ON SEMISIMPLICITY AND DEFORMATIONS OF QUASI-PRE-TANNAKIAN CATEGORIES

SOPHIE KRIZ

1. Introduction

In this paper, we give a new formalism for describing $\mathbb{C}$-linear additive pseudo-abelian categories with bilinear associative, commutative, unital tensor product and strong duality, generated by a single "basic object" X of dimension c. We call these *quasi-pre-Tannakian (QPT) categories* (see [12]). Among then, the most interesting examples are semisimple pre-Tannakian categories, which have been investigated extensively (see e.g. [2, 3, 4, 5, 8, 9, 10, 11, 13, 16]).

The point of considering QPT categories generated by a single object X is that they are easier to construct; in fact, QPT categories can be completely described by a universal algebra structure called *T_c-algebra*, which we described in [12, 13] (and will review in Section 3).

In this paper, we give a new reformulation of the T_c-algebra formalism for $c \notin \mathbb{N}_0$ and as an application, we prove a rigidity theorem. In [3], Chapter 10, P. Deligne introduced a semisimple pre-Tannakian category of "algebraic representations" of $GL(c)$ (see also [4]). In this paper, we will use a variant of this category which has coproducts, and which we denote by $\underline{Rep}(GL(c))$. Our first main result is

Theorem 1.1. *For $c \in \mathbb{C} \smallsetminus \mathbb{Z}$, there is an equivalence of categories Φ between the category of T_c-algebras and the category of associative, commutative, unital algebras in $\underline{Rep}(GL(c))$.*

As an application, we study deformations of T_c-algebras.

Theorem 1.2. *Let $c \in \mathbb{C} \smallsetminus \mathbb{Z}$. Let $\mathscr{C}$ be a QPT category whose corresponding $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$ is generated by a $\underline{Rep}(GL(c))$-object $\mathcal{G}$ with relations ideal $\mathcal{J}$. Then the $\mathbb{C}$-vector space of infinitesimal deformations of $\mathscr{C}$ is the $Y_{\emptyset,\emptyset}$ part of the dual of the kernel of the differential $\mathcal{A}$-module homomorphism from $\mathcal{A} \otimes_{Sym(\mathcal{G})} \mathcal{J}$ to the free $\mathcal{A}$-module on $\mathcal{G}$.*

This will be made precise in Section 4, see Theorem 4.5. The proof uses the methods of [7, 17].

Additionally, we investigate conditions for a QPT category $\mathscr{C}$ to be semisimple in terms of the $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$. We call an algebra in $\underline{Rep}(GL(c))$ *field-like* if it has no non-trivial ideal except 0 and itself. Further, we define a $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$ to be *étale* when the category of its finitely generated modules in $\underline{Rep}(GL(c))$ is semisimple. We call $\mathcal{A}$ *locally finite* if it has a finite dimensional $\mathbb{C}$-vector space of summands over each $Y_{\lambda,\mu}$. We call $\mathcal{A}$ *finitely presented* if it can be expressed as the quotient of the symmetric algebra on a finite set of generators over some chosen summands Y_{λ_i,μ_i}.

Theorem 1.3. *Suppose $\mathcal{A}$ is a locally finite field-like $\underline{Rep}(GL(c))$-algebra. Then the quasi-pre-Tannakian category determined by the T_c-algebra corresponding to $\mathcal{A}$ is a semisimple (and therefore, pre-Tannakian) category if and only if $\mathcal{A}$ is étale.*

In fact, we prove that these conditions are equivalent to the two categories being equivalent, which is equivalent to the category of finitely generated $\mathcal{A}$-modules having strong duality. (See Theorem 5.10 below).

In fact, we prove a finiteness theorem for infinitesimal deformations of semisimple T_c-algebras:

Theorem 1.4. *For a finitely-presented étale $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$, the module of deformations of $\mathcal{A}$ is finitely generated.*

We will see an example of a T_c-algebra satsifying the conditions of Theorem 1.4 where this space of deformations is non-trivial (see Example 5.11).

Conjecture 1.5. *Every étale $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$ is finitely presented.*

To prove Theorem 1.1, we use the category

$$FB_c^{\pm} = Rep_0(GL(c)) \tag{1.1}$$

defined by P. Deligne [3], Definition 10.2 (we denote it by $FB_c^{\pm}$ in the context of the categories considered in [14]). P. Deligne [3], Chapter 10, also introduced a category $Rep_1(GL(c))$ obtained from $Rep_0(GL(c))$ by adding finite direct sums formally and a category $Rep(GL(c))$ which is its pseudo-abelian envelope. However, we may also adjoin formal infinite sums (and morphism matrices with finitely many non-zero entries in each row) to obtain a variant $\underline{Rep}(GL(c))$ with infinite direct sums. (We will need this extension because algebras are typically not finitely generated additively.) $\underline{Rep}(GL(c))$ inherits a tensor product from $Rep_0(GL(c))$. Further, if we denote by $FB_c^{\pm}$-Mod the category of additive functors from (1.1) to $\mathbb{C}$-vector spaces, we have, essentially, for formal reasons, an equivalence of tensor categories

$$\underline{Rep}(GL(c)) \simeq FB_c^{\pm}\text{-Mod}, \tag{1.2}$$

where on the right-hand side, the tensor product is the "Day product." We then prove Theorem 1.1 by establishing an equivalence of categories between commutative algebras in the category $FB_c^{\pm}$-Mod and T_c-algebras, which amounts essentially to reinterpreting the axioms.

The remainder of the paper is based on the philosophy of doing algebraic geometry in the tensor category $\underline{Rep}(GL(c))$. We begin to compare this world to the classical situation and observe some similarities, but also some differences. For example, for general formal reasons, infinitesimal deformations are controlled by $Exalcomm$, which can be interpreted as first Quillen cohomology. On the other hand, unlike the classical context, the tensor category $\underline{Rep}(GL(c))$ is not generated by the unit object. In such tensor categories, commutative algebras can encode non-commutative structures, due to the tensor product of simple objects having several isomorphic summands. For example, when discussing the semisimplicity of a QPT category generated by a basic object X, this is equivalent to the semisimplicity of the algebras $End(X^{\otimes n})$, which are typically not commutative. Accordingly, the derived cotangent complex behaves somewhat differently than in the classical case. For example, say over $\mathbb{C}$, an étale algebra, which is the same as a semisimple algebra, is a finite product of copies of $\mathbb{C}$, so its derived cotangent complex is 0. This is no longer true in $\underline{Rep}(GL(c))$. Similarly, an étale algebra over $\mathbb{C}$ has no deformations (as is the case for semisimple non-commutative algebras). However, semisimple algebras in $\underline{Rep}(GL(c))$ can have deformations, for example, the algebra corresponding to the category $\underline{Rep}(GL(a)\times GL(c-a))$ for $a, c-a \notin \mathbb{Z}$ (Example 5.11). This makes the statement of Theorem 1.4 non-trivial.

The present paper is organized as follows: In Section 2, we review the definition of [3], Chapter 10, and explain the equivalence of tensor categories (1.2). We also discuss the equivalence between the full subcategory $\underline{Rep}(GL(c))_0 \subseteq \underline{Rep}(GL(c))$ on objects of weight 0 with the category of $FB^{\pm}_{c,0}$-modules where $FB^{\pm}_{c,0} \subseteq FB^{\pm}_c$ is the full subcategory on the objects (n, n).

In Section 3, we define T_c-algebras and prove an equivalence of categories between T_c-algebras (resp. graded T_c-algebras) and commutative algebras in the category of $FB^{\pm}_c$-modules (resp. $FB^{\pm}_{c,0}$-modules), completing the proof of Theorem 1.1.

In Section 4, we then use Theorem 1.1 to define and calculate the deformation module of a T_c-algebra using Quillen cohomology in the category $\underline{Rep}(GL(c))$.

In Section 5, we define locally finite, finitely presented, field-like, and étale $\underline{Rep}(GL(c))$-algebras. We use the calculations in Section 4 to conclude Theorem 1.4 and restate and prove the full statement of Theorem 1.3.

2. The Category of $FB^{\pm}_c$-Modules and $\underline{Rep}(GL(c))$

The purpose of this section is essentially to review the definitions of [3], Chapter 10 and to formulate the results in a context close to making the connection with T_c-algebras. In Subsection 2.1, we introduce the category (1.1), and the concept of an $FB^{\pm}_c$-module. In Subsection 2.2, we introduce the "Day product" (or convolution product) on $FB^{\pm}_c$-Mod. Theorem 2.3 below recapitulates the equivalence of tensor categories (1.2).

