ON COMPLETION AND THE EVENNESS CONJECTURE FOR HOMOTOPICAL EQUIVARIANT COBORDISM

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Abstract. We give counterexamples to the evenness conjecture for homotopical equivariant cobordism. To this end, we prove a completion theorem for certain complex cobordism modules which does not involve higher derived functors. A key step in the proof is provided by a certain new relation between Mackey and Borel cohomology.

1. Introduction

G-Equivariant stable homotopy theory [25] for G a finite or compact Lie group mainly studies G-spectra indexed over the complete universe, which represent fully stable G-equivariant generalized homology and cohomology theories with duality. All G-spectra considered in his paper will be indexed over the complete universe unless specified otherwise. Just as non-equivariantly, the most important equivariant spectra are ordinary homology HM [24], K_G-theory, and stable complex cobordism MU_G, represented by the G-equivariant complex Thom spectrum (see, e.g. [6, 25, 33]).

A conjecture dating back to the 1970s stated that the coefficient ring $(MU_G)_*$ is a free module over the non-equivariant cobordism coefficient ring MU_* on even-dimensional generators. It has been proved for abelian (compact Lie) groups by Löffler in [26] and Comezaña in [27], Chapter XXVIII. The main result of this paper shows that, in general, this conjecture is, in fact, false:

Theorem A. *Let G be the p-Sylow subgroup $P \subset GL_4(\mathbb{F}_p)$ for any prime $p > 2$. Then the coefficients $(MU_G)_*$ of the G-equivariant complex Thom spectrum do not form a flat MU_*-module concentrated in even degrees.*

To give some background, let us denote by $(\Omega_G)_*$ the G-equivariant complex cobordism ring of manifolds [32]. Then there is a map

$$(\Omega_G)_* \to (MU_G)_*, \tag{1}$$

which is not in general an isomorphism due to lack of G-equivariant transversality [35]. A geometric complement of Theorem A, i.e. a counterexample to the a version of the evenness conjecture [29, 34] for the geometric complex cobordism ring $(\Omega_G)_*$, was recently announced by Samperton and Uribe [30]. They observed that all their counterexamples have non-zero Bogomolov multiplier (cf. [5]). Thus, by [5], their groups also do provide a class of counterexamples to the Noether problem over $\mathbb{C}$ (asking if, for a faithful finite group G-representation V, the fixed point field $K(V)^G$, where $K(V)$ is the field of rational functions on V, is also isomorphic to a field of rational functions on a vector space).

The p-Sylow subgroup of $GL_4(\mathbb{F}_p)$, on the other hand, has Bogomolov multiplier 0. (Due to the fact that it has $p, p^2-1, p^3+p^2-2p, p^3-p^2-p+1$ conjugacy classes of sizes $1, p, p^2, p^3$, respectively, it is of type Φ_{31} or Φ_{32} according to the classification of James [18], and thus has Bogomolov multiplier 0 by the results of Chen and Ma [7].)

There is no immediate direct implication either way between the geometric and homotopical evenness conjectures. Bröcker and Hook [6] proved that $(MU_G)_*$ is a localization of a ring of the form

$$\bigoplus_V \widetilde{\Omega}^G_*(S^V). \tag{2}$$

Thus, the p-Sylow subgroups P of $GL_4(\mathbb{F}_p)$ also give counterexamples to the evenness conjecture for (2), which, however, is a stronger statement than the evenness conjecture of Ω^G_*. The Noether problem for the group P, as far as I know, is still open. This motivates the following

Question: *Does a positive solution to the Noether problem over $\mathbb{C}$ for a finite group G imply (any version of) the evenness conjecture for G-equivariant complex cobordism?*

A basic method for studying equivariant spectra is to use *completion theorems*. A completion theorem for equivariant K-theory was proved by Atiyah and Segal [1]. A completion theorem for MU_G was proved by Greenlees-May [14] (recently extended by La Vecchia [23]). A completion theorem for HM is false (which will be important to our discussion).

A key step in the proof of Theorem A is the following completion theorem. We formulate an assumption (Section 5, Assumption A) on a G-spectrum, requiring that the length of all its Atiyah-Hirzebruch spectral sequence differentials be uniformly bounded.

Theorem B. *Let E be a G-equivariant commutative ring spectrum for a finite group G such that for some faithful real G-representation V, there exists a unit of the $RO(G)$-graded E-coefficient ring*

$$\widetilde{e}_V \in E_{|V|-V}. \tag{3}$$

It E satisfies Assumption A (see Section 5), then there exists a decreasing ring filtration F^i on the $\mathbb{Z}$-graded coefficient ring E_* such that $F^0E_* = E_*^{\{e\}}$, F^1E_* is the augmentation ideal

$$\mathrm{Ker}(E_* = E_*^G \to E_*^{\{e\}}),$$

and

$$F(EG_+, E)_* \cong \lim_n E_*/F^nE_*.$$

For any E for which a unit (3) exists, an analogous statement also holds for E-module spectra which statisfy Assumption A.

We will show that Theorem B applies to the MU_G-module spectrum $MU_G \wedge_{MU} K(n)$, which can be applied to reducing Theorem A to a known statement about Morava $K(n)$-theory.

Note that Theorem B does not immediately follow from the completion theorems [14, 23]. Its main feature is the absence of higher derived functors, which is key for our application.

As already mentioned, the ordinary cohomology spectrum HM fails the assumptions of Theorem B. It satisfies, however, a weaker property that is in fact a key step in the proof of Theorem B:

Theorem C. *Let M be a Mackey module over a Green functor R. Then there exists an orientable faithful G-equivariant representation V and a class*

$$e_V \in HR_{|V|-V}$$

such that

$$e_V^{-1}HM \sim F(EG_+, HM(G/\{e\})).$$

The following more specific statement, which is needed in the proof, is also of independent interest:

Theorem D. *Suppose V is an orientable finite-dimensional faithful G-representation and G is a finite group.*

(1) *If $|V| = n$, then*

$$H_n^G(S^V; \underline{\mathbb{Z}}) = \mathbb{Z}.$$

(2) *Let M be a Mackey functor where the G-action on $M_{G/\{e\}}$ is trivial, and let $m > 1$. Then we have*

$$\widetilde{H}^G_{m|V|}(S^{mV}; M) = M_{G/\{e\}}.$$

In some sense, Theorems C and D together give an analogue of the main result of [6] with Ω_G replaced by ordinary G-equivariant cohomology.

