.* 8 (1969), 127-150.