2.1. **The Category of $FB^{\pm}_c$-Modules.** We define the category $FB^{\pm}_c$ by taking the objects

$$Obj(FB^{\pm}_c) = \{(m, n) \mid m, n \in \mathbb{N}_0\},$$

and morphisms freely generated by bijections

$$Hom_{FB^{\pm}_c}((m, n), (p, q)) = \mathbb{C}Mor_{FB}([m] \amalg [q], [p] \amalg [n]) \tag{2.1}$$

We graphically represent a bijection

$$[m] \amalg [q] \to [p] \amalg [n] \tag{2.2}$$

corresponding to a generator of by drawing a set of m dots above a set of n dots on the left, and a set of p dots above a set of q dots on the right, and drawing lines or curves to connecting elements of the source

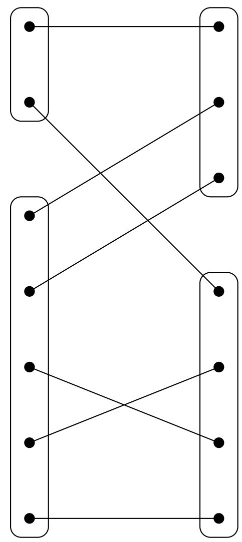

Figure 1. A bijection $[m] \amalg [q] \to [p] \amalg [n]$, for $m = 2$, $n = 4$, $p = 3$, $q = 5$. The words "source" and "target" refer to the corresponding morphism of $FB_c^{\pm}$.

of (2.2) with their images. A general morphism in $FB_c^{\pm}$ can then be described as a linear combination of such diagrams.

Composing two morphisms in $FB_c^{\pm}$ can then be described by placing their two corresponding diagrammatic representations side by side, connecting the corresponding points of the intermediate object with each other, composing all possible lines and curves, and replacing any loops in the resulting diagram by a coefficient of c, (as in the diagrammatic description of composition in the category $\underline{Rep}(GL(c))$ given in [3], Section 10.1). For a generalization of this construction with further applications, see [14].

Figure 1 shows an example of a bijection $[m] \amalg [q] \to [p] \amalg [n]$, and Figure 2 shows the graphical representation of its corresponding morphism of $FB_c^{\pm}$. (Note that, in particular, the Hom-space $Hom_{FB_c^{\pm}}((m,n),(p,q))$ is non-zero if and only if $m+q = n+p$.)

We define $FB_{c,0}^{\pm}$ to be the full subcategory of $FB_c^{\pm}$ on objects

$$Obj(FB_{c,0}^{\pm}) = \{(m,m) \mid m \in \mathbb{N}_0\} \subset Obj(FB_c^{\pm}).$$

Define $FB_c^{\pm}$-modules, resp. $FB_{c,0}^{\pm}$-modules, to be functors

$$FB_c^{\pm} \to Vect,$$

resp. functors

$$FB_{c,0}^{\pm} \to Vect.$$

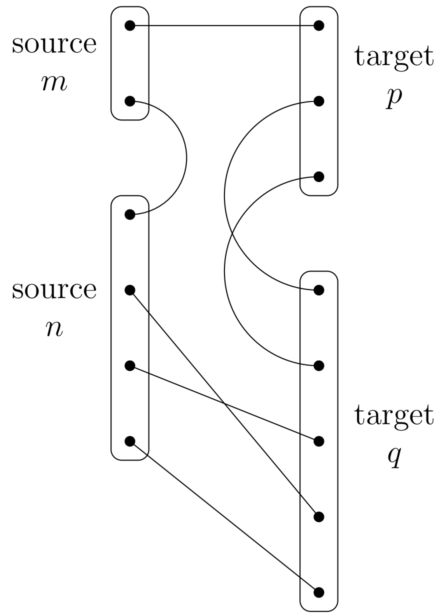

FIGURE 2. A generator of $Mor_{FB_c^{\pm}}((m,n),(p,q))$, for $m=2$, $n=4$, $p=3$, $q=5$, corresponding to the bijection pictured in Figure 1.

We define the category $FB_c^{\pm}$-Mod (resp. $FB_{c,0}^{\pm}$-Mod) to be the category of $FB_c^{\pm}$-modules (resp. $FB_{c,0}^{\pm}$-modules) and natural transformations.

2.2. **Structure as a Symmetric Tensor Category.** The category of $FB_c^{\pm}$-modules has a tensor category structure called the *Day product*. The Day product $M \otimes N$ of $FB_c^{\pm}$-modules M, N is defined as the left Kan extension

$$\begin{array}{ccc} FB_c^{\pm} \times FB_c^{\pm} & \xrightarrow{M \overline{\otimes} N} & Vect \\ \downarrow & \overset{M \otimes N}{\nearrow} & \\ FB_c^{\pm} & & \end{array}$$

where the vertical map is

$$FB_c^{\pm} \times FB_c^{\pm} \to FB_c^{\pm}$$
$$((m,n),(p,q)) \mapsto (m+p, n+q)$$

and the horizontal map is

$$FB_c^{\pm} \times FB_c^{\pm} \to Vect$$
$$((m,n),(p,q)) \mapsto M(m,n) \otimes_{\mathbb{C}} N(p,q)$$

Theorem 2.3. *We have an equivalence of tensor categories (1.2). In particular, the images $\mathcal{Y}_{\lambda,\mu}$ of $Y_{\lambda,\mu}$ under this equivalence multiply (with respect to the Day product) by the Littlewood-Richardson rule. Further, (1.2) restricts to an equivalence of categories between $FB^{\pm}_{c,0}$-Mod and the full subcategory $\underline{Rep}(GL(c))_0$ of $\underline{Rep}(GL(c))$ on objects $Y_{\lambda,\mu}$ with $|\lambda| = |\mu|$.*

Proof. To construct the equivalence (1.2), we send $Y_{\lambda,\mu}$ to the functor

$$(m,n) \mapsto Hom_{\underline{Rep}(GL(c))}(Y_{\lambda,\mu}, X^{\otimes m} \otimes (X^{\vee})^{\otimes n}) \tag{2.3}$$

(where X is the "basic object" [3]). The fact that the tensor product of $GL(c)$-representations corresponds to the Day product is due to the definition of matrix multiplication. The other statements are given by just reading off the definitions on either side of the correspondence. □

By an $FB^{\pm}_c$-*algebra* (resp. $\underline{Rep}(GL(c))$-algebra), we shall mean a commutative, associative, unital algebra in the symmetric $\otimes$-category $FB^{\pm}_c$-Mod (resp. $\underline{Rep}(GL(c))$. The category of $FB^{\pm}_c$-algebras (resp. $\underline{Rep}(GL(c))$) and homomorphisms will be denoted by $FB^{\pm}_c$-Alg (resp. $\underline{Rep}(GL(c))$-Alg). Theorem 2.3 therefore has an immediate

Corollary 2.4. *We have an equivalence of categories*

$$FB^{\pm}_c\text{-}Alg \simeq \underline{Rep}(GL(c))\text{-}Alg. \tag{2.4}$$

□

It is interesting to consider the semisimple $\mathbb{C}$-algebra

$$\Sigma^c_{m,n} := End_{FB^{\pm}_c}((m,n)).$$

Note that for $n = 0$, we have

$$\Sigma^c_{m,0} \cong \mathbb{C}\Sigma_m.$$

One has

$$dim(\Sigma^c_{m,n}) = (m+n)!$$

and $\mathcal{Y}_{\lambda,\mu}(m,n)$ with $|\lambda| - |\mu| = m - n$, $|\lambda| \leq m$, $|\mu| \leq n$ are the distinct simple $\Sigma^c_{m,n}$-representations.

3. T-Algebras and the Proof of Theorem 1.1

In this section, we discuss the category of T-algebras, which are the universal structures we use to model QPT categories, (i.e. $\mathbb{C}$-linear pseudo-abelian additive categories with an associative, commutative, unital tensor product and strong duality, which are generated by a single basic object). In Subsection 3.1, we define T-algebras and discuss their correspondence with categories. In Subsection 3.3, we restate and complete the proof of Theorem 1.1.

3.1. **The definition of T-algebras.** A category $\mathscr{C}$ linear over $\mathbb{C}$ with an associative, commutative, unital tensor product which has strong duality and is generated by some $X \in Obj(\mathscr{C})$ is equivalent to certain data, which we call a *T-algebra*, which describes the categorical structure given by $\mathscr{C}$ on the Hom-spaces

$$Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T})$$

for finite sets S, T.

Concretely, we define as follows:

Definition 3.2. *A T-algebra $\mathcal{T}$ is a collection of $\mathbb{C}$-vector spaces $\mathcal{T}_{S,T}$ indexed by pairs of finite sets S, T in addition to the following data:*

(1) For all finite sets S_1, S_2, T_1, T_2 and bijections

$$\phi : S_2 \to S_1$$

$$\psi : T_1 \to T_2,$$

a $\mathbb{C}$-linear map

$$\sigma_{\phi,\psi} : \mathcal{T}_{S_1,T_1} \to \mathcal{T}_{S_2,T_2}$$

which is functorial with respect to ϕ and ψ.

(2) For all finite sets S, T, for all subsets $S' \subseteq S$,$T' \subseteq T$, and choices of bijection

$$\phi : S' \to T',$$

a $\mathbb{C}$-linear partial trace *map*

$$\tau_\phi : \mathcal{T}_{S,T} \to \mathcal{T}_{S \smallsetminus S', T \smallsetminus T'}$$

which are functorial with respect to bijections to and from S' and T', respectively (acting on ϕ by composition).