The present paper is organized by proving the above theorems in reverse order: We prove Theorem D in Section 2. In Section 3, we restate Theorem C in more detail (Theorem 1, Corollaries 2, 3), and give a proof. Section 4 contains some background material on equivariant Whitehead towers of ring and module spectra. In Section 5, we give a proof of Theorem B and a counterexample for its statement after omitting Assumption A. In Section 6, we prove Theorem A.

2. Proof of Theorem D

We shall begin with a proof of Theorem D, which is done entirely on the chain level.

For a CW-complex X, we denote by $C_*(X)$ the cellular chain complex of X. We denote by $\mathscr{O}_G$ the orbit category of G. A G-coefficient system is a functor $\mathscr{O}_G^{Op} \to Ab$ (the category of abelian groups). Likewise, a G-co-coefficient system is a functor $\mathscr{O}_G \to Ab$. The category of finite G-sets and G-maps will be denoted by "f.G-Sets." For a G-CW complex X, let

$$C_G(X)(G/H) := C_G(X^H)$$

denote the cellular coefficient-system-valued chain complex of X. On the other hand, let

$$C^*_G(X)(G/H) := Hom_{Ab}(C_G(X)(G/H), \mathbb{Z})$$

denote the dual of $C_G(X)$. Since Hom_{Ab} is contravariant in the first variable, $C^*_G(X)$ is a co-coefficient system.

A *Mackey functor* is a pair consisting of a coefficient system and a co-coefficient system which agree on objects. There is a compatibility condition. (For details, see [8, 22].)

Proof of Theorem D. First, we shall prove (1). By definition, we have that

$$H_n^G(S^V;\underline{\mathbb{Z}}) = H_n(C_G(S^V)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}}).$$

Now, for any subgroup $\{e\}\neq H\subseteq G$, $V^H\subsetneq V$, since V is faithful. Thus, $dim(V^H)<dim(V)$. So, no top (i.e. n-dimensional) cell of S^V can be contained in such a V^H, by invariance of domain. So, every top cell must have isotropy $\{e\}$ and thus be free. Therefore, $C_n(S^V)$ must be of the form of a free $\mathbb{Z}[G]$-module, so we can write

$$C_n(S^V) = (\mathbb{Z}[G])^m,$$

for some m, where $\mathbb{Z}[G]$ denotes the group ring of G. In fact,

$$(C_G(S^V)_n)(G/H) = \mathbb{Z}[G]^m$$

for $H=\{e\}$ and is 0 else. Therefore,

$$C_G(S^V)_n\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} = C_0(S^V)_n(G/\{e\})\otimes_{\mathbb{Z}[G]}\mathbb{Z} = \mathbb{Z}^m,$$

since all other $C_0(S^V)_n(G/H)\otimes_{\mathbb{Z}[G]}\mathbb{Z}=0$ for $\{e\}\neq H\subseteq G$ and there are no identifications introduced by morphisms of $\mathscr{O}_G$ coming from inclusions of subgroups.

Now, so far, we have

$$\begin{array}{ccc} \mathbb{Z}^m = C_n(S^V)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} & & C_n(S^V) \\ \Big\downarrow{\scriptstyle d\otimes_{\mathscr{O}_G}\mathbb{Z}} & & \Big\downarrow{\scriptstyle d} \\ C_{n-1}(S^V)\otimes_{\mathscr{O}_G}\underline{\mathbb{Z}} & & C_{n-1}(S^V) \end{array}$$

The next step is to construct horizontal arrows completing the diagram. We shall define, for a functor $F:\mathscr{O}_G^{Op}\to Ab$, a map

$$F\otimes_{\mathscr{O}_G^{Op}}\underline{\mathbb{Z}}\to F(G/\{e\}), \tag{4}$$

and then use it on $F=C_G(S^V)_i$. We attempt to define this map by the one induced by

$$x\otimes\ell\mapsto \ell\cdot\sum_{f:G/e\to G/H} f^*x. \tag{5}$$

However, we need to show that this definition would be consistent. Consistency under composition is immediate. Thus, we need, in particular, to show for subgroups $H'\subseteq H$, and a map $\phi:G/H'\to G/H$, for $x\in F(G/H)$, the map (5) sends $\phi^*(x)\otimes 1\in F(G/H')\otimes_{\mathscr{O}_G^{Op}}\underline{\mathbb{Z}}(G/H')$

and $x\otimes |H|/|H'| \in F(G/H)\otimes_{\mathscr{O}_G^{Op}} \underline{\mathbb{Z}}(G/H)$ to the same element. From the given definition, it sends each to

$$\sum_{f:G/e\to G/H'} f^*\circ \phi^* x = \frac{|G/H'|}{|G/H|}\cdot \sum_{f:G/e\to G/H} f^* x.$$

and

$$|H|/|H'|\cdot \sum_{f:G/e\to G/H} f^* x,$$

respectively, which are equal since $|G/H'| = |H/H'|\cdot|G/H|$.

So, we can define

$$\Psi_i : C_i(S^V)\otimes_{\mathscr{O}_G^{Op}} \underline{\mathbb{Z}} \to C_i(S^V)$$

by (5).

We will show that Ψ_i is injective. This amounts to considering (4) when F is the free abelian group F_S on the representable functor by a G-set S:

$$F_S(G/H) = \mathbb{Z}Map_G(G/H, S).$$

Then (4) takes the form

$$\eta : \mathbb{Z}F_S \otimes_{\mathscr{O}_G^{Op}} \underline{\mathbb{Z}} \to \mathbb{Z}S.$$

Now, we can assume that $S = G/H$ is an orbit. Let $W(H) = N(H)/H$ be the Weyl group of H. Then we have a surjective map

$$\mathbb{Z}(G/H^H)\otimes_{\mathbb{Z}W(H)} \mathbb{Z} \twoheadrightarrow \mathbb{Z}F_{G/H}\otimes_{\mathscr{O}_G} \underline{\mathbb{Z}}.$$

By composing with η, we get a map

$$\mathbb{Z}(G/H^H)\otimes_{\mathbb{Z}W(H)} \mathbb{Z} \to \mathbb{Z}G/H.$$

Then we have $(G/H)^H = W(H)$, so this map is

$$\mathbb{Z} \to \mathbb{Z}(G/H)$$

$$1 \mapsto (1,\dots,1).$$

Therefore, the map must be injective. So, η is injective, too. So, in particular, the Ψ_k's must also be injective.

So, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^m & \xrightarrow{\Psi_n} & (\mathbb{Z}[G])^m \\ \downarrow{\scriptstyle d\otimes_{\mathscr{O}_G}\mathbb{Z}} & & \downarrow{\scriptstyle d} \\ C_{n-1}(S^V)\otimes_{\mathscr{O}_G} \underline{\mathbb{Z}} & \xrightarrow{\Psi_{n-1}} & C_{n-1}(S^V) \end{array} \qquad (6)$$

where the horizontal arrows are injective. By Poincaré duality, we have that $Ker(d) = \{\mu\}$, where

$$\mu \in C_n(S^V)^G = (\mathbb{Z}[G]^m)^G$$

since V is G-orientable. Moreover, μ is the sum of all the top cells with appropriate signs, which are G-compatible since V is orientable. Thus, $\mu = \sum_{g\in G} g(\mu_0) \in \mathbb{Z}[G]^m$, for some $\mu_0 \in \mathbb{Z}^m$. Then, $d \circ \Psi_n(\mu_0) = d(\mu) = 0$. So $x \in Ker(d \otimes_{\mathscr{O}_G} \mathbb{Z})$ if and only if $\Psi_{n-1} \circ (d \otimes_{\mathscr{O}_G} \mathbb{Z})(x) = 0$, which happens if and only if $d \circ \Psi_n(x) = 0$. So, $Ker(d \otimes_{\mathscr{O}_G} \mathbb{Z}) = Ker(d \circ \Psi_n) = \langle \mu_0 \rangle \cong \mathbb{Z}$. So,

$$H_n^G(S^V; \underline{\mathbb{Z}}) = Ker(d \otimes_{\mathscr{O}_G} \mathbb{Z})/im(d \otimes_{\mathscr{O}_G} \mathbb{Z}) = \mathbb{Z}/0 = \mathbb{Z}.$$

This concludes the proof of (1).

Now we will prove (2). Since V is faithful, for $|V| = n$ and $k > 1$, the sets of $(k \cdot n)$-cells and $(k \cdot n - 1)$-cells of S^{mV} are both free G-sets.

Now for a Mackey functor M where $M(G/\{e\})$ is G-fixed, then

$$\begin{array}{ccc} M(G/\{e\}) & \longrightarrow & M(G/\{e\})[G] \\ x & \mapsto & \sum_{g\in G} gx \end{array} \tag{7}$$

is injective, and thus, (7) can be used to construct injective Ψ_{kn}, Ψ_{kn-1} so that the diagram (6) commutes. Thus, the same argument applies. □

Comments:

(1) Note that the same argument works with $\underline{\mathbb{Z}}$ replaced by $\underline{R}$ for any commutative ring R and an R-oriented representation V, which in particular gives new information for $R = \mathbb{Z}/2$, since every representation is $\mathbb{Z}/2$-orientable.

(2) Note that part (1) of the statement of Theorem D also holds for a representation V which is not faithful since if H is the isotropy group, then

$$H_n^G(S^V; \underline{R}) \cong H_n^{G/H}(S^V; \underline{R}).$$

Similarly, part (2) of the statement remains valid when we replace $M_{G/\{e\}}$ with $M_{G/H}$.

3. Proof of Theorem C

In this section, we will state and prove a more precise version of Theorem C.

Let $\mathscr{A}$ denote the universal Mackey functor, i.e. the Burnside ring Green functor, where $\mathscr{A}(G/H)$ is the Burnside ring of H (the group completion of {isomorphism classes of finite H-sets} with respect to $\coprod$). For details, see [8, 12].

By applying Theorem D (2) to $M = \mathscr{A}$, we have

$$\mathbb{Z} \cong \mathscr{A}_{G/\{e\}} = \widetilde{H}^G_{k|V|}(S^{kV}; \mathscr{A}) = H\mathscr{A}_{k|V|-kV}.$$

Then we have

$$1 \in \mathbb{Z} \mapsto e_V \in \widetilde{H}^G_{k|V|}(S^{kV}; \mathscr{A}). \tag{8}$$

We shall call e_V the *weak orientation class*. We will study the ring obtained by inverting the generator e_V in the $RO(G)$-graded coefficients of $H\mathscr{A}$. Clearly, as a spectrum, $H\mathscr{A}$ is an E_∞-ring spectrum, and thus in particular a commutative ring spectrum. So, it is possible to invert e_V.

This construction can be entirely described on chain level (see [22]).

Theorem 1. *Using the notation of (8), for a faithful representation V and a Mackey functor M, we have*

$$e_V^{-1}HM = \operatorname*{hocolim}_m (HM \to \Sigma^{m|V|-mV} HM \to \Sigma^{2m|V|-2mV} HM \to \dots)$$
$$\sim F(EG_+, H(M(G/\{e\}))).$$

Here $H(M(G/\{e\}))$ is considered as a naive G-spectrum, i.e. a non-equivariant spectrum with a G-action.

