ON THE CANONICITY OF THE SINGULARITIES OF QUOTIENTS OF THE FULTON-MACPHERSON COMPACTIFICATION

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Abstract. We prove that quotients of the Fulton-MacPherson compactification of configuration spaces of smooth projective varieties of dimension > 1 by permutation groups have canonical singularities.

1. Introduction

Suppose X is a smooth projective variety over $\mathbb{C}$ of dimension N. Consider the ordered configuration space of n distinct points

$$F(X, n) \subset X^n.$$

Then we have the Fulton-MacPherson compactification $\overline{F(X,n)}$ (see [1]), which we will briefly recall in Section 2.

Now fix some subgroup $G \subseteq \Sigma_n$. Then we can construct the geometric invariant theory quotient

$$Z = \overline{F(X,n)}/G$$

by covering the scheme $\overline{F(X,n)}$ by affine open subsets preserved by the group action and then taking the rings of invariants of the corresponding coordinate rings.

The main result of this note is the following

Theorem 1. *When $N > 1$, the quotient singularities of Z are canonical, but not necessarily Gorenstein.*

We refer the reader to [5] for the definition of canonical and Gorenstein singularities. For $N = 1$, the statement of Theorem 1 is false for example when $X = \mathbb{P}^1$, $n = 3$. Note that if the singularities were Gorensitein, the statement of Theorem 1 would be trivial, since quotient singularities are rational and Gorenstein rational singularities are canonical, (see Corollaries 11.13, 11.14 of [3] and Proposition 5.13 of [4]). Counterexamples to the Gorenstein property will be clarified in the process of our proof.

Corollary 2. *Let X be a unirational smooth projective variety of dimension $N > 1$. For $m \in \mathbb{N}$ where mK_Z forms a Cartier divisor on Z (which exist since Z is a quotient of a smooth variety by a finite group, see for example,* [5]*), we have the following vanishing of global sections:*

$$\Gamma(Z, mK_Z) = 0.$$

Proof. Since $Z = \overline{F(X,n)}/G$ has canonical singularities by Theorem 1, for a resolution of singularities

$$f : Y \to Z,$$

we obtain

$$K_Y = f^*K_Z + \sum m_i E_i$$

where E_i denote exceptional divisors and $m \cdot m_i \in \mathbb{Z}_{\geq 0}$. Thus,

$$\Gamma(Z, mK_Z) = \Gamma(Y, f^*mK_Z) = \Gamma(Y, mK_Y - \sum m \cdot m_i E_i) \subseteq \Gamma(Y, mK_Y)$$

(the inclusion follows from including sections of divisor greater than or equal to $\sum m \cdot m_i E_i$, an effective divisor). Hence, since Y is unirational we have

$$\Gamma(Y, mK_Y) = 0$$

(see [6], Proposition 3.1). Thus,

$$\Gamma(Z, mK_Z) = 0.$$

□

Acknowledgment: I am very thankful to Professor Mircea Mustata for advice.

2. The Fulton-MacPherson Compactification

The Fulton-MacPherson compactification $\overline{F(X,n)}$ of [1] is obtained from the product X^n by a sequence of blow-ups of strict transforms of the diagonals in X^n performed in a suitable order. We can describe its closed points as follows:

Fix a point $x \in X$ and a finite set S with cardinality $|S| \geq 2$. Then we define an *S-screen at x* by induction on $|S|$. If $|S| = 2$, $S = \{s_1, s_2\}$, an S-screen consists of the data of a pair

$$(x_{s_1}, x_{s_2}) \in F(T_x, 2)/\mathbb{G}_m \ltimes T_x,$$

where we consider the action of the tangent space T_x on $F(T_x, 2)$ by shifting and the action of $\mathbb{G}_m$ on $F(T_x, 2)$ and T_x by scaling, therefore combining to a $\mathbb{G}_m \ltimes T_x$-action.