(3) For all finite sets S_1, S_2, T_1, T_2, a $\mathbb{C}$-linear product *map*

$$\pi : \mathcal{T}_{S_1,T_1} \otimes_{\mathbb{C}} \mathcal{T}_{S_2,T_2} \to \mathcal{T}_{S_1 \amalg S_2, T_1 \amalg T_2}$$

which is functorial respect to bijections to S_1 and S_2 and from T_1 and T_2 and a element

$$1 \in \mathcal{T}_{\emptyset,\emptyset}$$

such that π is associative, commutative, and unital with respect to 1, expressible by requiring that diagrams such as, for example,

$$\begin{array}{ccccc} \mathcal{T}_{S,T} \otimes_{\mathbb{C}} \mathcal{T}_{\emptyset,\emptyset} & \xleftarrow{Id \otimes 1} & \mathcal{T}_{S,t} & \xrightarrow{1 \otimes Id} & \mathcal{T}_{\emptyset,\emptyset} \otimes_{\mathbb{C}} \mathcal{T}_{S,T} \\ & {\scriptstyle\pi} \searrow & \downarrow {\scriptstyle Id} & \swarrow {\scriptstyle\pi} & \\ & & \mathcal{T}_{S,T}. & & \end{array}$$

commute. We also require that π commutes with the partial trace maps, meaning for all finite sets S_1, S_2, T_1, T_2 and subsets

$$S'_i \subseteq S_i, \qquad T'_i \subseteq T_i$$

with bijections

$$\phi_i : S'_i \to T'_i,$$

we have that the diagram

$$\begin{array}{ccc} \mathcal{T}_{S_1,T_1} \otimes_{\mathbb{C}} \mathcal{T}_{S_2,T_2} & \xrightarrow{\pi} & \mathcal{T}_{S_1 \amalg S_2, T_1 \amalg T_2} \\ {\scriptstyle \tau_{\phi_1} \otimes \tau_{\phi_2}} \downarrow & & \downarrow {\scriptstyle \tau_{\phi_1 \amalg \phi_2}} \\ \mathcal{T}_{S_1 \smallsetminus S'_1, T_1 \smallsetminus T'_1} \otimes_{\mathbb{C}} \mathcal{T}_{S_2 \smallsetminus S'_2, T_2 \smallsetminus T'_2} & \xrightarrow[\pi]{} & \mathcal{T}_{S_1 \amalg S_2 \smallsetminus (S'_1 \amalg S'_2), T_1 \amalg T_2 \smallsetminus (T'_1 \amalg T'_2)} \end{array}$$

commutes.

(4) An identity *element $\iota \in \mathcal{T}_{\{x\},\{x\}}$ such that for every $f \in \mathcal{T}_{\{y\},\{y\}}$, the product $\pi(\iota, f) \in \mathcal{T}_{\{x,y\},\{x,y\}}$ satisfies*

$$\tau_{\{x\}\to\{y\}}(\pi(\iota, f)) = \sigma_{Id_{\{x\}}, \{y\}\to\{x\}}(f)$$

and

$$\tau_{\{y\}\to\{x\}}(\pi(\iota, f)) = \sigma_{\{x\}\to\{y\}, Id_{\{x\}}}(f).$$

Note that on the right hand side of the above identities, the bijections acting on f are bijection on singletons and are only present for formal indexing reasons. The conditions can be abbreviated as

$$\iota \circ f = f \circ \iota = f,$$

by considering "composition" to be the appropriate combination of trace and product. For any $n \in \mathbb{N}_0$, let us denote by $\iota^n \in \mathcal{T}_{\{x_1,\dots,x_n\},\{x_1,\dots,x_n\}}$ the product of n copies of ι.

For T-algebras, $\mathcal{T}$, $\mathcal{S}$, we define a morphism

$$f : \mathcal{T} \to \mathcal{S}$$

as a collection of $\mathbb{C}$-linear maps

$$f_{S,T} : \mathcal{T}_{S,T} \to \mathcal{S}_{S,T}$$

for finite sets S, T preserving the T-algebra structure of $\mathcal{T}$ and $\mathcal{S}$. We can therefore form the category T-Alg of $\mathcal{T}$-algebras and their morphisms.

T-algebras are a universal algebra construction and are precisely equivalent to the data of a $\mathbb{C}$-linear category with associative, commutative, unital tensor product and strong duality generated by single object X. For such a category $\mathscr{C}$, the Hom-spaces

$$Hom_{\mathscr{C}}(X^{\otimes S}, X^{\otimes T})$$

form a T-algebra which we shall denote by $\mathscr{C}_{S,T}$, and for a T-algebra $\mathcal{T}$, we can construct the category $\mathscr{C}_{\mathcal{T}}$ by taking

$$Obj(\mathscr{C}_{\mathcal{T}}) = \{X^{\otimes m} \otimes (X^{\vee})^{\otimes n} \mid m, n \in \mathbb{N}_0\}$$

and

$$Hom_{\mathscr{C}_{\mathcal{T}}}(X^{\otimes m} \otimes (X^{\vee})^{\otimes n}, X^{\otimes p} \otimes (X^{\vee})^{\otimes q}) = \mathcal{T}_{[m]\amalg[q],[n]\amalg[p]}.$$

For a pair $m, n \in \mathbb{N}_0$, for a T-algebra $\mathcal{T}$, let us denote

$$\mathcal{T}_{m,n} = \mathcal{T}_{[m],[n]},$$

which is a $\Sigma_m \times \Sigma_n$-representation by functoriality. Let us call a T-algebra *graded* if for finite sets S, T, if $|S| \neq |T|$,

$$\mathcal{T}_{S,T} \cong \mathcal{T}_{|S|,|T|} = 0.$$

Define, also, a *T_c-algebra* to be a T-algebra $\mathcal{T}$ such that $\mathcal{T}_{\emptyset,\emptyset} = \mathbb{C}$ and the trace of the identity element is

$$\tau_{Id_{\{x\}}}(\iota) = c.$$

Let T_c-Alg denote the full subcategory of T-Alg of T_c-algebras, and let T_c-Alg^{gr} denote the full subcategory of graded T-Alg of T_c-algebras.

3.3. **The Proof of Theorem 1.1.** Using the notation introduced in Subsection 3.1, we may now restate Theorem 1.1 more explicitly:

Theorem 3.4. *For a T_c-algebra $\mathcal{T}$, there is a natural $FB_c^{\pm}$-algebra $\mathcal{M}_{\mathcal{T}}$ such that for every $(m, n) \in \mathbb{N}_0^2$,*

$$\mathcal{M}_{\mathcal{T}}(m, n) = \mathcal{T}_{m,n}. \tag{3.1}$$

On the other hand, for an $FB_c^{\pm}$-module $\mathcal{M}$, there exists a natural T_c-algebra $\mathcal{T}(\mathcal{M})$ such that for finite sets S, T

$$\mathcal{T}(\mathcal{M})_{S,T} = \mathcal{M}(|S|, |T|). \tag{3.2}$$

This defines an equivalence of categories

$$T_c\text{-}Alg \to FB_c^{\pm}\text{-}Alg. \tag{3.3}$$

Given this theorem, denote the composition of (3.3) with (2.4) by

$$\Phi : \mathrm{T}_c\text{-Alg} \to \underline{Rep}(GL(c))\text{-Alg}. \tag{3.4}$$

This Φ is then the claimed equivalence of categories in Theorem 1.1. Denote by Ψ the inverse (up to natural isomorphism) functor

$$\Psi : \underline{Rep}(GL(c))\text{-Alg} \to \mathrm{T}_c\text{-Alg}.$$

We also have the following

Corollary 3.5. *Restricting (3.3) to the full subcategory of graded T_c-algebras gives an equivalence of categories*

$$T_c\text{-}Alg^{gr} \to FB_{c,0}^{\pm}\text{-}Alg.$$

□

Proof of Theorem 3.4. Suppose we are given a T_c-algebra $\mathcal{T}$. To give (3.1) the $FB_c^{\pm}$-algebra, we must describe a functorial map

$$\mathcal{M}_{\mathcal{T}}(f) : \mathcal{M}_{\mathcal{T}}(m, n) \to \mathcal{M}_{\mathcal{T}}(p, q)$$

corresponding to a given morphism

$$f \in Hom_{FB_c^{\pm}}((m, n), (p, q)). \tag{3.5}$$

We shall define $\mathcal{M}_{\mathcal{T}}(f)$ only for the generating morphisms f which correspond to bijections

$$\phi_f : [m] \amalg [q] \to [p] \amalg [n],$$

and extend $\mathbb{C}$-linearly to define $\mathcal{M}_{\mathcal{T}}(f)$ for every f in (3.5). First, we will factor f as a composition $FB_c^{\pm}$ morphisms f_θ, f_η, and f_ζ which will correspond to performing partial trace, the action of a bijection, and product with identity elements, respectively.