Proof. First, for a finite G-set X,

$$H_{k+nm|V|-nmV}(X; M) = \widetilde{H}_{k+mn|V|}(S^{nmV} \wedge X_+; M) =$$
$$= H^G_{k+mn|V|}(\widetilde{C}_G(S^{mV} \wedge X_+) \otimes_{\mathscr{O}_G} M).$$

We also have that $S^{nmV} \wedge X_+$ is free in degrees k where $mn(|V|-1) < k \leq mn|V|$ and 0 in degrees $> mn|V|$. Thus, for $k > -mn$,

$$H^G_{k+mn|V|}(\widetilde{C}_G(S^{mV} \wedge X_+) \otimes_{\mathscr{O}_G} M) =$$
$$= H^G_{k+mn|V|}(\widetilde{C}(S^{mV} \wedge X_+) \otimes_{\mathbb{Z}[G]} M(G/\{e\})) =$$
$$= H^G_{k+mn|V|}((\widetilde{C}(S^{mnV}) \otimes C(X)) \otimes_{\mathbb{Z}[G]} M(G/\{e\})).$$

Furthermore, in degrees $> mn(|V|-1)$, $\widetilde{C}(S^{mnV})$ coincides with

$$Hom_{\mathbb{Z}}(R, \mathbb{Z})[mn|V|] = Hom_{\mathbb{Z}[G]}(R, \mathbb{Z}[G])[mn|V|]$$

for a finite type free $\mathbb{Z}[G]$-resolution R of $\mathbb{Z}$. Therefore, for $k > -mn+1$,

$$\begin{aligned} (9) \qquad & H^G_{k+mn|V|}((\widetilde{C}(S^{mnV}) \otimes C(X)) \otimes_{\mathbb{Z}[G]} M(G/\{e\})) = \\ & = H_{k+mn|V|}((C(X) \otimes Hom_{\mathbb{Z}}(R,\mathbb{Z})) \otimes_{\mathbb{Z}[G]} M(G/\{e\})) = \\ & = H_{k+mn|V|}(Hom_{\mathbb{Z}[G]}(C^*X \otimes R, M(G/\{e\})). \end{aligned}$$

Mapping to higher n induces an isomorphism in homology in the given range.

Now on the level of spectra, we have a map

$$(10) \qquad HM \to F(EG_+, HM) \sim F(EG_+, HM(G/\{e\})).$$

By the fact that

$$\widetilde{C}(S^{mV}) \otimes Hom_{\mathbb{Z}}(R,\mathbb{Z}) = Hom_{\mathbb{Z}}(R',\mathbb{Z})[m|V|]$$

for another finite type free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$, we have an equivalence

$$F(EG_+, M(G/\{e\})) \sim e_V^{-1} F(EG_+, M(G/\{e\})),$$

so (10) induces a map

$$e_V^{-1} HM \to F(EG_+, M(G/\{e\})),$$

which, by (9) applied to $X = G/H$, is an equivalence.

□

Corollary 2. *For a finite G-spectrum X,*

$$\begin{aligned} \mathrm{colim}_n\, H^G_{k+nm|V|-nmV}(X;M) = H^{-k}_{G,Borel}(DX; M(G/\{e\})) = \\ = F(EG_+ \wedge DX, HM)_k \end{aligned}$$

where DX is the Spanier-Whitehead dual of X.

The proof of Theorem 1 also gives the following

Corollary 3. *Let M be a Green functor and let*

$$e_V \in H_{|V|-V}(*;M) = H^{V-|V|}(*;M)$$

be a class which restricts to a unit in the ring

$$H^{V-|V|}(G/\{e\};M) = M(G/\{e\}).$$

Then the canonical morphism below is a G-equivalence

$$e_V^{-1} HM \xrightarrow{\sim} F(EG_+, HM) = F(EG_+, HM(G/\{e\})).$$

4. Whitehead Towers of Ring and Module Equivariant Spectra

In this section, we treat some preliminary material needed in the proof of Theorem B. We use the language of triangulated categories and t-structures [2], and also discuss some multiplicative properties.

Recall that the derived category of G-spectra is a triangulated category [2, 25]. Suppose G is a finite group and E is a G-spectrum. Then the k-th homotopy groups of E form a Mackey functor

$$\underline{\pi}_k(E)(G/H) = \pi_k(E^H).$$

This is the "homology theory" associated with a t-structure. Then we have

$$\underline{\pi}_k(\tau_{\geq n}E) = \underline{\pi}_k E$$

for $k \geq n$ and is 0 for $k < n$, and

$$\underline{\pi}_k(\tau_{<n}E) = \underline{\pi}_k E$$

for $k < n$ and is 0 for $k \geq n$.

We also have a distinguished triangle

$$\tau_{\geq 0}E \to E \to \tau_{<0}E \to \tau_{\geq 0}E[1].$$

Also, note that, while $[\tau_{<k}F, \tau_{\geq k}E]$ can be non-trivial,

$$[\tau_{\geq k}E, \tau_{<k}F] = 0, \tag{11}$$

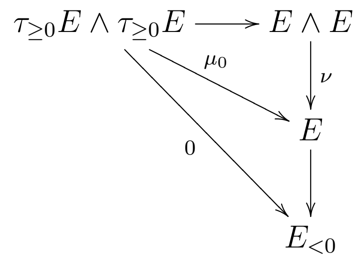
(where $[?, ?]$ denotes the abelian group of morphisms in our triangulated category, i.e. the derived category of G-spectra). In addition, for $k > \ell$, we have

$$\tau_{<k}\tau_{<\ell} = \tau_{<\ell}. \tag{12}$$

Lemma 4. *(See also Dugger [9], §4) If E is a ring spectrum, then $\tau_{\geq 0}E$ is also a ring spectrum and $\tau_{\geq n}E$ is a $\tau_{\geq 0}E$-module spectrum (in this paper, only commutative ring spectra are considered). If E is a ring spectrum and K is an E-module spectrum, then $\tau_{\geq n}K$ is a $\tau_{\geq 0}E$-module spectrum.*

Proof. In the derived category of G-spectra, we have a diagram

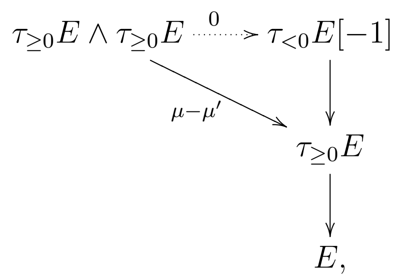
(13)



by (12), where ν denotes the operation of E. Thus, μ_0 lifts to a map

$$\mu : \tau_{\geq 0}E \wedge \tau_{\geq 0}E \to \tau_{\geq 0}E.$$

Note that μ is unique. This is because if there is another such lift μ', then we have the diagram