If S'-screens at x have been defined for $|S'| < |S|$, then an *S-screen at x* consists of a system of non-empty disjoint sets

$$S_1 \amalg \cdots \amalg S_m = S,$$

points

$$(x_1, \ldots, x_m) \in F(T_x, m)/\mathbb{G}_m \ltimes T_x$$

(where, again, the $\mathbb{G}_m \ltimes T_x$-action on the mth configuration space $F(T_x, m)$ is given by the actions of $\mathbb{G}_m$ and T_x by scaling and shifting, respectively), and for each $i \in \{1, \ldots, m\}$ with $|S_i| > 1$, an S_i-screen at x. (Note that for any point $y \in T_x$, the tangent space of T_x at y is again T_x).

Now the closed points of $\overline{F(X,n)}$ correspond bijectively to the data consisting of non-empty disjoint sets $S_1, \ldots, S_m$ such that

$$S_1 \amalg \cdots \amalg S_m = \{1, \ldots, n\}, \tag{1}$$

and an m-tuple

$$(x_1, \ldots, x_m) \in F(X, m) \tag{2}$$

and for each $i \in \{1, \ldots, m\}$ with $|S_i| > 1$, an S_i-screen at x_i.

The algebraic variety $\overline{F(X,n)}$ is defined by performing blow-ups of diagonals in a precisely defined order [1]. For the purposes of the present paper, we only need the following statement:

Proposition 3. *([1]) Locally, analytically, a neighborhood of a point of $\overline{F(X,n)}$ given by (1), (2) and S_i-screens at x_i whenever $|S_i| \geq 2$, is obtained by a sequence of blow-ups of*

$$\prod_{i=1}^{m} T_{x_i}^{S_i}$$

at strict transforms E_S of the loci

$$\Delta_S = \{(x_1, \ldots, x_n) | x_s = x_t \textit{ for } s, t \in S\}$$

over all sets S involved in the definition of any of the S_i-screens, in any order such that E_S is blown up before E_T when $T \subsetneq S$.

$\square$

3. Proof of Theorem 1

We begin by discussing the isotropy groups G which can arise in the action of the symmetric group Σ_n on $\overline{F(X,n)}$. Non-trivial isotropy can occur when permutation action coincides with identification by the actions of $\mathbb{G}_m \ltimes T_x$. The most general case when this happens is as follows:

For a subgroup $H \subseteq \Sigma_S$ (where Σ_S denotes the symmetric group on S), we define an H-*symmetrical* S-screen at x as follows: If $|S| = 2$, then any S-screen at x is Σ_S-symmetrical. If H'-symmetrical S'-screen have been defined for $2 \leq |S'| < |S|$, an H-symmetrical S-screen at x occurs when we have an embeddings

$$\mathbb{Z}/M \subset \mathbb{G}_m \ltimes T_x \tag{3}$$

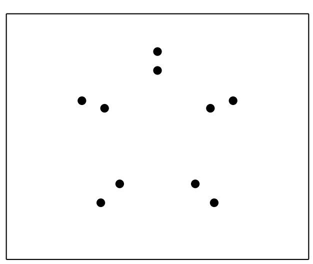
$$\mathbb{Z}/M \subseteq \Sigma_m \tag{4}$$

where the image of the generator (4) consists of only M-cycles and at most one 1-cycle (i_0) acting on the points x_i in the same way as (3), and H_i-symmetrical S_i-screens for representatives of the orbits of (4) transformed to the remaining S_j-screens by (3), where

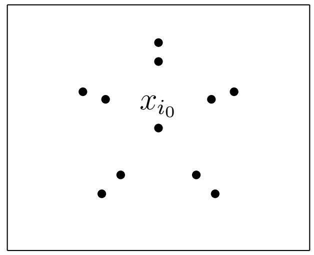
$$G \subseteq \mathbb{Z}/M \wr \prod_{(i)} H_i$$

resp.

$$G \subseteq \left(\mathbb{Z}/M \wr \prod_{(i)} H_i \right) \times H_{i_0} :$$



or



By induction, it follows that

Proposition 4. *A point of $\overline{F(X,n)}$ given by (1), (2) and S_i-screens at x_i, $i \in J$ is G-fixed if and only if*

$$G \subseteq \prod_{i \in J} G_i$$

where each of the S_i-screens at x_i is G_i-symmetrical.