By definition, ϕ_f restricts to bijections

$$[m] \smallsetminus \phi_f^{-1}([n]) \xrightarrow{\cong} [p] \smallsetminus \phi_f([q]), \tag{3.6}$$

$$[q] \smallsetminus \phi_f^{-1}([p]) \xrightarrow{\cong} [n] \smallsetminus \phi_f([m]). \tag{3.7}$$

Let us define

$$a = |[m] \smallsetminus \phi_f^{-1}([n])| = |[p] \smallsetminus \phi_f([q])|$$
$$b = |[q] \smallsetminus \phi_f^{-1}([p])| = |[n] \smallsetminus \phi_f([m])|.$$

We can define a bijection

$$\theta : [m] \amalg [b] \to [a] \amalg [n]$$

given as the disjoint union of the restriction

$$\phi_f|_{\phi_f^{-1}(n)} : [m] \cap \phi_f^{-1}([n]) \to [n] \cap \phi_f([m])$$

and the identity on the rest of the source, identifying $[a]$ with $[m] \smallsetminus \phi_f^{-1}([n])$ and $[b]$ with $[n] \smallsetminus \phi_f([m])$.

We can define a bijection

$$\eta : [a] \amalg [b] \to [a] \amalg [b]$$

given as the disjoint union of the bijections (3.6) and (3.7), where in the source of η we identify $[a]$ with $[m]\smallsetminus\phi_f^{-1}([n])$ and $[b]$ with $[q]\smallsetminus\phi_f^{-1}([p])$, and in the target of η we identify $[a]$ with $[p] \smallsetminus \phi_f([q])$ and $[b]$ with $[n] \smallsetminus \phi_f([m])$.

Finally, we can define a bijection

$$\zeta : [a] \amalg [q] \to [p] \amalg [b]$$

given by the disjoint union of the restriction

$$\phi_f|_{\phi_f^{-1}(p)} : [q] \cap \phi_f^{-1}([p]) \to [p] \cap \phi_f([q])$$

and the identity on the rest of source, identifying $[a]$ with $[p] \smallsetminus \phi_f([q])$ and $[b]$ with $[q] \smallsetminus \phi_f^{-1}([p])$.

Denote by

$$f_\theta \in Hom_{FB_c^\pm}((m,n),(a,b))$$
$$f_\eta \in Hom_{FB_c^\pm}((a,b),(a,b))$$

and

$$f_\zeta \in Hom_{FB_c^\pm}((a,b),(p,q))$$

the corresponding morphisms of $FB_c^\pm$. We then have

$$f = f_\zeta \circ f_\eta \circ f_\theta,$$

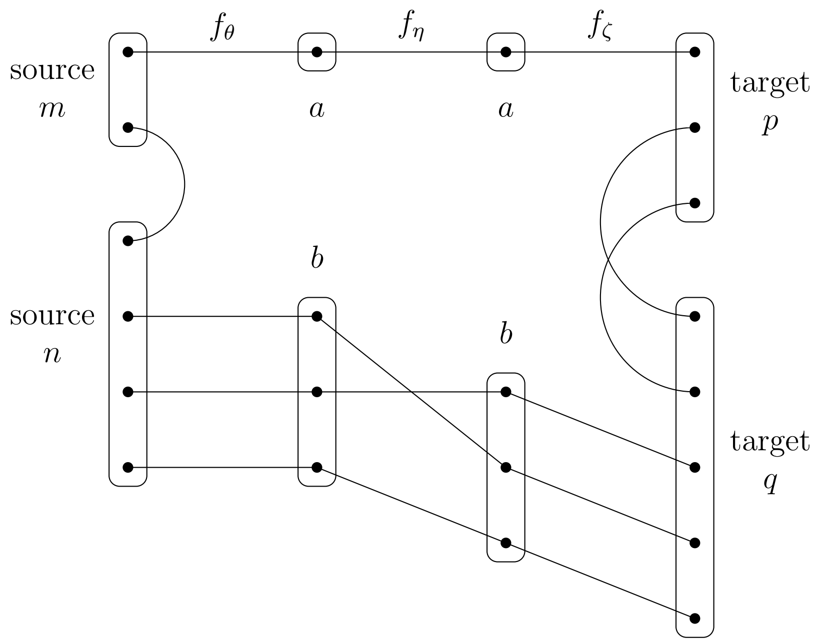

FIGURE 3. The decomposition of the generator of $Mor_{FB_c^\pm}((m,n),(p,q))$, for $m=2$, $n=4$, $p=3$, $q=5$ in Figure 2, which gives $a=1$ and $b=3$.

as morphisms of $FB_c^\pm$. Figure 3 shows the graphical representations of f_θ, f_η, and f_ζ for the example of f given in Figure 2.

We then define

$$\mathcal{M}_\mathcal{T}(f_\theta) : \mathcal{M}_\mathcal{T}(m,n) = \mathcal{T}_{m,n} \to \mathcal{T}_{a,b} = \mathcal{M}_\mathcal{T}(a,b)$$

as the partial trace matching the coordinates corresponding to the elements of $[m]$ and $[n]$ matched by f_θ. More concretely, we take

$$\mathcal{M}_\mathcal{T}(f_\theta) = \tau_{\phi_f|_{[m]\cap\phi^{-1}([n])}}$$

along the bijection

$$\phi_f|_{[m]\cap\phi^{-1}([n])} : [m] \cap \phi_f^{-1}([n]) \to [n] \cap \phi_f([m])$$

again identifying $[a]$ with $[m] \smallsetminus \phi_f^{-1}([n])$ and $[b]$ with $[n] \smallsetminus \phi_f([m])$.

We define

$$\mathcal{M}_\mathcal{T}(f_\eta) : \mathcal{M}_\mathcal{T}(a,b) = \mathcal{T}_{a,b} \to \mathcal{T}_{a,b} = \mathcal{M}_\mathcal{T}(a,b)$$

as the action of the bijection, permuting the coordinates of the source and target according to f_η. More concretely, we take

$$\mathcal{M}_\mathcal{T}(f_\eta) = \sigma_{\phi_f^{-1}|_{[p]\smallsetminus\phi_f([q])}, \phi_f^{-1}|_{[n]\smallsetminus\phi_f([m])}}$$

along the bijection

$$\phi_f^{-1}|_{[m]\smallsetminus\phi_f^{-1}([n])} : [a] \to [a]$$

identifying $[a]$ with $[p] \smallsetminus \phi_f([q])$ in the source identifying $[a]$ with $[m] \smallsetminus \phi_f^{-1}([n])$ in the target, and the bijection

$$\phi_f^{-1}|_{[n]\smallsetminus\phi_f([m])} : [b] \to [b]$$

identifying $[b]$ with $[n] \smallsetminus \phi_f([m])$ in the source and identifying $[b]$ with $[q] \smallsetminus \phi_f^{-1}([p])$ in the target.

Finally, we define

$$\mathcal{M}_{\mathcal{T}}(f_\zeta) : \mathcal{M}_{\mathcal{T}}(a,b) = \mathcal{T}_{a,b} \to \mathcal{T}_{a,b} = \mathcal{M}_{\mathcal{T}}(p,q)$$

as taking the product with copies of the identity element, permuted into the places corresponding to the elements of $[p]$ and $[q]$ matched by f_ζ. More concretely, we take

$$\mathcal{M}_{\mathcal{T}}(f_\zeta) = \sigma_{\alpha,\beta}(\pi(\phi_f|_{[m]\cap\phi^{-1}([n])}, \iota^{|\phi_f([q])\cap[p]|}))$$

where α denotes the order-preserving permutation of the last $|\phi_f([q]) \cap [p]|$ coordinates (the original source of the identity elements after the product) into the order where there is an order-preserving bijection of the total set of source coordinates with $[p]$ sending these coordinates to the subset $\phi_f([q]) \cap [p] \subseteq [p]$, and similarly β denotes the order-preserving permutation of the last $|\phi_f([q]) \cap [p]| = |\phi_f^{-1}([p]) \cap [q]|$ coordinates (the original target of the identity elements after the product) into the order where there is an order-preserving bijection of the total set of target coordinates with $[q]$ sending these coordinates to the subset $\phi_f^{-1}([p]) \cap [q]$.

We therefore define

$$\mathcal{M}_{\mathcal{T}}(f) = \mathcal{M}_{\mathcal{T}}(f_\zeta) \circ \mathcal{M}_{\mathcal{T}}(f_\eta) \circ \mathcal{M}_{\mathcal{T}}(f_\theta).$$

Figure 4 shows the action of $\mathcal{M}_{\mathcal{T}}(f)$ on an element $x \in \mathcal{M}_{\mathcal{T}}(m,n) = \mathcal{T}_{m,n}$ in the case of the example pictured in Figures 2 and 3.