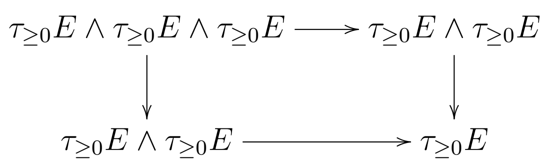


by the distinguished triangle

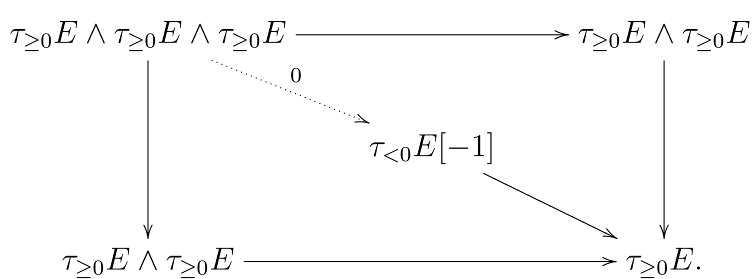
$$\tau_{<0}E[-1] \to \tau_{\geq 0}E \to E \to \tau_{<0}E$$

and the fact that $[\tau_{\geq 0}E \wedge \tau_{\geq 0}E, \tau_{<0}E[-1]] = 0$. The associativity diagram

(14)



commutes by a similar reason using the diagram



The cases of the unit and commutativity are analogous.

The case of $\tau_{\geq n}E$ is analogous.

The argument for modules precisely mimics the proof of Lemma 4 with one E-coordinates replaced by K. $\square$

Suppose that V is an orientable faithful representation of G.

Lemma 5. *Any morphism of G-spectra*

$$\widetilde{e}_V : S^0 \to S^{V-|V|} \wedge E$$

lifts to a morphism of G-spectra

$$S^0 \to (S^{V-|V|} \wedge \tau_{\geq 0}E).$$

Proof. We have a distinguished triangle

$$S^{V-|V|} \wedge \tau_{\geq 0}E \to S^{V-|V|} \wedge E \to S^{V-|V|} \wedge \tau_{<0}E \to S^{V-|V|}\tau_{\geq}E[1]. \tag{15}$$

Then we have

$$\begin{array}{ccc} & & S^{V-|V|} \wedge \tau_{\geq 0}E \\ & & \downarrow \\ S^0 & \xrightarrow{\widetilde{e}_V} & S^{V-|V|} \wedge E \\ & \searrow^{\varphi} & \downarrow \\ & & S^{V-|V|} \wedge \tau_{<0}E. \end{array}$$

Then $\varphi = 0$, by (11), since $S^{V-|V|} \wedge \tau_{<0}E$ only has cells in degrees ≤ 0.

So, by the long exact sequence in homotopy classes corresponding to (15), we get a lifting

$$\begin{array}{ccc} & \nearrow & S^{V-|V|} \wedge \tau_{\geq 0}E \\ & & \downarrow \\ S^0 & \xrightarrow{\widetilde{e}_V} & S^{V-|V|} \wedge E. \end{array}$$

$\square$

Remark: J. P. C. Greenlees pointed out that if G is a (finite) group acting freely on a sphere $S(V)$ and $a_V : S^0 \to S^V$ is the map defined by sending the non-base point to the point at infinity, we can write

$$F(EG_+, HM) \simeq HM^{\wedge}_{(a_V)} \tag{16}$$

which, in view of Corollary 3, leads to the curious formula

$$e_V^{-1}HM \simeq F(EG_+, HM) \simeq HM^{\wedge}_{(a_V)}. \tag{17}$$

The right hand side of (16) is more precisely defined to be the homotopy limit over n of the mapping spectrum of the homotopy fiber of

$$a_V^n : S^0 \to S^{nV},$$

which is $S(nV)_+$, into HM, so (16) follows from the fact that

$$EG_+ = \operatorname*{hocolim}_n S(nV)_+.$$

On the chain level, since $C_*S(nV)$ is a chain complex of free $\mathbb{Z}[G]$-modules, we obtain a chain map

$$C_*(S(nV)) \to C_*(EG) = C_*(S(\infty V))$$

and hence a map

$$C^*(EG) \to C^*(S(nV)) \tag{18}$$

which is, in fact, (using this model of EG) an isomorphism in degrees $> -n|V|$, and thus will be an isomorphism after taking limits. On the other hand, the chain complex $\widetilde{C}_*S^{nV}$ is $\mathbb{Z}[G]$-free in degrees > 0 and we have also constructed a map

$$\widetilde{C}_*(S^{nV})[-nV] \to C^*(EG) \tag{19}$$

which then becomes an equivalence after taking (homotopy) colimits.

To see why in (17) we have an equivalence between a homotopy limit and homotopy colimit, recall that for a general chain complex C, we have chain maps

$$\tau_{\geq m}C \to C \to C_{\geq m} \tag{20}$$

where $C_{\geq m}$ is the "stupid filtration," i.e. taking terms of degree $\geq m$ and 0 elsewhere, while $\tau_{\geq m}C$ is the "Whitehead filtration" which induces an isomorphism in homology in degrees $\geq m$. We will discuss the Whitehead filtration in more detail in the next section. In (20), C becomes a quasi-isomorphism when taking the homotopy colimit of the left hand term $\tau_{\geq m}C$ as well as the homotopy limit of the right hand term $C_{\geq m}$ with $m \to -\infty$. In the current setting, (19) represents (up to quasi-isomorphism) the Whitehead filtration $\tau_{\geq -n|V|+1}C^*(EG)$ while (20) represents the "stupid filtration" $C^*(EG)_{\geq -n|V|+1}$.

5. Proof of Theorem B

We shall now restate Theorem B in more detail and give a proof. First, let us state our assumption. Let L be a G-spectrum.

Assumption A: There exists an $r < \infty$ such that for every G-space X, the equivariant Atiyah-Hirzebruch spectral sequence converging to L_*X collapses to E^r.