$\square$

Proposition 3 implies that any element of order 2 in G acting on a G-symmetrical screen is a quasi-reflection (since it acts trivially on the blow-up coordinates). Thus, we have

Proposition 5. *Suppose a point $x \in \overline{F(X,n)}$ is G-fixed for $G \subseteq \Sigma_n$. Then the 2-Sylow subgroup $G_{(2)} \subseteq G$ is normal and the small group associated with G in the sense of* [2] *is $G/G_{(2)}$.*

$\square$

Now let the 1-dimensional complex representation of $\mathbb{Z}/M$ where the generator acts by ζ_M^k be denoted by z_M^k. Then the Theorem of Reid, Shephard-Barron, and Tai ([2], Theorem 2.3 (ii)), a quotient singularity of a small group G is canonical if and only if each element $g \in G$ of order M acts by

$$z_M^{k_0} \oplus \cdots \oplus z_M^{k_\ell}$$

where

$$k_0 + \cdots + k_\ell \geq M. \tag{5}$$

Since this condition cannot be spoiled by adding more coordinates, it suffices to consider the case of a G-symmetrical screen where $G = \mathbb{Z}/M$ in (4) acting with only one orbit. Additionally, it suffices to assume that all sets S_i satisfy $|S_i| = 1$ (since otherwise, again, we would only add coordinates).

By Proposition 5, in this case, we have a sum of regular representations

$$\bigoplus_N \mathbb{C}[\mathbb{Z}/M]$$

in which we are blowing up the trivial subrepresentation $\bigoplus_N \mathbb{C}$. This is equivalent to blowing up the origin 0 in the sum of reduced regular representations

$$\bigoplus_N \widetilde{\mathbb{C}[\mathbb{Z}/M]}. \tag{6}$$

Denoting the coordinates of (6) by

$$x_1, \dots, x_{N(M_1)}, \tag{7}$$

the blow-up coordinates can be chosen as

$$x_1, \frac{x_2}{x_1}, \dots, \frac{x_{N(M-1)}}{x_1}. \tag{8}$$

Thus, if we choose the coordinates (7) to be invariant under the $\mathbb{Z}/M$ action, they will represent (6) as

$$\bigoplus_N \bigoplus_{1\leq k\leq M-1} z_M^k. \tag{9}$$

Assuming x_1 correspond to z_M^j, (8) then is represented by

$$z_M^j \oplus \bigoplus_{1\leq k\leq M-1, k\neq j} z_M^{(k-j)\ MOD\ M} \oplus \bigoplus_{N-1} \left(\bigoplus_{1\leq k\leq M-1} z_M^{(k-j)\ MOD\ M} \right). \tag{10}$$

Note that in the first summand of (10), z_M^{M-j} is missing from $\widetilde{\mathbb{C}[\mathbb{Z}/m]}$ (since there is no z_M^0 summand in (9)), and there is an extra z_M^j summand. In the remaining $N-1$ summands of (10), again the summand z_M^{M-j} is missing from $\widetilde{\mathbb{C}[\mathbb{Z}/m]}$ (and the summand z_M^0 occurs instead). This produces the lowest value of $k_1 + \dots k_{N(M-1)}$ when $j = 1$. Even in the case $j = 1$, however, the sum is always $\geq M$ except when $M = 3$ and $N = 1$ (which leads to the counterexample mentioned in the Introduction). One also sees that $k_1 + \dots k_{N(M-1)}$ is not necessarily divisible by M (even in the case $N > 1$), which is the condition for the singularity being Gorenstein by a Theorem of Khinich and Watanabe ([2], Theorem 2.3 (i)). This completes the proof of Theorem 1.

References

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