The commutation of the trace and product maps and their functoriality guarantee that $\mathcal{M}_{\mathcal{T}}$ defined in this way indeed forms a $FB_c^{\pm}$-algebra

$$\mathcal{M}_{\mathcal{T}} : FB_c^{\pm} \to Vect.$$

The T-algebra product determines a product

$$\pi : \mathcal{M}_{\mathcal{T}}(m_1,n_1) \otimes_{\mathbb{C}} \mathcal{M}_{\mathcal{T}}(m_2,n_2) \to \mathcal{M}_{\mathcal{T}}(m_1+m_2, n_1+n_2),$$

which induces a map of $FB_c^{\pm}$-algebras from the Day product $\mathcal{M}_{\mathcal{T}} \otimes \mathcal{M}_{\mathcal{T}}$ to $\mathcal{M}_{\mathcal{T}}$. Associativity and commutativity are guaranteed by the

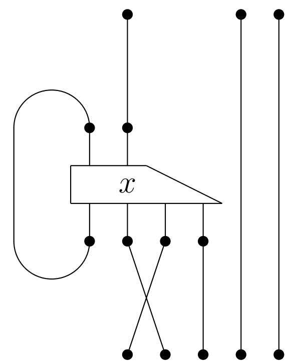

Figure 4. The image $\mathcal{M}_{\mathcal{T}}(f)(x) \in \mathcal{M}_{\mathcal{T}}(p,q) = \mathcal{T}_{p,q}$ of an element $x \in \mathcal{M}_{\mathcal{T}}(m,n) = \mathcal{T}_{m,n}$ for the generator f of $Mor_{FB_c^{\pm}}((m,n),(p,q))$ for $m=2$, $n=4$, $p=3$, $q=5$ shown in Figure 2

associativity and commutativity of π. Hence, we have defined a functor

$$\begin{aligned} \mathrm{T}_c\text{-Alg} &\to FB_c^{\pm}\text{-Alg} \\ \mathcal{T} &\mapsto \mathcal{M}_{\mathcal{T}}. \end{aligned} \tag{3.8}$$

To prove that the functor (3.8) is in fact an equivalence of categories, we must construct an inverse functor

$$FB_c^{\pm}\text{-Alg} \to \mathrm{T}_c\text{-Alg}$$
$$\mathcal{M} \mapsto \mathcal{T}(\mathcal{M})$$

satisfying (3.2). This construction is done precisely symmetrically to the construction of (3.8) above: Partial traces, functoriality, and the product with identity all correspond exactly to morphisms of $FB_c^{\pm}$, and their compatibility is proved by the consistency of the category $FB_c^{\pm}$. Again, an associative, commutative, unital multiplication map

$$\mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$$

determines the product π on $\mathcal{T}(\mathcal{M})$. The functor defined in this way will be, by definition, an inverse functor to (3.8).

□

4. Deformation Theory for T-Algberas

The goal of this section is to use the equivalence of categories between T_c-algebras and algebras in $\underline{Rep}(GL(c))$ to give a description of the module of deformations of a T_c-algebras (see Theorem 4.5, below). In Subsection 4.1 we describe some algebraic background in the category $\underline{Rep}(GL(c))$. In Subsection 4.4, we define deformations and state Theorem 4.5, making an explicit algebraic claim about the module of deformations of any algebra in $\underline{Rep}(GL(c))$. In Subsection 4.6, we prove Theorem 4.5 using calculations in Quillen cohomology.

4.1. **Elements, ideals, and modules for a $\underline{Rep}(GL(c))$-algebra.** In particular, for any algebra $\mathcal{A}$ in $\underline{Rep}(GL(c))$, we may write a decomposition

$$\mathcal{A} = \bigoplus_{(\lambda,\mu)} \bigoplus_{i\in I_{\lambda,\mu}} Y_{\lambda,\mu} \tag{4.1}$$

where the first sum runs over all pairs of Young diagrams λ and μ, and the second sum of copies of $Y_{\lambda,\mu}$ is indexed by a (possible infinite) set $I_{\lambda,\mu}$.

Example: Consider the category $\underline{Rep}(GL(c))$, which is an example of a $\mathbb{C}$-linear category with associative, commutative, unital tensor product and strong duality, generated by a basic object X (which has categorical dimension c). We may therefore consider its corresponding T_c-algebra $\mathcal{T}^{\underline{Rep}(GL(c))}$ defined by

$$\mathcal{T}^{\underline{Rep}(GL(c))}_{S,T} = Hom_{\underline{Rep}(GL(c))}(X^{\otimes S}, X^{\otimes T})$$

In this case, the T_c-algebra is graded and in fact

$$dim(\mathcal{T}^{\underline{Rep}(GL(c))}_{S,T}) = \begin{cases} (|S|)! & \text{if } |S| = |T| \\ 0 & \text{if } |S| \neq |T|. \end{cases} \tag{4.2}$$

We have in particular,

$$dim(\mathcal{T}^{\underline{Rep}(GL(c))}_{\emptyset,\emptyset}) = 1.$$

Since the only terms of a decomposition (4.1) which will generate nontrivial elements in the T_c-algebra at the pair of finite sets $(\emptyset, \emptyset)$ (since

$$dim(\mathcal{Y}_{\lambda,\mu}(0,0)) \neq 0$$

only if $\lambda = \mu = \emptyset$). Therefore, the decomposition of $\Phi(\mathcal{T}^{\underline{Rep}(GL(c))})$ must contain a copy of $Y_{\emptyset,\emptyset}$.

Note now that for every n,

$$dim(\mathcal{Y}_{\emptyset,\emptyset}(n,n)) = dim(\underline{(0,0)}(n,n)) = n!,$$

exactly matching the dimensions of (4.2).

Therefore, the T_c-algebra corresponding to the category $\underline{Rep}(GL(c))$ is decomposed (in the sense of (4.1)) as

$$\Phi(\mathcal{T}^{\underline{Rep}(GL(c))}) = Y_{\emptyset,\emptyset}.$$

For a general $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$ i, given the decomposition (4.1), let us write

$$\mathcal{A}(\lambda,\mu) := \bigoplus_{i\in I_{\lambda,\mu}} Y_{\lambda,\mu} = \mathbb{C}^{\oplus I_{\lambda,\mu}} \otimes_{\mathbb{C}} Y_{\lambda,\mu} \tag{4.3}$$

(where $\mathbb{C}^{\oplus I_{\lambda,\mu}} = \bigoplus_{i\in I_{\lambda,\mu}} \mathbb{C}$). We refer to vectors in the $\mathbb{C}$-vector space

$$\bigoplus_{\lambda,\mu} \mathbb{C}^{\oplus I_{\lambda,\mu}} \tag{4.4}$$

as the "*elements*" of $\mathcal{A}$. We further refer to vectors in the $\mathbb{C}$-vector space $\mathbb{C}^{\oplus I_{\lambda,\mu}}$ as the "*homogeneous elements of type* (λ,μ)" in $\mathcal{A}$.

We also define modules and ideals for an algebra $\mathcal{A}$ in $\underline{Rep}(GL(c))$ in the usual way:

Definition 4.2. *For a $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$, we may consider an $\mathcal{A}$-module in $\underline{Rep}(GL(c))$ to be an object $\mathcal{M}$ with the structure of a multiplication map*

$$\mathcal{A}\otimes\mathcal{M}\to\mathcal{M}$$

satisfying associativity and unitality.

Definition 4.3. *We define an* ideal $\mathcal{J}$ *in an algebra $\mathcal{A}$ over the category $\underline{Rep}(GL(c))$ as a submodule of $\mathcal{A}$ which is closed under the product, meaning that we are given the data of a product*

$$\mathcal{J}\otimes\mathcal{A}\to\mathcal{A},$$

and the images of the compositions of inclusions and multiplication

$$\mathcal{J}\otimes\mathcal{A} \xrightarrow{\subseteq\otimes Id_{\mathcal{A}}} \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}$$

$$\mathcal{A}\otimes\mathcal{J} \xrightarrow{Id_{\mathcal{A}}\otimes\subseteq} \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}$$

are contained in $\mathcal{J}$.

For an ideal $\mathcal{J}$ in an algebra $\mathcal{A}$ over $\underline{Rep}(GL(c))$ which is not equal to $\mathcal{A}$, by Zorn's Lemma, there exists an ideal $\mathcal{J}' \subsetneq \mathcal{A}$ which is maximal with respect to inclusion. We then call such an ideal $\mathcal{J}$ in $\mathcal{A}$ *maximal*.

For a $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$ and a set of generators $\mathcal{G} = \{g_i \mid i \in I\}$ (for some indexing set I) where we give each g_i a chosen type Y_{λ_i,μ_i} for pairs of Young diagrams λ_i, μ_i, we may also define the *free $\mathcal{A}$-module* $\mathcal{A}\mathcal{G}$ as the object of $\underline{Rep}(GL(c))$

$$\mathcal{A}\mathcal{G} := \bigoplus_{i\in I} \mathcal{A} \otimes Y_{\lambda_i,\mu_i} \tag{4.5}$$

with multiplication given by multiplication on the first tensor factor of $\mathcal{A}$ in each summand.