Theorem 6. *Suppose there exists an* $\widetilde{e}_V \in \widetilde{E}_{|V|}(S^V) = E_{|V|-V}(*)$ *such that*

$$(\nu \wedge S^{V-|V|}) \circ (E \wedge \widetilde{e}_V) : E \xrightarrow{\sim} E \wedge S^{V-|V|}$$

is an equivalence. If $L = E$ *satisfies Assumption A, then there exists a decreasing filtration* F^i *on* $\pi_*^G E$ *such that*

(1)
$$\lim_i \pi_*^G E / \pi_*^G F^i E \cong \pi_*^G F(EG_+, E).$$

(2)
$$F^0 \pi_*^G E = \pi_*^G E$$

and $F^1 \pi_*^G E$ *is the augmentation ideal of* $\pi_*^G E$ *(i.e. the kernel of the restriction* $\pi_*^G E \to \pi_*^{\{e\}} E$*).*

(3) F *is a filtration of rings (i.e.* $F^i \cdot F^j \subseteq F^{i+j}$*).*

Suppose K *is an* E*-module spectrum. If* $L = K$ *satisfies Assumption A (without necessarily assuming Assumption A for* $L = E$*), then there exists a decreasing filtration* F^i *on* $\pi_*^G K$ *such that*

(4)
$$\lim_i \pi_*^G K / \pi_*^G F^i K \cong \pi_*^G F(EG_+, K).$$

(5)
$$F^0 \pi_*^G K = \pi_*^G K$$

(6) F *is a filtration of* E*-modules (i.e.*

$$F^i(\pi_* E) \cdot F^j(\pi_* K) \subseteq F^{i+j}(\pi_* K)).$$

We will first discuss the ring case. Observe that we have a homotopy decreasing filtration

$$\cdots \to \tau_{\geq n+1} E \to \tau_{\geq n} E \to \tau_{\geq n-1} E \to \dots \tag{21}$$

Its homotopy limit is 0 because, by taking cofibers of the canonical maps to E, we obtain a sequence

$$\to \tau_{<n+1} E \to \tau_{<n} E \to \tau_{<n-1} E \to \dots$$

whose homotopy limit is E. Put

$${}_{(0)}\mathscr{F}^i = \tau_{<i} E.$$

Then by applying $\widetilde{e}_V$ to the filtration (21) repeatedly we get a sequence of filtrations

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\geq n+1}E & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n+1}E)\wedge S^{V-|V|} & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n+1}E)\wedge S^{2(V-|V|)} \to \dots \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\geq n}E & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n}E)\wedge S^{V-|V|} & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n}E)\wedge S^{2(V-|V|)} \to \dots \\
\downarrow & & \downarrow & & \downarrow \\
\tau_{\geq n-1}E & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n-1}E)\wedge S^{V-|V|} & \xrightarrow{\widetilde{e}_V} & (\tau_{\geq n-1}E)\wedge S^{2(V-|V|)} \to \dots \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

Now each vertical homotopy filtration individually has homotopy limit 0, since again by taking cofibers of the canonical maps into $E = E \wedge S^{j(V-|V|)}$, we obtain a sequence

$$_{(j)}\mathscr{F}^i = (\tau_{<i}E) \wedge S^{j(V-|V|)}$$

whose homotopy limit in i is E.

Lemma 7. *We have*

$$\begin{aligned} _{(\infty)}\mathscr{F}^i &= \operatorname*{hocolim}_j((\tau_{<i}E)\wedge S^{j(V-|V|)}) \\ &\sim F(EG_+, \tau_{<i}E). \end{aligned}$$

Proof. This follows from the fact that EG is a G-CW-complex of finite type. Thus, the canonical map from the left to the right induces an isomorphism of homotopy groups by Theorem 1. □

Proof of Theorem 6. Denote by $_{(j)}E$ the spectral sequence in homotopy groups based on the homotopy filtrations $_{(j)}\mathscr{F}$ (which is a shift of the regrading of the equivariant Atiyah-Hirzebruch spectral sequence for S^V following the argument of [13], Appendix B). Then $_{(j)}E$ collapses to some $_{(j)}E^r_{*,*}$ for the same r by Assumption A, and hence $_{(\infty)}E$ also collapses to E^r.

By Lemma 7, however, $_{(\infty)}E$ converges conditionally to $\pi_*F(EG_+, E)$, and thus converges strongly to $\pi_*F(EG_+, E)$ (see [4]).

In other words,

$$\pi^G_n F(EG_+, E) = \lim_i \operatorname{Im}(\pi^G_n F(EG_+, E) \to \pi^G_n(_{(\infty)}\mathscr{F}^i)).$$

On the other hand, $\mathrm{Im}(\pi_n^G F(EG_+, E) \to \pi_n^G({}_{(\infty)}\mathscr{F}^i))$ is an extension of finitely many terms ${}_{(\infty)}E^{\infty}_{p,q}$ and thus is equal to

$$\operatorname*{colim}_j(\mathrm{Im}(\pi_n^G E \to \pi_n^G({}_{(j)}\mathscr{F}))) = \mathrm{Im}(\pi_n^G E \to \pi_n^G({}_{(\infty)}\mathscr{F})).$$

The multiplicativity on the level of spectral sequences follows from the arguments of Blumberg-Mandell [3], Section 4. See also Dugger [9], §4. The paper [15] by Hedenlund, Krause, Nikolaus contains a treatment in the language of ∞-categories.

This concludes the proof of (1), (2), (3). The proof of (4), (5), (6) is analogous by replacing E by K in Lemma 7.

□

Example: Now we will present a counterexample to the statement of Theorem 6 when we omit Assumption A. Let $G = \mathbb{Z}/2$. By the results of Löffler [26] and Comezaña [27], the coefficient ring $(MU_{\mathbb{Z}/2})_*$ is a flat MU_*-module concentrated in even degrees. Hence

$$(MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)_*^{\mathbb{Z}/2}$$

is concentrated in even degrees, while

$$F(E\mathbb{Z}/2_+, MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)_* = H^*(B\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2[a], \tag{22}$$

where the degree of a is 1. In particular, (22) is non-zero in odd degrees and hence the conclusion of Theorem 6 is false.