On the other hand, for a set of generators $\mathcal{G} = \{g_i \mid i \in I\}$ (for some indexing set I) where each element g_i is of a chosen type Y_{λ_i,μ_i}, we can define the *symmetric algebra on* $\mathcal{G}$ in $\underline{Rep}(GL(c))$ as

$$Sym(\mathcal{G}) := Sym(\bigoplus_{i\in I} Y_{\lambda_i,\mu_i}).$$

For any $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$, there then exists a set of generators $\mathcal{G}$ such that we can write $\mathcal{A}$ as the quotient

$$\mathcal{A} = Sym(\mathcal{G})/\mathcal{J} \tag{4.6}$$

for an ideal $\mathcal{J}$ in $Sym(\mathcal{G})$. We call $\mathcal{G}$ and $\mathcal{J}$ a choice of a *set of generators* and an *ideal of relations* for $\mathcal{A}$.

4.4. **Deformations of a T_c-algebras.** We shall investigate deformations of T_c-algebras by using a version of Quillen cohomology for their corresponding algebras over $\underline{Rep}(GL(c))$. We define Quillen homology and cohomolgoy for algebras over $\underline{Rep}(GL(c))$ entirely analogously to its definition for commutative rings, which is explicitly described, for example, in [17].

Specifically, the deformations of an algebra $\mathcal{A}$ in $\underline{Rep}(GL(c))$, and therefore its corresponding T_c-algebra and QPT category, will then be classified by

$$Excalcomm_{Y_{\emptyset,\emptyset}}(\mathcal{A},\mathcal{A}) = H^1_{Y_{\emptyset,\emptyset}}(\mathcal{A},\mathcal{A}),$$

where for the remainder of this paper we consider $Y_{\emptyset,\emptyset}$ to be an algebra by recalling that $Y_{\emptyset,\emptyset}$ is the unit with respect to tensor in $\underline{Rep}(GL(c))$

and taking the algebra structure

$$Y_{\emptyset,\emptyset} \otimes Y_{\emptyset,\emptyset} = Y_{\emptyset,\emptyset} \xrightarrow{Id} Y_{\emptyset,\emptyset}$$

(note that, as in the above Example, this is the T_c-algebra corresponding to the category $\underline{Rep}(GL(c))$ itself).

We claim the following

Theorem 4.5. *For an* $\underline{Rep}(GL(c))$*-algebra* $\mathcal{A}$*, let us write*

$$\mathcal{A} = Sym(\mathcal{G})/\mathcal{J},$$

for a set of generators $\mathcal{G}$ *and an ideal of relations* $\mathcal{J}$*. Then the module of infinitesimal deformations of* $\mathcal{A}$ *is*

$$Hom_{\mathcal{A}}(Ker(d : \mathcal{A} \otimes_{Sym(\mathcal{G})} \mathcal{J} \to \mathcal{A}\mathcal{G}), \mathcal{A}). \tag{4.7}$$

Recall that (similarly as classically), d in (4.7) is a map of $\mathcal{A}$-modules: Suppose $v \in \mathcal{J}$, $u \in \mathcal{A}$. Then

$$d(uv) = u \cdot (dv) + (du) \cdot v,$$

but the second term is $0 \in \mathcal{A}$, since $v \in \mathcal{J}$.

4.6. **The Proof of Theorem 4.5.** For a $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$, consider a choice of a set of generators $\mathcal{G}$ and an ideal of relations $\mathcal{J}$ giving an expression (4.6) for $\mathcal{A}$. Let us write

$$\mathcal{B} := Sym(\mathcal{G}). \tag{4.8}$$

By definition, we have a surjection in $\underline{Rep}(GL(c))$ mapping

$$\mathcal{B} \to \mathcal{A},$$

with kernel $\mathcal{J}$. On the other hand, we have an inclusion

$$Y_{\emptyset,\emptyset} \to \mathcal{B}$$

(taking $Y_{\emptyset,\emptyset}$ to be the 0th degree of the symmetric algebra (4.8)).

To prove Theorem 4.5, it suffices to calculate

$$H^1_{Y_{\emptyset,\emptyset}}(\mathcal{A}, \mathcal{A}). \tag{4.9}$$

Let us begin by recalling the notation of Kähler differentials, taking, for a $\underline{Rep}(GL(c))$ algebras $\mathcal{X}$ and an algebra $\mathcal{Y}$ over $\mathcal{X}$,

$$\Omega_{\mathcal{Y}/\mathcal{X}} = \mathcal{X}\{dx \mid x \in \mathcal{Y}\}/\mathcal{Z}$$

(as $\mathcal{X}$-modules) where $\mathcal{Z}$ denotes the $\mathcal{X}$-module generated by

$$d(x \cdot y) - x \cdot dy - y \cdot dx \text{ for } x, y \in \mathcal{Z}$$

and

$$dx \text{ for } x \in \mathcal{X}$$

(where all x and y are considered as elements of $\mathcal{X}$ or $\mathcal{Z}$ in the sense described using vector spaces (4.4)).

We may then write a long exact sequence

(4.10)

$$\begin{array}{ccccccc} \dots \mathcal{A}\otimes_{\mathcal{B}}\Omega^1_{\mathcal{B}/Y_{\emptyset,\emptyset}} & \longrightarrow & \Omega^1_{\mathcal{A}/Y_{\emptyset,\emptyset}} & & & & \\ & & \downarrow & & & & \\ & & \Omega^1_{\mathcal{A}/\mathcal{B}} & & & & \\ & & \downarrow & & & & \\ & & \mathcal{A}\otimes_{\mathcal{B}}\Omega_{\mathcal{B}/Y_{\emptyset,\emptyset}} & \longrightarrow & \Omega_{\mathcal{A}/Y_{\emptyset,\emptyset}} & \longrightarrow & \Omega_{\mathcal{A}/\mathcal{B}} \longrightarrow 0 \end{array}$$

Since $\mathcal{B}$ can be considered as a "polynomial algebra" over $Y_{\emptyset,\emptyset}$, it has no deformations, and therefore $\Omega^1_{\mathcal{B}/Y_{\emptyset,\emptyset}}$ vanishes. Hence, (4.10) gives that $\Omega^1_{\mathcal{A}/Y_{\emptyset,\emptyset}}$ is the kernel of the map

$$\Omega^1_{\mathcal{A}/\mathcal{B}} \to \mathcal{A}\otimes_{\mathcal{B}}\Omega_{\mathcal{B}/Y_{\emptyset,\emptyset}}. \tag{4.11}$$

(Recall that, by definition,

$$H^1_{Y_{\emptyset,\emptyset}}(\mathcal{A},\mathcal{A}) = Hom_{\mathcal{A}}(\Omega^1_{\mathcal{A}/Y_{\emptyset,\emptyset}},\mathcal{A}).) \tag{4.12}$$

First note that by Theorem 6.3 of [17] (which has an obvious analogue here over a semisimple pre-Tannakian category), we have

$$\Omega^1_{\mathcal{A}/\mathcal{B}} \cong Tor_1^{\mathcal{B}}(\mathcal{A},\mathcal{A}). \tag{4.13}$$

We can further express

$$Tor_1^{\mathcal{B}}(\mathcal{A},\mathcal{A}) \cong \mathcal{J}\otimes_{\mathcal{B}}\mathcal{A}.$$

The map (4.11) then corresponds to differentiation

$$d : \mathcal{J}\otimes_{\mathcal{B}}\mathcal{A} \to \Omega_{\mathcal{B}/Y_{\emptyset,\emptyset}}\otimes_{\mathcal{B}}\mathcal{A}.$$

Again, since $\mathcal{B}$ is a polynomial algebra on the generators $\mathcal{G}$, we have

$$\Omega_{\mathcal{B}/Y_{\emptyset,\emptyset}} \cong \mathcal{B}\mathcal{G},$$

(identifying the generators dg on the left hand side with g on the right has side for $g \in \mathcal{G}$), and therefore, we have

$$\Omega_{\mathcal{B}/Y_{\emptyset,\emptyset}}\otimes_{\mathcal{B}}\mathcal{A} \cong \mathcal{A}\mathcal{G}$$

giving the claim of Theorem 4.5 by (4.12).

□

5. Geometric Conditions for $\underline{Rep}(GL(c))$-Algebras

In this section, we will define some "geometric" conditions on algebras in $\underline{Rep}(GL(c))$, such as being *locally finite*, *finitely presented*, *field-like*, and *étale*. We shall also prove Theorems 1.4 and 1.3 stated in the Introduction.

In Subsection 5.1, we define locally finite and finitely presented $\underline{Rep}(GL(c))$-algebras. In Subsection 5.4, we define étale $\underline{Rep}(GL(c))$-algebras by studying modules in $\underline{Rep}(GL(c))$ and conclude Theorem 1.4. In Subsection 5.6, we construct a fully faithful tensor functor from the quasi-pre-Tannakian category obtained from the T_c-algebra corresponding to a $\underline{Rep}(GL(c))$-algebra to its category of finitely generated modules. In Subsection 5.8, we define field-like $\underline{Rep}(GL(c))$-algebras, and restate and prove Theorem 1.3.

5.1. **Notions of Finiteness for $\underline{Rep}(GL(c))$-algebras.** Let us recall, for a $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$, the decomposition (4.1):

$$\mathcal{A} = \bigoplus_{(\lambda,\mu)} \bigoplus_{i \in I_{\lambda,\mu}} Y_{\lambda,\mu}.$$

Definition 5.2. *We call a* $\underline{Rep}(GL(c))$*-algebra* $\mathcal{A}$ locally finite *if for every pair of Young diagrams* λ*,* μ*, the indexing set* $I_{\lambda,\mu}$ *in the decomposition (4.1) is finite.*

Another notion of finiteness is the following:

Definition 5.3. *We call a* $\underline{Rep}(GL(c))$*-algebra* $\mathcal{A}$ finitely presented *if there exists a finite set of generators* $\mathcal{G}$ *and a finitely generated ideal of relations* $\mathcal{J} \subset Sym(\mathcal{G})$ *such that*

$$\mathcal{A} = Sym(\mathcal{G})/\mathcal{J}.$$

(Note that a finitely presented $\underline{Rep}(GL(c))$-algebra may not necessarily be locally finite: For example, the symmetric algebra on a single generator of type $\lambda = (1)$, $\mu = (1)$, i.e.

$$Sym(Y_{1,1}) = \bigoplus_{n \in \mathbb{N}_0} Sym^n(Y_{1,1})$$

contains infinitely many copies of $Y_{1,1}$.)

5.4. **Étale $\underline{Rep}(GL(c))$-algebras and Theorem 1.4.** Recall Definition 4.2 of a module over an algebra in $\underline{Rep}(GL(c))$.

For any $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$, we may construct a class of examples of $\mathcal{A}$-modules corresponding to objects of $Rep(GL(c))$ by considering, for a simple object $Y_{\lambda,\mu}$, the object

$$\mathcal{A} \otimes Y_{\lambda,\mu} \in \underline{Rep}(GL(c))$$

with the multiplication map

$$\mu_{\mathcal{A}} \otimes Id_{Y_{\lambda,\mu}} : \mathcal{A} \otimes \mathcal{A} \otimes Y_{\lambda,\mu} \to \mathcal{A} \otimes Y_{\lambda,\mu} \tag{5.1}$$

where $\mu_{\mathcal{A}}$ denotes the algebra multiplication of $\mathcal{A}$. (These are precisely the free $\mathcal{A}$-modules on a single generator $\mathcal{G} = \{g_{\lambda,\mu}\}$ of type $Y_{\lambda,\mu}$, see (4.5).)

We define an $\mathcal{A}$-module $\mathcal{M}$ to be *finitely generated* if it can be expressed as a quotient of a finite direct sum of $\mathcal{A}$-modules of the form $\mathcal{A} \otimes Y_{\lambda,\mu}$. (In other words, $\mathcal{M}$ is finitely generated if there exists a finite set of generators $\mathcal{G}$ and a surjection

$$\mathcal{A}\mathcal{G} \twoheadrightarrow \mathcal{M}$$

of $\mathcal{A}$-modules.)

Denote by $\mathcal{A}$-Mod the category of $\mathcal{A}$-modules and morphisms which preserve $\mathcal{A}$-multiplication. Let $\mathcal{A}$-Mod^{fg} be the full subcategory on finitely generated modules. The category $\mathcal{A}$-Mod has a tensor product $\otimes_{\mathcal{A}}$ defined in the natural way, making $\mathcal{A}$ (considered as a module over itself via its algebra structure) the unit. This makes $\mathcal{A}$-Mod into an (abelian) tensor category. $\mathcal{A}$-Mod^{fg} is an additive subcategory closed under the tensor product.

Definition 5.5. *We call a $Rep(GL(c))$-algebra $\mathcal{A}$* étale *when the category $\mathcal{A}$-Mod^{fg} is semisimple.*

The statement of Theorem 1.4 in the Introduction is now precise, and follows from Theorem 4.5:

Proof of Theorem 1.4. Suppose $\mathcal{A}$ is a finitely presented, étale $\underline{Rep}(GL(c))$-algebra. By Theorem 4.5, it suffices to prove that

$$Hom_{\mathcal{A}}(Ker(d : \mathcal{A} \otimes_{Sym(\mathcal{G})} \mathcal{J} \to \mathcal{A}\mathcal{G}), \mathcal{A}) \tag{5.2}$$

is finitely generated.

Since $\mathcal{A}$ is finitely presented, we may assume that $\mathcal{G}$ is a finite set and that $\mathcal{J}$ is a finitely generated ideal. Therefore, the source and target of the map

$$d : \mathcal{A} \otimes_{Sym(\mathcal{G})} \mathcal{J} \to \mathcal{A}\mathcal{G} \tag{5.3}$$

are finitely generated $\mathcal{A}$-modules.

Since $\mathcal{A}$ is étale, the category of finitely generated $\mathcal{A}$-modules is semisimple, and therefore the kernel of the map (5.3) is also a finitely generated $\mathcal{A}$-module. Since, again, the category of finitely generated $\mathcal{A}$-modules is semisimple, the Hom space (5.2) is finitely generated.

□

5.6. **A functor from a QPT category to modules over its corresponding $\underline{Rep}(GL(c))$-algebras.** Let us consider the QPT category constructed from the T_c-algebra $\mathcal{T}$ which corresponds to $\mathcal{A}$, i.e. satisfies $\Phi(\mathcal{T}) = \mathcal{A}$, (which is well-defined up to natural isomorphism). To simplify notation, we shall denote this category by

$$\mathscr{C}_\mathcal{A} := \mathscr{C}_\mathcal{T}$$

for the remainder of this section.

Now we can construct a functor

$$\Xi : \mathscr{C}_\mathcal{A} \to \mathcal{A}\text{-Mod}^{fg},$$

defined by sending the basic object X generating $\mathscr{C}_\mathcal{A}$ to the $\mathcal{A}$-module

$$X \mapsto \mathcal{A} \otimes Y_{(1),\emptyset} \in Obj(\mathcal{A}\text{-Mod}^{fg})$$

and its dual to

$$X^\vee \mapsto \mathcal{A} \otimes Y_{\emptyset,(1)} \in Obj(\mathcal{A}\text{-Mod}^{fg}).$$

We therefore have

$$X^{\otimes m} \otimes (X^\vee)^{\otimes n} \mapsto \mathcal{A} \otimes X^{\otimes m} \otimes (X^\vee)^{\otimes n} \in Obj(\mathcal{A}\text{-Mod}^{fg})$$

where on the left hand side X denotes the basic object of $\mathscr{C}_\mathcal{A}$, while on the right hand side it denotes the basic object of $\underline{Rep}(GL(c))$.

It remains to define maps

$$\begin{aligned} &Hom_{\mathscr{C}_\mathcal{A}}(X^{\otimes m} \otimes (X^\vee)^{\otimes n}, X^{\otimes k} \otimes (X^\vee)^{\otimes \ell}) \to \\ &\to Hom_{\mathcal{A}\text{-Mod}^{fg}}(\mathcal{A} \otimes X^{\otimes m} \otimes (X^\vee)^{\otimes n}, \mathcal{A} \otimes X^{\otimes k} \otimes (X^\vee)^{\otimes \ell}) \end{aligned} \tag{5.4}$$

for $m, n, k, \ell \in \mathbb{N}_0$. By adjunction, the source of (5.4) is

$$Hom_{\mathcal{A}\text{-Mod}^{fg}}(\mathcal{A} \otimes X^{\otimes m} \otimes (X^\vee)^{\otimes n}, \mathcal{A} \otimes X^{\otimes k} \otimes (X^\vee)^{\otimes \ell}) =$$
$$= Hom_{\underline{Rep}(GL(c))}(X^{\otimes m} \otimes (X^\vee)^{\otimes n}, \mathcal{A} \otimes X^{\otimes k} \otimes (X^\vee)^{\otimes \ell}),$$

which, by the definition of $\mathscr{C}_\mathcal{A}$, is identified with the source of (5.4).

Lemma 5.7. *The constructed functor*

$$\Xi : \mathscr{C}_\mathcal{A} \to \mathcal{A}\text{-}Mod^{fg}$$

is fully faithful.

Proof. Faithfulness is clear. Ξ is full since morphisms in the target category are quotients of morphisms from free modules. By adjunction, those correspond to morphisms from $Y_{\lambda,\mu}$ in $\underline{Rep}(GL(c))$, which can be realized in $\mathscr{C}_\mathcal{A}$ by picking out morphisms transforming under the given $\Sigma^c_{m,n}$-simple module. □

5.8. **Field-like $\underline{Rep}(GL(c))$-algebras and Theorem 1.3.** Recall Definition 4.3 of ideals in a $\underline{Rep}(GL(c))$-algebra.