To explain precisely what is happening, we have ([31, 16])

$$(MU_{\mathbb{Z}/2})_* = MU_*[b_{i,j}, q_j | i, j \in \mathbb{N}_0]/\sim$$

where, writing $u = b_{0,0}$, the relations are

$$\begin{gathered} q_0 = 0 \\ b_{0,1} = 1, b_{0,j} = 0 \text{ for all } j \geq 2 \\ q_j - c_j = uq_{j+1} \\ b_{i,j} - a_{i,j} = ub_{i,j+1}. \end{gathered}$$

Here we write

$$x +_F y = \sum_{i,j \geq 0} a_{i,j} x^i y^j$$

$$c_k = \sum_{i+j=k} a_{i,j}.$$

(so $[2]_F x = \sum_{j \geq 0} c_j x^j$), where F is the universal formal group law.

Hence,

$$(MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)_* = \mathbb{Z}/2[u, q_j, b_{i,j} | i \geq 1, j \geq 0]/\sim$$

where the relations are

$$q_0 = 0,\ q_j = uq_{j+1}$$
$$b_{i,j} = ub_{i,j+1}.$$

We also have (for example by the arguments of [13])

$$\begin{aligned} F(E\mathbb{Z}/2_+, MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2) = (MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)^{\wedge}_{u} = \\ = \operatorname*{holim}_{s}(MU_{\mathbb{Z}/2} \wedge_{MU} H\mathbb{Z}/2)/u^s \end{aligned} \tag{23}$$

The generators $b_{i,j}$, $i \geq 1$ are uniquely divisible by u, and thus will not contribute to (23); for this reason, we shall ignore them. Factoring them out (which we could also do on the spectral level), we obtain the ring

$$R = \mathbb{Z}/2[u, q_{i,j} | j \geq 0]/(q_0 = 0,\ uq_{j+1} = q_j)$$

which additively is

$$\mathbb{Z}/2[u] \oplus \mathbb{Z}/2\{q_j | j > 0\}. \tag{24}$$

(We have degrees $|u| = -2$, $|q_j| = 2j$.)

Non-equivariantly, we just have $\mathbb{Z}/2$ in degree 0, so on the Mackey functor level, we have (24) corresponds to

$$\underline{\mathbb{Z}/2} \oplus \Phi\{u^j | j \geq 0\} \oplus \Phi\{q_j | j > 0\}$$

where Φ is the Mackey functor given by

$$\Phi(\mathbb{Z}/2/\mathbb{Z}/2) = \mathbb{Z}/2,\ \Phi(\mathbb{Z}/2/\{e\}) = 0.$$

Let α be the real sign representation. We have ([17], Section 6)

$$\begin{aligned} (H\underline{\mathbb{Z}/2} \wedge S^{2j\alpha - 2j})_\ell &= \begin{cases} \mathbb{Z}/2, & \text{if } -2j \leq \ell \leq 0 \\ 1, & \text{else} \end{cases} \\ (H\Phi \wedge S^{2j\alpha - 2j})_\ell &= \begin{cases} \mathbb{Z}/2, & \text{if } \ell = -2j \\ 1, & \text{else} \end{cases} \end{aligned} \tag{25}$$

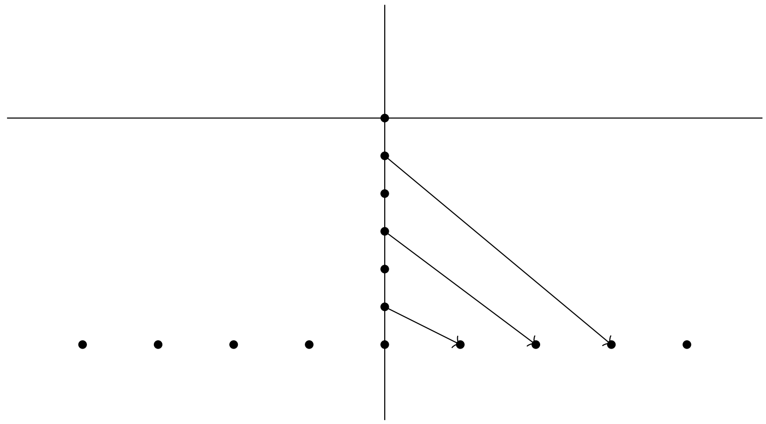
(note that $H\Phi$ is α-periodic since $H\Phi_{\{e\}} = 0$). Thus, the spectral sequence corresponding to ${}_{(j)}\mathscr{F}$ has

$$E_2^{-2j,2\ell} = \mathbb{Z}/2, \text{ for } \ell \in \mathbb{Z}$$
$$E_2^{0,\ell} = \mathbb{Z}/2, \text{ for } -2j \leq \ell \leq 0$$
$$E_2^{p,q} = 0 \text{ else.}$$

Since it converges to (24), we have

$$d_{2s} : E_2^{0,2s-1-2j} \xrightarrow{\cong} E_2^{2s-2j}$$

for $1 \leq s \leq j$. In the case of $j = 3$, $E_2^{p,q}$ looks like



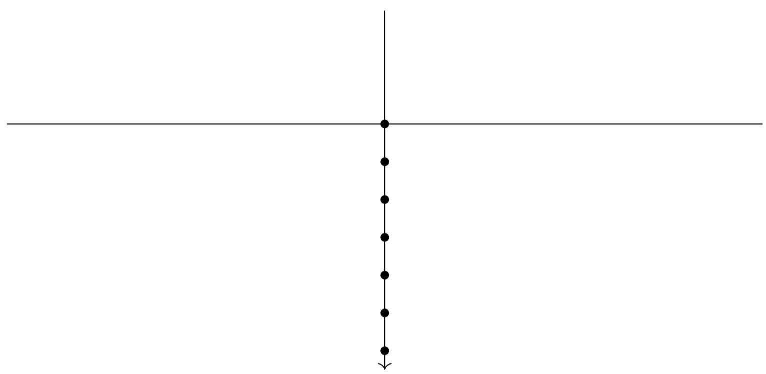
where q is the vertical axis and p is the horizontal one.

We see that in fact the colimit ${}_{(\infty)}\mathscr{F}$ has

$$E_2^{0,\ell} = \mathbb{Z}/2 \text{ for } \ell \leq 0$$

$$E_2^{p,q} = 0 \text{ else}$$

which collapses to $H(B\mathbb{Z}/2;\mathbb{Z}/2)$. $E_2^{p,q}$ looks like



Due to the increasing lengths of differentials in this example, colimit of a sequence of spectral sequences does not commute with abutment.

Comment: The Example well illuminates the fact that Theorem B without Assumption A only fails due to the fact that the passage from ${}_{(j)}\mathscr{F}$ to ${}_{(\infty)}\mathscr{F}$ fails to preserve convergence and abutment. The odd-degree elements in the above example seem like a derived functor of the process. Thus, we expect that some verion of Theorem B without Assumption A holds where higher derived functors are taken into account, but we do not pursue that here.

6. The Proof of Theorem A

To prove Theorem A, the main step in the following

Proposition 8. *The MU_G-module spectrum $MU_G \wedge_{MU} K(n)$ satisfies Assumption A.*

First, let us state a general observation about lengths of differentials:

Lemma 9. *Let E be a G-spectrum and let*

$$X \to Y \to Z \tag{26}$$

be a cofibration sequence for G-spaces. If the E-Atiyah-Hirzebruch (homological or cohomological) spectral sequence for X, Z collapse to E^r, resp E^s, then the E-Atiyah-Hirzebruch spectral sequence for Y collapses to E^{r+s}.

Proof. For any spectral sequence associated to a homological exact couple (D, E), the maximum length of a differential is equal to the maximum difference $p_2 - p_1$ where $p_1 \leq p_2$ and there exists an element $x \in D_{p_2}$ which is in the image of D_{p_1} but not in the image of D_{p_2-1}.

To prove the statement of the Lemma, we can assume that (Y, X) is a CW-pair where the filtration is induced by the skeletal filtration. Considering, say, the cohomological spectral sequence, the non-trivial case is an element of D supported by a cell of X_p. If the element extends to $X_{p'}$ but not $X_{p'+1}$, but does extend to $Y_{p'+1}$, then this determines an element y of the Z-Atiyah-Hirzebruch spectral sequence supported by a $(p'+1)$-cell. Therefore, the target of a differential on x will be supported by a Z-cell and hence the target of a differential of y.

$\square$

Now we also have the following

Lemma 10. *Assumption A holds for a G-spectrum E if it holds for the $W(H)$-spectra E^H for every $H \subseteq G$ and every free $W(H)$-space Y.*

Proof. This is by Lemma 9 using isotropy separation (see, e.g. [21], Propositions 4,5). $\square$

Finally, we claim

Lemma 11. *Let L be a G-equivariant module over the associative ring spectrum $K(n)$ (with trivial G-action) where G is a finite p-group. Then there exists a constant r such that for a free G-CW complex X, the L-cohomology Atiyah-Hirzebruch spectral sequence collapses to the E_r-term and is 0 in filtration degrees $p > r$.*

Proof. First, we consider the case $L = K(n)$. Considering an extension of the form

$$1 \to H \to G \to \mathbb{Z}/p \to 1,$$

we obtain a fibration sequence

$$X \times_H EH \longrightarrow X \times_G EG \xrightarrow{f} B\mathbb{Z}/p \tag{27}$$

Ravenel [28] proves that the periodicity element $x \in H^2(B\mathbb{Z}/p)$ in the cohomological Serre spectral sequence corresponding to (27) satisfies $x^{p^n} = 0$.

Considering the fact that the Serre spectral sequence is obtained from the fibration $f^{-1}B_n$ where B_n is the cellular filtration of the base, and further considering the cellular filtration of the pre-image of each cell, we can reduce the statement for $X \times_G EG$ to the above stated result of [28] and to the same statement for $X \times_H EH$. This gives the required result by induction.

Now recall that if V is any $\mathbb{Z}/p$-representation in characteristic p, and we put recursively

$$V_{i+1} = V_i/(V_i)^G,$$

then there exists a constant N only dependent of G such that $V_N = 0$. (It suffices to prove this for $G = \mathbb{Z}/p$, and to prove the constant is uniform for V finite dimensional. In this case, extend scalars to $\overline{\mathbb{F}_p}$ and consider the Jordan decomposition of the generator; the eigenvalue is 1 and all Jordan blocks are well-known to decompose into Jordan blocks of size $\leq p$ as representations.)

Now for a general L, the L-Atiyah-Hirzebruch spectral sequence for X gives a module over the $K(n)$-cohomology Atiyah-Hirzebruch spectral sequence. On the level of E_2-terms, further, from the above results, we see that it is generated by the corresponding $K(n)$-E_2-term in filtration degrees $\leq N$. Thus, the result follows from the statement for $K(n)$.

□

We derive the Proposition as follows:

Proof of Proposition 8. By Lemma 10, this is reduced to the case of free G-spaces X. Due to duality, it does not matter whether we work in homology or cohomology.

Now the standard transfer argument reduces the statement too the case when G is a p-group (by replacing G by its p-Sylow subgroup). Now note that $(MU_G \wedge_{MU} K(n))^H$, as a naive $W(H)$-spectrum (over the trivial universe) is a module over the fixed spectrum $K(n)$. Thus, the Proposition follows from Lemma 11.

□

Then Theorem A results as follows:

Proof of Theorem A. We will proceed by contradiction. Assume that the P-equivariant coefficients $(MU_P)_*$ form a flat MU_*-module concentrated in even degrees. Consider the ring spectrum MU_P and the MU_P-module spectrum $K(n) \wedge_{MU} MU_P$, where $K(n)$ denotes Morava K-theory. (This can be formed since MU_P is an E_∞-algebra over the pushforward of the E_∞-ring spectrum MU, over which $K(n)$ is an E_∞-module.) We then obtain by our flatness assumption that

$$(K(n) \wedge_{MU} MU_P)_* = K(n)_* \otimes_{MU_*} (MU_P)_*$$

is concentrated in even degrees.

On the other hand, by Propostion 8, $K(n) \wedge_{MU} MU_P$ satisfies Assumption A, and hence, we can apply Theorem 6, to obtain there exists a suitable completion satisfying

$$((K(n) \wedge_{MU} MU_P)_*)^\wedge \cong K(n)^* MU$$

Therefore, $K(n)^* MU$ is also concentrated in even degrees. However, this is a contradiction with the results of [19, 20], which state that $K(2)^* MU$ contains non-zero elements in odd degrees for G the p-Sylow subgroup of $GL_4(\mathbb{F}_p)$ for $p > 2$.

□

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