Definition 5.9. *We call an algebra $\mathcal{A}$ in $\underline{Rep}(GL(c))$* field-like *if it contains no non-zero ideals.*

In particular, for an algebra $\mathcal{A}$ over $\underline{Rep}(GL(c))$ which has a maximal ideal $\mathcal{J}$, the quotient $\mathcal{A}/\mathcal{J}$ is a field. We say a T_c-algebra is *field-like* if its corresponding algebra over $\underline{Rep}(GL(c))$ is a field.

The condition of $\mathcal{A}$ being field-like implies that the $\emptyset$, $\emptyset$ part of $\mathcal{A}$, meaning the algebra

$$\bigoplus_{i \in I_{\emptyset,\emptyset}} \mathbb{C},$$

corresponding to the term $\mathcal{A}(\emptyset, \emptyset)$ (see (4.3)), is a field. Additionally, it implies that the category $\mathscr{C}_\mathcal{A}$ corresponding to $\mathcal{A}$ contains no negligible morphisms (for a summary of negligible morphisms, see for example, [6]).

Now, we may finally use Ξ to give the following restatement of Theorem 1.3:

Theorem 5.10. *For a locally finite field-like $Rep(GL(c))$-algebra $\mathcal{A}$, the following are equivalent*

(1) *The category* $\mathscr{C}_{\mathcal{A}}$ *is semisimple*
(2) *The functor*
$$\Xi : \mathscr{C}_{\mathcal{A}} \to \mathcal{A}\text{-}Mod^{fg}$$
is an equivalence of categories
(3) *The category* $\mathcal{A}$*-Mod*^{fg} *has strong duality*
(4) *The category* $\mathcal{A}$*-Mod*^{fg} *is pre-Tannakian.*
(5) $\mathcal{A}$ *is an étale Rep*$(GL(c))$*-algebra*

Proof. The implication that (1) $\Rightarrow$ (2) is clear since, by Lemma 5.7, Ξ is fully faithful.

To prove (2) $\Rightarrow$ (5), we note that $\mathcal{A}$-Mod^{fg} has colimits. Thus,
$$x^n = 0 \Rightarrow tr(x) = 0$$
(Exercise 8.18.9 of [5]). Therefore, the semisimplification (see [6]) of $\mathcal{A}$-Mod^{fg} is semisimple. However, by our assumption of $\mathcal{A}$ being field-like, $\mathscr{C}_{\mathcal{A}}$ has no negligible morphisms, and therefore, since we are assuming that Ξ is an equivalence of categories, neither does $\mathcal{A}$-Mod^{fg}. Hence, $\mathcal{A}$-Mod^{fg} is semisimple already.

It is clear that (5) $\Rightarrow$ (3).

We see that (3) $\Rightarrow$ (4) since (3) implies that in $\mathcal{A}$-Mod^{fg} we have cokernels and strong duality, and therefore we also have kernels.

Finally, to prove (4) $\Rightarrow$ (1) we note that (4) implies that $\mathscr{C}_{\mathcal{A}}$ embeds into a pre-Tannakian category. Hence, again, by [5], Exercise 8.18.9, it is semisimple, since $\mathscr{C}_{\mathcal{A}}$ has no negligible morphisms because $\mathcal{A}$ is field-like.

□

Example 5.11. Consider the semisimple pre-Tannakian category
$$\mathscr{C} = Rep(GL(a) \times GL(c-a)) \tag{5.5}$$
where $c, a, c-a \notin \mathbb{Z}$. Denote the identity on the basic object X by ι. Then we have disjoint idempotents $r, r' \in End(X)$ where $r + r' = \iota$, $tr(r) = a$, $tr(r') = c - a$, $tr(\iota) = c$. Then the element
$$x = \sqrt{\frac{c-a}{a}}r - \sqrt{\frac{a}{c-a}}r'$$

is over $Y_{1,1}$. One has

$$Sym^2(Y_{1,1}) = Y_{\emptyset,\emptyset} \oplus Y_{1,1} \oplus Y_{2,2} \oplus Y_{(1^2),(1^2)}. \tag{5.6}$$

The first two summands in (5.6) give us products

$$\cdot : Y_{1,1} \otimes Y_{1,1} \to Y_{\emptyset,\emptyset},\ \star : Y_{1,1} \otimes Y_{1,1} \to Y_{1,1}.$$

Proposition 5.12. *The $\underline{Rep}(GL(c))$-algebra $\mathcal{A}$ corresponding to the category $\mathscr{C}$ of (5.5) is given by*

$$\mathcal{A} = Sym(Y_{1,1})/(x \cdot x - c,\ x \star x - \frac{c-2a}{\sqrt{a(c-a)}}x) \tag{5.7}$$

Proof. The relations are easily verified. To show that these are the only relations, one notes that $End(X^{\otimes n})$ can be represented as the free module on permutations σ on $\{1, \dots, n\}$ where each pair $(i, \sigma(i))$ is colored black (corresponding to ι) or red (corresponding to x). The elements $x \cdot x$, $x \star x$ then calculate $tr(x \circ x)$, and the traceless component of the composition $x \circ x$, respectively. We see therefore that these determine all the relations in the algebra $\mathcal{A}$. □

Now one has

$$d(x \cdot x - c) = 2x \cdot dx \tag{5.8}$$

$$d(x \star x - \frac{c-2a}{\sqrt{a(c-a)}}x) = (dx) \star x + x \star (dx) - \frac{c-2a}{\sqrt{a(c-a)}}dx. \tag{5.9}$$

We see that by applying $? \cdot x$ to the relation (5.9), and adding a suitable multiple of the relation (5.8), we get 0, thus constructing a non-trivial element in $\Omega^1_{\mathcal{A}/Y_{\emptyset,\emptyset}}$ over $Y_{\emptyset,\emptyset}$. By semisimplicity of $\mathcal{A}$ (which follows from Theorem 5.10), this gives a non-zero element

$$\alpha \in Hom_{\mathcal{A}}(\Omega^1_{\mathcal{A}/Y_{\emptyset,\emptyset}}, \mathcal{A}). \tag{5.10}$$

(In fact, it is not difficult to check by examining $\mathcal{A}$ in low degrees that every other element of (5.10) is its multiple.) The element α corresponds to the deformation of $\mathcal{A}$ corresponding to varying the number a.

References

[1] T. Church, J. S. Ellenberg, B. Farb: FI-modules and stability for representations of symmetric groups, *Duke Math. J.* 164 (2015), no. 9, 1833-1910.

[2] P. Deligne. Catégories Tensorielles. *Mosc. Math. J.* 2, 227-248, 2002.

[3] P. Deligne. La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel. *Algebraic groups and homogeneous spaces*, 209-273, Tata Inst. Fund. Res. Stud. Math., 19, Tata Inst. Fund. Res., Mumbai, 2007.

[4] P. Deligne, J. S. Milne. Tannakian categories, in *Hodge Cycles, Motives and Shimura Varieties*, Lecture Notes in Math. 900, Springer Verlag, 101-228, 1982.

[5] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik: *Tensor categories.* Math. Surveys Monogr., 205 American Mathematical Society, Providence, RI, 2015, xvi+343 pp.

[6] P. Etingof, V. Orstrik. On semisimplification of tensor categories. arXiv:1801.04409, 2019.

[7] A. Grothendieck, J. Dieudonné: Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Première partie. *Publications Mathématiques de l'IHÉS.* 20 (1964). 65.

[8] N. Harman, A. Snowden. Oligomorphic groups and tensor categories. arXiv:2204.04526, 2022.

[9] N. Harman, A. Snowden, N. Snyder. The Delannoy Category. arXiv:2211.15392, 2023.

[10] F. Knop. A construction of semisimple tensor categories. *C. R. Math. Acad. Sci. Paris C.* 343, 2006.

[11] F. Knop. Tensor Envelopes of Regular Categories. *Adv. Math.* 214, 2007.

[12] S. Kriz. Arbitrarily High Growth in Quasi-Pre-Tannakian Categories, 2024. Available at https://krizsophie.github.io/ACUCategory23085.pdf

[13] S. Kriz: Quantum Delannoy Categories, 2023. Available at https://krizsophie.github.io/SquaredGrowth23088.pdf

[14] S. Kriz. Interpolation of General Affine Groups and Semidirect Products of Symplectic Groups with Heisenberg Groups via Representation Stability, 2023. Available at https://krizsophie.github.io/FIPMModules23109.pdf

[15] A. Snowden. Algebraic Structures in Representation Stability, lecture notes, 2010.

[16] A. Snowden. Some Fast-Growing Tensor Categories. arXiv:2305.18230, 2023.

[17] D. G. Quillen. On the (co-)homology of commutative rings, *Proc. Symp. Pure Math.* 17 (1970), 65-87.